Journal of Mathematical Physics

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Cite as: J. Math. Phys. **58**, 071707 (2017); https://doi.org/10.1063/1.4990795
Submitted: 06 March 2017 . Accepted: 16 June 2017 . Published Online: 26 July 2017

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JOURNAL OF MATHEMATICAL PHYSICS **58**, 071707 (2017)

The second bosonization of the CKP hierarchy

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(Received 6 March 2017; accepted 16 June 2017; published online 26 July 2017)

In this paper we discuss the second bosonization of the Hirota bilinear equation for the CKP hierarchy introduced in the work of Date *et al.* [J. Phys. Soc. Jpn. **50**(11), 3813–3818 (1981)]. We show that there is a second, untwisted, Heisenberg action on the Fock space, in addition to the twisted Heisenberg action suggested by Date *et al.* [J. Phys. Soc. Jpn. **50**(11), 3813–3818 (1981)] and studied in the work of van de Leur *et al.* [SIGMA **8**, 28 (2012)]. We derive the decomposition of the Fock space into irreducible Heisenberg modules under this action. We show that the vector space spanned by the highest weight vectors of the irreducible Heisenberg modules has a structure of a super vertex algebra, specifically the symplectic fermion vertex algebra. We complete the second bosonization of the CKP Hirota equation by expressing the generating field via exponentiated boson vertex operators acting on a polynomial algebra with two infinite sets of variables. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4990795]

I. INTRODUCTION

The Kadomtsev-Petviashvili (KP) hierarchy is famously associated with the boson-fermion correspondence, a vertex algebra isomorphism between the charged free fermions super vertex algebra and the lattice super vertex algebra of the rank one odd lattice (see, e.g., Ref. 19). One of the aspects of the boson-fermion correspondence is the equivalence between the KP hierarchy of differential equations in the bosonic space and the algebraic Hirota bilinear equation on the fermionic space. Namely, the KP hierarchy can be defined by the following Hirota bilinear equation:

$$Res_z\Big(\psi^+(z)\otimes\psi^-(z)\Big)(\tau\otimes\tau)=0,$$

where $\psi^+(z)$ and $\psi^-(z)$ are the two fermionic fields generating the charged free fermion super vertex algebra (following the notation of Ref. 19) and τ is an element of the Fock space of states of this super vertex algebra (the charge 0 subspace, to be exact). But the KP hierarchy is a hierarchy of differential equations, hence to demonstrate the equivalence with the Hirota bilinear approach one needs to bosonize the fields $\psi^+(z)$ and $\psi^-(z)$, i.e., write them in terms of bosonic (differential) operators. This bosonization was one side of the isomorphism known as the boson-fermion correspondence (there is a vast literature on this, as well as other aspects of the boson-fermion correspondence, see, e.g., Refs. 18, 19, and 24 among many others).

In Refs. 11 and 10, Date, Jimbo, Kashiwara, and Miwa introduced two new hierarchies related to the KP hierarchy: the BKP and the CKP hierarchies. As was the case for the KP hierarchy as well, the BKP and the CKP hierarchies were initially defined via a Lax form instead of Hirota bilinear equation:

$$\frac{\partial L}{\partial x_n}=[(L^n)_+,L],$$

where L is a certain pseudo-differential operator of the form $L=\partial+u_1(x)\partial^{-1}+u_1(x)\partial^{-2}+\cdots$ (see, e.g., Refs. 9 and 24 for details). The connection between the Hirota bilinear equation and the Lax

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forms is given by

$$u_1 = \frac{\partial^2}{\partial x_1} \ln \tau(x).$$

Specifically, both the BKP and the CKP hierarchies were defined as reductions from the KP hierarchy, by assuming conditions on the pseudo-differential operator L used in the Lax form. For both of them, Date, Jimbo, Kashiwara, and Miwa suggested a Hirota bilinear equation, i.e., an operator approach. The Hirota equation approach was later completed for the BKP hierarchy (see Ref. 30 among others). There were no surprises encountered for the BKP case, and similarly to the KP case, the bosonization of the BKP hierarchy was shown to be one of the sides of the boson-fermion correspondence of type B (Refs. 11 and 30), which was later interpreted as an isomorphism of certain twisted vertex (chiral) algebras (Refs. 3 and 4).

In Ref. 10, Date, Jimbo, Kashiwara, and Miwa suggested the following Hirota equation for the CKP hierarchy:

$$Res_z(\chi(z) \otimes \chi(-z))(\tau \otimes \tau) = 0,$$

where the field $\chi(z)$ is actually itself bosonic, with operator product expansion (OPE)

$$\chi(z)\chi(w) \sim \frac{1}{z+w}.$$

Even though the field $\chi(z)$ is bosonic (and thus the algebra generated by its operator coefficients is a Lie algebra, see Sec. II), we still need to bosonize it further in terms of Heisenberg algebra operators, in order to recover the connection with the Lax approach. In Ref. 10, Date, Jimbo, Kashiwara, and Miwa suggested an approach to bosonization, via a twisted Heisenberg field defined by the normal ordered product $\frac{1}{2} : \chi(z)\chi(-z) :$, but did not complete the bosonization. In Ref. 27, van de Leur, Orlov, and Shiota completed this suggested bosonization and derived further properties and applications. The CKP hierarchy though held several surprises, with more yet to come perhaps. The most consequential one so far, and the one we address in this paper, is that the CKP hierarchy admits two different actions of two different Heisenberg algebras, one twisted and one untwisted, and thus two bosonizations of the Hirota equation are possible. The existence of these two different Heisenberg actions was discovered in Ref. 6. The twisted Heisenberg algebra was the one used in Ref. 27. In this paper, we complete the second bosonization, initiated by the second, untwisted, Heisenberg algebra action. We will study the further properties and applications of this bosonization in a consequent paper, but here in this paper we concentrate on the necessary steps to complete the bosonization.

There are 3 stages to any bosonization:

(1) Construct a bosonic Heisenberg current from the generating fields, hence obtaining a field representation of the Heisenberg algebra on the Fock space;
(2) Decompose the Fock space into irreducible Heisenberg modules;
(3) Express the original generating fields in terms of exponential boson fields, if possible.

This paper completes these three stages. In Sec. II, we introduce the required notation, recall the two Heisenberg actions (further on in this paper we will only be concerned with the untwisted Heisenberg action), and introduce two necessary gradings. We then follow through with the decomposition of the Fock space into irreducible Heisenberg modules. Herein lies the second surprise of the CKP hierarchy: although similarly to the KP case there is a charge decomposition of the Fock space (via the charge grading induced by the action of h_0, the 0 component of the untwisted Heisenberg field), unlike for the KP Fock space the charge decomposition is not the same as the decomposition into irreducible modules. Specifically, unlike in the KP case, none of the charge components is irreducible as a Heisenberg module. Here indeed the Fock space is completely reducible, but the vector space spanned by the highest weight vectors of the Heisenberg modules has a much more detailed and fine structure. (This is true for the first bosonization completed in Ref. 27 as well.) We show in Proposition 2.3 that the indexing set $\mathfrak{P}_{tdo}$ for the highest weight vectors (and thus for the irreducible Heisenberg modules in the decomposition) consists of the distinct partitions with a triangular part plus a distinct subpartition of odd half integers, namely,

$$\mathfrak{P}_{tdo} = \{\mathfrak{p} = (T_m, \lambda_1, \lambda_2, \ldots, \lambda_k) \mid T_m - \text{triangular number}, \lambda_1 > \lambda_2 > \cdots > \lambda_k, \lambda_i \in \frac{1}{2} + \mathbb{Z}_{\geq 0}, i = 1, \ldots, k\}.$$

Further, as we show in Sec. III, the space of highest weight vectors has a structure of a super vertex algebra, and specifically a structure realizing the symplectic fermion super vertex algebra, introduced first in Refs. 21 and 22 (see also triplet vertex algebra and, e.g., Refs. 28, 1, and 2). Finally, in Sec. IV, by using the known embedding of the symplectic fermion vertex algebra as a subalgebra (the "small" subalgebra) of the charged free fermion vertex algebra (following Refs. 14 and 21), and thus via the boson-fermion correspondence to a lattice vertex algebra, we can express the generating field $\chi(z)$ via exponentiated boson vertex operators acting on a polynomial algebra with two infinite sets of variables.

II. HEISENBERG ACTION AND MODULE DECOMPOSITION

We will use common concepts and technical tools from the areas of vertex algebras and conformal field theory, such as the notions of field, locality, Operator Product Expansions (OPEs), normal ordered products, Wick's Theorem, for which we refer the reader to any book on the topic (such as Refs. 12 and 19). We will also use the extension of these technical tools to the case of N-point locality, as introduced in Ref. 7.

The starting point is the twisted neutral boson field $\chi(z)$,

$$\chi(z) = \sum_{n\in\mathbb{Z}+1/2} \chi_n z^{-n-1/2}, \tag{2.1}$$

with OPE

$$\chi(z)\chi(w) \sim \frac{1}{z+w}. \tag{2.2}$$

This OPE determines the commutation relations between the modes χ_n, $n \in \mathbb{Z} + 1/2$, as

$$[\chi_m, \chi_n] = (-1)^{m-\frac{1}{2}}\delta_{m,-n}1. \tag{2.3}$$

The modes of the field $\chi(z)$ form a Lie algebra which we denote by L_χ. Let F_χ be the Fock module of L_χ with vacuum vector $|0\rangle$, such that $\chi_n|0\rangle = 0$ for $n > 0$. The vector space F_χ has a basis

$$\{(\chi_{-j_k})^{m_k} \dots (\chi_{-j_2})^{m_2}(\chi_{-j_1})^{m_1}|0\rangle \mid j_k > \dots > j_2 > j_1 > 0,\ j_i \in \mathbb{Z}+\frac{1}{2},\ m_i > 0, m_i \in \mathbb{Z},\ i = 1, 2, \dots, k\}. \tag{2.4}$$

Thus with our indexing F_χ is isomorphic to the Fock space F of Ref. 27.

We use here an indexing of the field $\chi(z)$ typical of vertex algebra fields (as opposed to Ref. 10, where it was introduced originally, or Ref. 27). The field $\chi(z)$ is related to the double-infinite rank Lie algebra c_∞ (see, e.g., Refs. 20, 29, and 7); consequently, it is denoted by $\phi^C(z)$ in Ref. 7.

In Ref. 10, Date, Jimbo, Kashiwara, and Miwa introduced the CKP hierarchy through a reduction of the KP hierarchy and suggested the following algebraic Hirota bilinear equation associated with it:

$$Res_z(\chi(z) \otimes \chi(-z))(\tau \otimes \tau) = 0. \tag{2.5}$$

Here τ is an element of the Fock space F_χ (in fact, τ may need to be an element of a certain completion of F_χ, as we will discuss in a consequent paper about the solutions to this Hirota equation).

In order to relate this purely algebraic Hirota equation to a system of differential equations, we need to bosonize it. As outlined in the Introduction, the bosonization will proceed in 3 stages. The first surprise presented by the CKP case is that, as we showed in Ref. 6, there is a second Heisenberg field generated by the field $\chi(z)$ and its descendant field $\chi(-z)$, and therefore two different bosonizations of the algebraic Hirota equation are possible:

Proposition 2.1. I. Let $h_\chi^{\mathbb{Z}+1/2}(z) = \frac{1}{2} :\chi(z)\chi(-z):$. *We have* $h_\chi^{\mathbb{Z}+1/2}(-z) = h_\chi^{\mathbb{Z}+1/2}(z)$, *and we index* $h_\chi^{\mathbb{Z}+1/2}(z)$ *as* $h_\chi^{\mathbb{Z}+1/2}(z) = \sum_{n\in\mathbb{Z}+1/2} h_n^{\mathbb{Z}+1/2} z^{-2n-1}$. *The field* $h_\chi^{\mathbb{Z}+1/2}(z)$ *has OPE with itself given by*

$$h_\chi^{\mathbb{Z}+1/2}(z)h_\chi^{\mathbb{Z}+1/2}(w) \sim -\left(\frac{z^2+w^2}{2(z^2-w^2)}\right)^2 \sim -\frac{1}{4}\frac{1}{(z-w)^2} - \frac{1}{4}\frac{1}{(z+w)^2}, \tag{2.6}$$

and its modes, $h_n^{\mathbb{Z}+1/2}$, $n \in \mathbb{Z} + 1/2$, *generate a* ***twisted*** *Heisenberg algebra* $\mathcal{H}_{\mathbb{Z}+1/2}$ *with relations* $[h_m^{\mathbb{Z}+1/2}, h_n^{\mathbb{Z}+1/2}] = -m\delta_{m+n,0}1$, $m, n \in \mathbb{Z} + 1/2$.

II. Let $h^{\mathbb{Z}}_{\chi}(z) = \frac{1}{4z}(:\chi(z)\chi(z): - :\chi(-z)\chi(-z):)$*. We have* $h^{\mathbb{Z}}_{\chi}(-z) = h^{\mathbb{Z}}_{\chi}(z)$*, and we index* $h^{\mathbb{Z}}_{\chi}(z)$ *as* $h^{\mathbb{Z}}_{\chi}(z) = \sum_{n\in\mathbb{Z}} h^{\mathbb{Z}}_{n} z^{-2n-2}$*. The field* $h^{\mathbb{Z}}_{\chi}(z)$ *has OPE with itself given by*

$$h^{\mathbb{Z}}_{\chi}(z)h^{\mathbb{Z}}_{\chi}(w) \sim -\frac{1}{(z^2-w^2)^2}, \tag{2.7}$$

and its modes, $h^{\mathbb{Z}}_{n},\ n\in\mathbb{Z}$*, generate an* ***untwisted*** *Heisenberg algebra* $\mathcal{H}_{\mathbb{Z}}$ *with relations* $[h^{\mathbb{Z}}_{m}, h^{\mathbb{Z}}_{n}] = -m\delta_{m+n,0}1,\ m,n\in\mathbb{Z}$*.*

The bosonization initiated by the twisted Heisenberg current from the above proposition is studied in Ref. 27. In this paper we study the second bosonization, initiated by the untwisted Heisenberg current. For simplicity, from now on, we will denote the untwisted field $h^{\mathbb{Z}}_{\chi}(z)$ by $h_{\chi}(z)$ and its modes by $h_n,\ n\in\mathbb{Z}$.

For the second step in the bosonization process, we first need to show that the Heisenberg algebra representation on F_{χ} is in fact completely reducible. It is immediate that the representation is a bounded field representation (see, e.g., Theorem 3.5 of Ref. 19), and we just need to show that h_0 is diagonalizable. To that effect, we need to introduce various gradings on F_{χ}. There are at least two types of natural gradings: the first one necessarily derived from the Heisenberg field, specifically from the action of h_0, and the second from one of the families of Virasoro fields that we discussed in Ref. 6.

We first introduce a normal ordered product $:\chi_m\chi_n:$ on the modes χ_m of the field $\chi(z)$, compatible with the normal ordered product of fields, by

$$:\chi(z)\chi(w): = \sum_{m,n\in\mathbb{Z}+1/2} :\chi_m\chi_n: z^{-m-1/2}w^{-n-1/2} = \sum_{m,n\in\mathbb{Z}} :\chi_{-m-\frac{1}{2}}\chi_{-n-\frac{1}{2}}: z^m w^n, \tag{2.8}$$

and thus for $m,n\in\mathbb{Z}$, this results in the usual "move annihilation operators to the right" approach,

$$:\chi_{-m-\frac{1}{2}}\chi_{-n-\frac{1}{2}}: = \chi_{-m-\frac{1}{2}}\chi_{-n-\frac{1}{2}} \quad \text{for } m+n\neq 1,$$
$$:\chi_{-m-\frac{1}{2}}\chi_{-n-\frac{1}{2}}: = \chi_{-m-\frac{1}{2}}\chi_{-n-\frac{1}{2}} - (-1)^{m-\frac{1}{2}} = \chi_{-n-\frac{1}{2}}\chi_{-m-\frac{1}{2}} \quad \text{for } m+n=-1, n\geq 0,$$
$$:\chi_{-m-\frac{1}{2}}\chi_{-n-\frac{1}{2}}: = \chi_{-m-\frac{1}{2}}\chi_{-n-\frac{1}{2}} \quad \text{for } m+n=-1, m\geq 0.$$

Hence we can express the modes of the field $h_{\chi}(z) = \sum_{n\in\mathbb{Z}} h_n z^{-2n-2}$ as follows:

$$h_n = \frac{1}{2}\sum_{k\in\mathbb{Z}+1/2} :\chi_k\chi_{2n-k}: = \frac{1}{2}\sum_{i\in\mathbb{Z}} :\chi_{-i-\frac{1}{2}}\chi_{2n+i+\frac{1}{2}}:. \tag{2.9}$$

In particular, we have

$$h_0 = \sum_{k\in\mathbb{Z}_{\geq 0}+1/2} :\chi_{-k}\chi_k: = \chi_{-\frac{1}{2}}\chi_{\frac{1}{2}} + \chi_{-\frac{3}{2}}\chi_{\frac{3}{2}} + \cdots. \tag{2.10}$$

Hence it follows that on a monomial $(\chi_{-j_k})^{m_k}\dots(\chi_{-j_2})^{m_2}(\chi_{-j_1})^{m_1}|0\rangle$ in F_{χ}, we have

$$h_0((\chi_{-j_k})^{m_k}\dots(\chi_{-j_2})^{m_2}(\chi_{-j_1})^{m_1}|0\rangle) = \Big(\sum_{j_i\in 2\mathbb{Z}+1/2} m_i - \sum_{j_i\in 2\mathbb{Z}-1/2} m_i\Big) \times ((\chi_{-j_k})^{m_k}\dots(\chi_{-j_2})^{m_2}(\chi_{-j_1})^{m_1}|0\rangle). \tag{2.11}$$

This shows that h_0 is diagonalizable, and thus the Heisenberg algebra representation on F_{χ} is completely reducible. It also gives F_{χ} a $\mathbb{Z}$ grading, which we will call *charge* and denote *chg* (as it is similar to the charge grading in the usual boson-fermion correspondence of type A, i.e., the bosonization related to the KP hierarchy),

$$chg(|0\rangle) = 0; \quad chg\Big(\big(\chi_{-j_k}\big)^{m_k}\dots\big(\chi_{-j_2}\big)^{m_2}\big(\chi_{-j_1}\big)^{m_1}|0\rangle\Big) = \sum_{j_i\in 2\mathbb{Z}+1/2} m_i - \sum_{j_i\in 2\mathbb{Z}-1/2} m_i. \tag{2.12}$$

Example: $chg(\chi_{-\frac{1}{2}}|0\rangle) = 1$; $chg(\chi_{-\frac{3}{2}}|0\rangle) = -1$; $chg(\chi_{-\frac{3}{2}}\chi_{-\frac{1}{2}}|0\rangle) = 0$.

Denote the linear span of monomials of charge n by $F^{(n)}_{\chi}$. The Fock space F_{χ} has a charge decomposition

$$F_{\chi} = \oplus_{n\in\mathbb{Z}} F^{(n)}_{\chi}.$$

In the usual boson-fermion correspondence (of type A), the charge decomposition is in fact the decomposition of the Fock space in terms of irreducible Heisenberg modules (see, e.g., Theorem 5.1 of Ref. 19 as well as the more detailed descriptions in Refs. 18 and 24), i.e., each charge component $F_{\chi}^{(n)}$ is in fact a Heisenberg irreducible module. This is not the case here, for example, the vector

$$v_{4;0} := 2\chi_{-\frac{7}{2}}\chi_{-\frac{1}{2}}|0\rangle - 2\chi_{-\frac{5}{2}}\chi_{-\frac{5}{2}}|0\rangle - \chi^2_{-\frac{3}{2}}\chi^2_{-\frac{1}{2}}|0\rangle \tag{2.13}$$

is of charge 0, but we can also directly check that $h_n v_{4;0} = 0$ for any $n > 0$. Thus $v_{4;0}$ is another highest weight vector of charge 0 for the action of the Heisenberg algebra, besides the vacuum $|0\rangle$. Therefore the charge 0 component $F_{\chi}^{(0)}$ is not irreducible as a Heisenberg module, in contrast to the usual boson-fermion correspondence (of type A). Similarly, neither are the other charge components, as we shall see.

Next, there is a $\frac{1}{2}\mathbb{Z}$ grading on F_{χ}, we will call *degree* and denote by *deg*, which we obtain by using one of the three families of Virasoro fields that were discussed in Ref. 6. In Ref. 6 ,we introduced the descendent fields $\beta_{\chi}(z^2), \gamma_{\chi}(z^2)$ defined by

$$\beta_{\chi}(z^2) = \frac{\chi(z) - \chi(-z)}{2z}, \quad \gamma_{\chi}(z^2) = \frac{\chi(z) + \chi(-z)}{2}. \tag{2.14}$$

These fields have OPEs,

$$\beta_{\chi}(z^2)\beta_{\chi}(w^2) \sim 0, \quad \gamma_{\chi}(z^2)\gamma_{\chi}(w^2) \sim 0, \quad \beta_{\chi}(z^2)\gamma_{\chi}(w^2) \sim \frac{1}{z^2 - w^2}, \quad \gamma_{\chi}(z^2)\beta_{\chi}(w^2) \sim -\frac{1}{z^2 - w^2}. \tag{2.15}$$

In particular, we have

$$\beta_{\chi}(z^2) = \sum_{m\in\mathbb{Z}} \chi_{-2m+\frac{1}{2}}(z^2)^{m-1}, \quad \gamma_{\chi}(z^2) = \sum_{m\in\mathbb{Z}} \chi_{-2m-\frac{1}{2}}(z^2)^{m}. \tag{2.16}$$

Hence, we can translate the following Virasoro field (Ref. 6) from the $\beta - \gamma$ system

$$L_2^{\beta\gamma;\,(\lambda,\mu)}(z) = \lambda : (\partial_z\beta(z))\,\gamma(z) : + (\lambda+1) : \beta(z)\,(\partial_z\gamma(z)) : + \frac{\mu}{z} : \beta(z)\gamma(z) : + \frac{(2\lambda+1)\mu - \mu^2}{2z^2} \tag{2.17}$$

into a Virasoro action on F_{χ}. For simplicity, we will consider only the case $\mu = 0$, and we have

$$L^{\lambda}(z^2) = \sum_{n\in\mathbb{Z}} L_n (z^2)^{-n-2} = -\sum_{n\in\mathbb{Z}} \Big(\sum_{k+l=n} (\lambda(k+1) + (\lambda+1)l) : \chi_{2k+\frac{1}{2}}\chi_{2l-\frac{1}{2}} : \Big) (z^2)^{-n-2}, \tag{2.18}$$

in particular,

$$L_0^{\lambda} = -\sum_{k\in\mathbb{Z}} (\lambda + k) : \chi_{-2k+\frac{1}{2}}\chi_{2k-\frac{1}{2}} : . \tag{2.19}$$

We can further vary λ ($\lambda = -\frac{1}{2}$ is usually chosen in conformal field theory), but a useful choice here is $\lambda = -\frac{1}{4}$. In that case, we have a central charge $c = -\frac{1}{4}$, with

$$L_0 = \frac{1}{2}\Big(\frac{1}{2} : \chi_{-\frac{1}{2}}\chi_{\frac{1}{2}} : - \frac{3}{2} : \chi_{-\frac{3}{2}}\chi_{\frac{3}{2}} : + \frac{5}{2} : \chi_{-\frac{5}{2}}\chi_{\frac{5}{2}} : - \cdots\Big). \tag{2.20}$$

Hence

$$\begin{aligned} L_0\Big((\chi_{-j_k})^{m_k} \ldots (\chi_{-j_2})^{m_2}(\chi_{-j_1})^{m_1}|0\rangle\Big) &= \frac{1}{2}(m_k \cdot j_k + \cdots m_2 \cdot j_2 + m_1 \cdot j_1) \\ &\quad \times \Big((\chi_{-j_k})^{m_k} \ldots (\chi_{-j_2})^{m_2}(\chi_{-j_1})^{m_1}|0\rangle\Big). \end{aligned} \tag{2.21}$$

Discarding the factor of $\frac{1}{2}$, we have the *deg* grading on F_{χ} (also used in Ref. 27),

$$deg(|0\rangle) = 0, \ deg\chi_{-j}|0\rangle = j, \ deg\Big(\big(\chi_{-j_k}\big)^{m_k} \ldots \big(\chi_{-j_2}\big)^{m_2}\big(\chi_{-j_1}\big)^{m_1}|0\rangle\Big) = m_k \cdot j_k + \cdots m_2 \cdot j_2 + m_1 \cdot j_1, \tag{2.22}$$

where $j_k > \cdots > j_2 > j_1 > 0$, $j_i \in \mathbb{Z} + \frac{1}{2}$, $m_i > 0, m_i \in \mathbb{Z}$, $i = 1, 2, \ldots, k$. Consequently we have a degree decomposition of F_χ as in Ref. 27, which now we know is actually derived from the Virasoro operator component L_0. The formal character is given by

$$dim_q F_\chi := tr_{F_\chi} q^{2L_0} = \sum_{k\in\frac{1}{2}\mathbb{Z}} dim\Big(span\{v \in F_\chi \mid deg(v) = k\}\Big) q^k. \tag{2.23}$$

We can also form the character with respect to both the L_0 and h_0 grading operators (they are both diagonalizable),

$$dim_{q,z} F_\chi := tr_{F_\chi} q^{2L_0} z^{h_0}. \tag{2.24}$$

Now observing that acting by $\chi_{-2j-\frac{1}{2}}$, $j \geq 0$, on a monomial $\left(\chi_{-j_k}\right)^{m_k} \ldots \left(\chi_{-j_2}\right)^{m_2} \left(\chi_{-j_1}\right)^{m_1} |0\rangle$ will produce a factor of $z^{+1} q^{2j+\frac{1}{2}}$, and acting by $\chi_{-2j+\frac{1}{2}}$, $j \geq 1$, will produce a factor of $z^{-1} q^{2j-\frac{1}{2}}$, it is immediate that

$$dim_{q,z} F_\chi = \frac{1}{\prod_{j\in\mathbb{Z}_+} (1 - z q^{2j-\frac{3}{2}})(1 - z^{-1} q^{2j-\frac{1}{2}})}. \tag{2.25}$$

The formula

$$dim_q F_\chi = \frac{1}{\prod_{j\in\mathbb{Z}_+} (1 - q^{\frac{2j-1}{2}})} \tag{2.26}$$

of Ref. 27 then follows from setting $z = 1$ in (2.25).

Lemma 2.2. The following relations hold:

$$[L_0, h_m] = -m h_m, \quad \forall\ m \in \mathbb{Z},$$

and thus for any $v \in F_\chi$, we have

$$deg(h_{-m} v) = 2m + deg(v), \quad \forall\ m \in \mathbb{Z}_+. \tag{2.27}$$

Proof. By using the relation with the $\beta\gamma$ system, we can calculate the OPE between $L(z^2) = -\frac{1}{4} : \left(\partial_{z^2} \beta(z^2)\right) \gamma(z^2) : + \frac{3}{4} : \beta(z^2) \left(\partial_{z^2} \gamma(z^2)\right) :$ and $h_\chi(z) =: \beta(z^2)\gamma(z^2) :$ via Wick's Theorem. The calculations are straightforward. □

Since the conditions of Theorem 3.5 of Ref. 19 are satisfied, the Heisenberg module F_χ is completely reducible, and is a direct sum of irreducible highest weight Heisenberg modules, each isomorphic to

$$\mathbb{C}[h_{-1}, h_{-2}, \ldots, h_{-n}, \ldots] \cdot v$$

for some highest weight vector v, for which $h_n v = 0$ for any $n > 0$. It is a well known fact (see, e.g., Refs. 18 and 12) that any irreducible highest weight module of the Heisenberg algebra $\mathcal{H}_\mathbb{Z}$ introduced in Sec. II is isomorphic to the polynomial algebra with infinitely many variables $B_\lambda \cong \mathbb{C}[x_1, x_2, \ldots, x_n, \ldots]$ where $v \mapsto 1$ and

$$h_n \mapsto i\partial_{x_n}, \quad h_{-n} \mapsto i n x_n \cdot, \quad \text{for any } n \in \mathbb{N}, \quad h_0 \mapsto \lambda\cdot,\ \lambda \in \mathbb{C}. \tag{2.28}$$

In fact, we can introduce an arbitrary re-scaling $s_n \neq 0$, $s_n \in \mathbb{C}$, for $n \neq 0$ only, so that

$$h_n \mapsto i s_n \partial_{x_n}, \quad h_{-n} \mapsto i s_n^{-1} n x_n \cdot, \quad \text{for any } n \in \mathbb{N}, \quad h_0 \mapsto \lambda \cdot . \tag{2.29}$$

Thus each of the irreducible modules in our Heisenberg decomposition is isomorphic to $B_\lambda \cong \mathbb{C}[x_1, x_2, \ldots, x_n, \ldots]$ for some $\lambda \in \mathbb{C}$ determined by the charge of the highest weight vector generating the module. Now if v is a highest weight vector, which induces an irreducible module $V = \mathbb{C}[h_{-1}, h_{-2}, \ldots, h_{-n}, \ldots] \cdot v \cong B_\lambda$, then as a consequence of (2.27) V has graded dimension

$$dim_q V = \frac{q^{deg(v)}}{\prod_{i\in\mathbb{Z}_+} (1 - q^{2i})}.$$

Since F_χ is a direct sum of such irreducible modules, we have

$$dim_q F_\chi = \frac{\sum_{\mathfrak{p}\in\mathfrak{P}_{tdo}} q^{deg(v_\mathfrak{p})}}{\prod_{i\in\mathbb{Z}_+} (1 - q^{2i})},$$

where the summation is over an as yet unknown indexing set $\mathcal{P}_{tdo}$. By comparing this formula for the graded dimension with (2.26), we have

$$\sum_{\mathfrak{p}\in\mathfrak{P}_{tdo}} q^{deg(v_\mathfrak{p})} = \frac{\prod_{i\in\mathbb{Z}_+}(1-q^{2i})}{\prod_{i\in\mathbb{Z}_+}(1-q^{\frac{2i-1}{2}})} = \frac{\prod_{i\in\mathbb{Z}_+}(1-q^{2i})\prod_{i\in\mathbb{Z}_+}(1+q^{\frac{2i-1}{2}})}{\prod_{i\in\mathbb{Z}_+}(1-q^{2i-1})}.$$

Now using the Jacobi triple product identity in one of its forms

$$\prod_{i=1}^{\infty}(1-q^i)(1+zq^{i-1})(1+z^{-1}q^i) = \sum_{m\in\mathbb{Z}} z^m q^{\frac{m(m-1)}{2}}, \tag{2.30}$$

we have, by setting $z = 1$,

$$2\sum_{m\in\mathbb{Z}_{\geq 0}} q^{T_m} = 2\prod_{i=1}^{\infty}(1-q^{2i})(1+q^i) = 2\frac{\prod_{i=1}^{\infty}(1-q^{2i})(1-q^{2i-1})(1+q^i)}{\prod_{i=1}^{\infty}(1-q^{2i-1})} = 2\frac{\prod_{i=1}^{\infty}(1-q^i)(1+q^i)}{\prod_{i=1}^{\infty}(1-q^{2i-1})},$$

where T_m denotes the mth triangular number—$T_m := 1 + 2 + \cdots + m = \frac{m(m+1)}{2}$, with $T_0 = 0$. And so we re-derived a known formula for the triangular numbers

$$\sum_{m\in\mathbb{Z}_{\geq 0}} q^{T_m} = 1 + q + q^3 + q^6 + q^{10} + \cdots + q^{T_m} + \cdots = \frac{\prod_{i=1}^{\infty}(1-q^{2i})}{\prod_{i=1}^{\infty}(1-q^{2i-1})}.$$

We knew this identity went far back in time but could not find the original reference for this formula until the referee pointed it out: this identity can be found on page 185 of the original manuscript by Jacobi, Ref. 17. For some reason this identity keeps being re-derived, including by better number theorists than this author, see e.g. Proposition 1 of Ref. 25; we continued this trend by re-deriving it here.

Using this formula, we have

$$\sum_{\mathfrak{p}\in\mathfrak{P}_{tdo}} q^{deg(v_\mathfrak{p})} = \Big(\sum_{m\in\mathbb{Z}_{\geq 0}} q^{T_m}\Big)\cdot\prod_{i\in\mathbb{Z}_+}(1+q^{\frac{2i-1}{2}}). \tag{2.31}$$

Since the right-hand side is now a sum with positive coefficients, it determines the indexing set $\mathfrak{P}_{tdo}$, namely, it consists of distinct partitions of the type

$$\mathfrak{P}_{tdo} = \{\mathfrak{p} = (T_m, \lambda_1, \lambda_2, \ldots, \lambda_k) \mid T_m - \text{triangular number}, \lambda_1 > \lambda_2 > \cdots > \lambda_k, \lambda_i \in \frac{1}{2}+\mathbb{Z}_{\geq 0},\ i = 1, \ldots, k\}. \tag{2.32}$$

As usual, the weight $|\mathfrak{p}|$ of such a partition $\mathfrak{p}$ is the sum of its parts, $|\mathfrak{p}| := T_m + \lambda_1 + \lambda_2 + \cdots + \lambda_k$.

Hence we arrive at the following proposition, which provides the decomposition of F_χ into irreducible Heisenberg modules, thus completing the second step in the process of bosonization.

Proposition 2.3. For the action of the Heisenberg algebra $\mathcal{H}_\mathbb{Z}$ on F_χ, the number of highest weight vectors of degree $n \in \frac{1}{2}\mathbb{Z}$ equals the number of partitions $\mathfrak{p} \in \mathfrak{P}_{tdo}$ of weight n. Thus as Heisenberg modules

$$F_\chi \cong \oplus_{\mathfrak{p}\in\mathfrak{P}_{tdo}}\mathbb{C}[x_1, x_2, \ldots, x_n, \ldots]. \tag{2.33}$$

Example 2.4. We can calculate the highest weight vectors of given degree by brute force. For the first few degrees, we have

$$\begin{aligned}\sum_{\mathfrak{p}\in\mathfrak{P}_{tdo}} q^{deg(v_\mathfrak{p})} &= \Big(\sum_{m\in\mathbb{Z}_{\geq 0}} q^{T_m}\Big)\cdot\prod_{i\in\mathbb{Z}_+}(1+q^{\frac{2i-1}{2}})\\ &= (1+q+q^3+q^6+\cdots)(1+q^{\frac{1}{2}}+q^{\frac{3}{2}}+q^2+q^{\frac{5}{2}}+q^3+q^{\frac{7}{2}}+2q^4+2q^{\frac{9}{2}}+\cdots)\\ &= 1+q^{\frac{1}{2}}+q+2q^{\frac{3}{2}}+q^2+2q^{\frac{5}{2}}+3q^3+3q^{\frac{7}{2}}+3q^4+4q^{\frac{9}{2}}+\cdots.\end{aligned}$$

The corresponding highest weight vectors are (in each degree, the maximum charge of the highest weight vectors starts at twice that degree, and also the charges inside each degree are equivalent modulo 4):

$$
\begin{array}{r|l}
1 & |0\rangle \\
q^{\frac{1}{2}} & \chi_{-\frac{1}{2}}|0\rangle \\
q & \chi^2_{-\frac{1}{2}}|0\rangle \\
2q^{\frac{3}{2}} & \chi^3_{-\frac{1}{2}}|0\rangle, \quad \chi_{-\frac{3}{2}}|0\rangle \\
q^2 & \chi^4_{-\frac{1}{2}}|0\rangle \\
2q^{\frac{5}{2}} & \chi^5_{-\frac{1}{2}}|0\rangle, \quad \chi_{-\frac{3}{2}}\chi^2_{-\frac{1}{2}}|0\rangle + 2\chi_{-\frac{5}{2}}|0\rangle \\
3q^3 & \chi^6_{-\frac{1}{2}}|0\rangle, \quad \chi_{-\frac{3}{2}}\chi^3_{-\frac{1}{2}}|0\rangle + 3\chi_{-\frac{5}{2}}\chi_{-\frac{1}{2}}|0\rangle, \quad \chi^2_{-\frac{3}{2}}|0\rangle \\
3q^{\frac{7}{2}} & \chi^7_{-\frac{1}{2}}|0\rangle, \quad \chi_{-\frac{3}{2}}\chi^4_{-\frac{1}{2}}|0\rangle + 4\chi_{-\frac{5}{2}}\chi^2_{-\frac{1}{2}}|0\rangle, \quad \chi^2_{-\frac{3}{2}}\chi_{-\frac{1}{2}}|0\rangle - 2\chi_{-\frac{7}{2}}|0\rangle \\
3q^4 & \chi^8_{-\frac{1}{2}}|0\rangle, \quad \chi_{-\frac{3}{2}}\chi^5_{-\frac{1}{2}}|0\rangle + 5\chi_{-\frac{5}{2}}\chi^3_{-\frac{1}{2}}|0\rangle, \quad 2\chi_{-\frac{7}{2}}\chi_{-\frac{1}{2}}|0\rangle - 2\chi_{-\frac{5}{2}}\chi_{-\frac{3}{2}}|0\rangle - \chi^2_{-\frac{3}{2}}\chi^2_{-\frac{1}{2}}|0\rangle \\
4q^{\frac{9}{2}} & \chi^9_{-\frac{1}{2}}|0\rangle, \quad \chi_{-\frac{3}{2}}\chi^6_{-\frac{1}{2}}|0\rangle + 6\chi_{-\frac{5}{2}}\chi^4_{-\frac{1}{2}}|0\rangle, \quad \chi^2_{-\frac{3}{2}}\chi^3_{-\frac{1}{2}}|0\rangle + 6\chi_{-\frac{9}{2}}|0\rangle + 6\chi_{-\frac{5}{2}}\chi_{-\frac{3}{2}}\chi_{-\frac{1}{2}}|0\rangle, \quad \chi^3_{-\frac{3}{2}}|0\rangle
\end{array}
.
$$

There are several families of highest weight vectors, for instance, one can easily check that $\chi^n_{-\frac{1}{2}}|0\rangle$, $\chi^n_{-\frac{3}{2}}|0\rangle$, and $\chi_{-\frac{3}{2}}\chi^{n+2}_{-\frac{1}{2}}|0\rangle + (n+2)\chi_{-\frac{5}{2}}\chi^n_{-\frac{1}{2}}|0\rangle$ are highest weight vectors for any $n \geq 0$. Also, observe that at any given weight $\frac{n}{2}$ for $n \in \mathbb{Z}$, $\chi^n_{-\frac{1}{2}}|0\rangle$ is the highest weight vector of the highest charge (n) with that degree.

Remark 2.5. It would be interesting to derive a formula giving a correspondence between a partition $\mathfrak{p} \in \mathfrak{P}_{tdo}$ of weight n and the highest weight vector corresponding to that partition, or even the charge of that highest weight vector. As the weights of the partitions grow, the charges are less straightforward to calculate. For example, at weight $\frac{13}{2}$, there are 7 partitions from $\mathfrak{P}_{tdo}$ and one can calculate by brute force that there is a highest weight vector of charge 13, a highest weight vector of charge 9, two highest weight vectors of charge 5, two highest weight vectors of charge 1, and a highest weight vector of charge −3.

Denote by F^{hwv}_{χ} the vector space spanned by all the highest weight vectors for the Heisenberg action. To accomplish the third step in the bosonization process, in Sec. III we will first show that F^{hwv}_{χ} has a structure realizing the symplectic fermion super vertex algebra.

III. SYMPLECTIC FERMIONS: VERTEX ALGEBRA STRUCTURE ON THE SPACE SPANNED BY THE HEISENBERG HIGHEST WEIGHT VECTORS

As usual, for a rational function $f(z, w)$, with poles only at $z = 0$, $z = \pm w$, we denote by $i_{z,w}f(z, w)$ the expansion of $f(z, w)$ in the region $|z| \gg |w|$ (the region in the complex z plane outside the points $z = 0, \pm w$), and correspondingly for $i_{w,z}f(z, w)$.

Lemma 3.1. The following OPEs hold:

$$h_\chi(z)\beta_\chi(w^2) \sim -\frac{1}{z^2 - w^2}\beta_\chi(w^2), \quad h_\chi(z)\gamma_\chi(w^2) \sim \frac{1}{z^2 - w^2}\gamma_\chi(w^2). \tag{3.1}$$

Proof. By direct application of Wick's Theorem. □

Denote

$$V^-(z) = \exp\Big(-\sum_{n>0}\frac{1}{n}h_n z^{-2n}\Big), \qquad V^+(z) = \exp\Big(\sum_{n>0}\frac{1}{n}h_{-n}z^{2n}\Big). \tag{3.2}$$

Consequently, we will write

$$V^-(z)^{-1} = \exp\Big(\sum_{n>0}\frac{1}{n}h_n z^{-2n}\Big), \qquad V^+(z)^{-1} = \exp\Big(-\sum_{n>0}\frac{1}{n}h_{-n}z^{2n}\Big). \tag{3.3}$$

Lemma 3.2. The following commutation relations hold:

$$[h_\chi(z),V^-(w)]=-\Big(\sum_{n>0}\frac{z^{2n-2}}{w^{2n}}\Big)V^-(w)=-i_{w,z}\frac{1}{w^2-z^2}V^-(w)=i_{w,z}\frac{1}{z^2-w^2}V^-(w), \tag{3.4}$$

$$[h_\chi(z),V^+(w)]=-\Big(\sum_{n>0}\frac{w^{2n}}{z^{2n+2}}\Big)V^+(w)=-i_{z,w}\frac{w^2}{z^2(z^2-w^2)}V^+(w), \tag{3.5}$$

$$V^-(z)V^+(w)=i_{z,w}\frac{z^2}{z^2-w^2}V^+(w)V^-(z), \tag{3.6}$$

$$V^-(z)^{-1}V^+(w)^{-1}=i_{z,w}\frac{z^2}{z^2-w^2}V^+(w)^{-1}V^-(z)^{-1}, \tag{3.7}$$

$$V^-(z)^{-1}V^+(w)=\frac{z^2-w^2}{z^2}V^+(w)V^-(z)^{-1}, \tag{3.8}$$

$$V^-(z)V^+(w)^{-1}=\frac{z^2-w^2}{z^2}V^+(w)^{-1}V^-(z). \tag{3.9}$$

Proof. The proof is by direct calculation on the first two relations. On the other four, we apply the Baker-Campbell-Hausdorff formula, and we will only show it for one of the relations,

$$\begin{aligned}V^-(z)V^+(w)&=\exp\left(-[\sum_{m>0}\frac{1}{m}h_m z^{-2m},\sum_{n>0}\frac{1}{n}h_{-n}w^{2n}]\right)\cdot V^+(w)V^-(z)\\&=\exp\left(\sum_{m>0}\frac{1}{m}\frac{w^{2m}}{z^{2m}}\right)\cdot V^+(w)V^-(z)=i_{z,w}\exp\left(-\ln\left(1-\frac{w^2}{z^2}\right)\right)\cdot V^+(w)V^-(z).\end{aligned}$$

□

Observe that $V^-(z)$ and $V^+(z)$ are actually functions of z^2. With that in mind, we introduce the following fields, which are necessary to complete the bosonization.

Definition 3.3. Define the following fields on F_χ:

$$H^\beta(z^2)=V^+(z)^{-1}\beta_\chi(z^2)z^{-2h_0}V^-(z)^{-1},\qquad H^\gamma(z^2)=V^+(z)\gamma_\chi(z^2)z^{2h_0}V^-(z). \tag{3.10}$$

Thus we have

$$\beta_\chi(z^2)=V^+(z)H^\beta(z^2)V^-(z)z^{2h_0},\qquad \gamma_\chi(z^2)=V^+(z)^{-1}H^\gamma(z^2)V^-(z)^{-1}z^{-2h_0}. \tag{3.11}$$

Remark 3.4. We want to mention that in the case of the usual boson-fermion correspondence (for the KP hierarchy, also known as of type A), one introduces an invertible operator u from the subspace of charge m to the subspace of charge $m+1$ (see, e.g., Ref. 19, Sec. 5.2), mapping the unique—in that case—highest weight vector of charge m to the unique highest weight vector of charge $m+1$. That operator u is then used to define the (simpler) counterparts of $H^\beta(z^2)$ and $H^\gamma(z^2)$. As we saw in Sec. II, in our case the charge components $F_\chi^{(m)}$ are not irreducible, and therefore such an invertible operator u does not exist, at least not as an invertible operator sending a highest weight vector to highest weight vector.

Lemma 3.5. The following commutation relations hold:

$$[h_\chi(z),H^\beta(w^2)]=-\frac{1}{z^2}H^\beta(w^2),\qquad [h_\chi(z),H^\gamma(w^2)]=\frac{1}{z^2}H^\gamma(w^2). \tag{3.12}$$

Therefore $H^\beta(z^2)$ and $H^\gamma(z^2)$ can be considered fields (vertex operators) on F_χ^{hwv}, i.e., *for each $v\in F_\chi^{hwv}$, we have $H^\beta(z^2)v\in F_\chi^{hwv}((z^2))$ and $H^\gamma(z^2)v\in F_\chi^{hwv}((z^2))$.*

Proof. We have

$$[h_\chi(z), H^\beta(w^2)] = [h_\chi(z), V^+(w)^{-1}\beta(w^2)w^{-2h_0}V^-(w)^{-1}]$$
$$= \Big(\sum_{n>0}\frac{z^{2n-2}}{w^{2n}} - i_{z,w}\frac{1}{z^2-w^2} - i_{w,z}\frac{1}{w^2-z^2} + \sum_{n>0}\frac{w^{2n}}{z^{2n+2}}\Big)H^\beta(w^2) = -\frac{1}{z^2}H^\beta(w^2).$$

Hence we observe that

$$[h_n, H^\beta(w^2)] = 0, \quad \text{for any} \quad n\neq 0, \quad [h_0, H^\beta(w^2)] = -H^\beta(w^2). \tag{3.13}$$

We can similarly observe that

$$[h_\chi(z), H^\gamma(w^2)] = [h_\chi(z), V^+(w)\gamma(w^2)w^{2h_0}V^-(w)]$$
$$= \Big(-\sum_{n>0}\frac{z^{2n-2}}{w^{2n}} + i_{z,w}\frac{1}{z^2-w^2} + i_{w,z}\frac{1}{w^2-z^2} - \sum_{n>0}\frac{w^{2n}}{z^{2n+2}}\Big)H^\gamma(w^2) = \frac{1}{z^2}H^\gamma(w^2).$$

Thus

$$[h_n, H^\gamma(w^2)] = 0, \quad \text{for any} \quad n\neq 0, \quad [h_0, H^\gamma(w^2)] = H^\gamma(w^2). \tag{3.14}$$

Now let v be a highest weight vector, i.e., $v\in F_\chi^{hwv}$; from (3.13), it is clear that

$$h_nH^\beta(z^2)v = H^\beta(z^2)h_nv = 0, \quad \text{for any} \quad n>0.$$

Hence the coefficients of $H^\beta(z^2)v$ are in fact highest weight vectors themselves, i.e., $H^\beta(z^2)v\in F_\chi^{hwv}((z^2))$ (instead of the more general $F_\chi((z))$). Therefore we can view the field $H^\beta(z^2)$ as a field on F_χ^{hwv}, instead of more generally on F_χ, and similarly for $H^\gamma(z^2)$. □

As mentioned above, in the case of the boson-fermion correspondence of type A (the bosonization of the KP hierarchy), the counterparts of the fields $H^\beta(z^2)$ and $H^\gamma(z^2)$ are the simple operators u^{-1} and u, see, e.g., Ref. 19, Sec. 5.2 (which can be identified with $e^{-\alpha}$ and e^α if one identifies the vector space of highest weight vectors in that case with $\mathbb{C}[e^\alpha, e^{-\alpha}]$). In particular, there the operators u^{-1} and u are actually independent of z. This is not the case for the fields $H^\beta(z^2)$ and $H^\gamma(z^2)$, as we will show.

Proposition 3.6. The following commutation relations hold:

$$\{H^\beta(z^2), H^\gamma(w^2)\} = i_{z,w}\frac{1}{(z^2-w^2)^2} - i_{w,z}\frac{1}{(w^2-z^2)^2}, \tag{3.15}$$

$$\{H^\beta(z^2), H^\beta(w^2)\} = 0, \quad \{H^\gamma(z^2), H^\gamma(w^2)\} = 0. \tag{3.16}$$

Here we use the notation $\{A, B\} := AB + BA$ *for two operators* A, B.

If we use the delta function notation (see Ref. 19),

$$\delta(z,w) := \sum_{n\in\mathbb{Z}}\frac{z^n}{w^{n+1}} = i_{z,w}\frac{1}{z-w} + i_{w,z}\frac{1}{w-z},$$

the nontrivial commutation relation in the proposition above can be written as

$$\{H^\beta(z^2), H^\gamma(w^2)\} = \partial_{w^2}\delta(z^2,w^2); \quad \{H^\gamma(z^2), H^\beta(w^2)\} = \partial_{z^2}\delta(z^2,w^2).$$

For the proof of this proposition, we need the following.

Lemma 3.7. The following commutation relations hold:

$$\beta_\chi(z^2)V^+(w) = i_{z,w}\frac{z^2}{z^2-w^2}V^+(w)\beta_\chi(z^2), \quad \beta_\chi(z^2)V^+(w)^{-1} = \frac{z^2-w^2}{z^2}V^+(w)^{-1}\beta_\chi(z^2), \tag{3.17}$$

$$\beta_\chi(z^2)V^-(w) = \frac{w^2-z^2}{w^2}V^-(w)\beta_\chi(z^2), \quad \beta_\chi(z^2)V^-(w)^{-1} = i_{w,z}\frac{w^2}{w^2-z^2}w^2V^-(w)^{-1}\beta_\chi(z^2), \tag{3.18}$$

$$\gamma_\chi(z^2)V^+(w) = \frac{z^2-w^2}{z^2}V^+(w)\gamma_\chi(z^2), \quad \gamma_\chi(z^2)V^+(w)^{-1} = i_{z,w}\frac{z^2}{z^2-w^2}V^+(w)^{-1}\gamma_\chi(z^2), \tag{3.19}$$

$$\gamma_\chi(z^2)V^-(w) = \frac{w^2-z^2}{w^2}V^-(w)\gamma_\chi(z^2), \quad \gamma_\chi(z^2)V^-(w)^{-1} = i_{w,z}\frac{w^2}{w^2-z^2}V^-(w)^{-1}\gamma_\chi(z^2). \tag{3.20}$$

Proof. From the definition of $H^{\beta}(z^2)$, we have

$$\beta_{\chi}(z^2)V^+(w) = V^+(z)H^{\beta}(z^2)z^{2h_0}V^-(z)V^+(w) = V^+(z)H^{\beta}(z^2)z^{2h_0}i_{z,w}\frac{z^2}{z^2-w^2}V^+(w)V^-(z)$$
$$= i_{z,w}\frac{z^2}{z^2-w^2}V^+(w)V^+(z)H^{\beta}(z^2)z^{2h_0}V^-(z) = i_{z,w}\frac{z^2}{z^2-w^2}V^+(w)\beta_{\chi}(z^2).$$

Here we used both Lemma 3.5, namely, that $H^{\beta}(z^2)$ commutes with both $V^+(w)$ and $V^-(z)$, and Lemma 3.2. Similarly

$$\gamma_{\chi}(z^2)V^-(w) = V^+(z)^{-1}H^{\gamma}(z^2)z^{-2h_0}V^-(z)^{-1}V^-(w) = V^+(z)^{-1}V^-(w)H^{\gamma}(z^2)z^{-2h_0}V^-(z)^{-1}$$
$$= \left(1-\frac{z^2}{w^2}\right)V^-(w)V^+(z)^{-1}H^{\gamma}(z^2)z^{-2h_0}V^-(z)^{-1} = \frac{w^2-z^2}{w^2}V^-(w)\gamma_{\chi}(z^2).$$

The other relations are proved similarly. □

We now return to the proof of the proposition.

Proof. We will prove the first of the nontrivial relations, the other is proved similarly. We use the commutation relations from Lemma 3.2 and commute successively the annihilating $V^-(z)^{-1}$ to the right and the creating $V^+(w)$ to the left,

$$H^{\beta}(z^2)H^{\gamma}(w^2) = V^+(z)^{-1}\beta_{\chi}(z^2)z^{-2h_0}V^-(z)^{-1}V^+(w)\gamma_{\chi}(w^2)w^{2h_0}V^-(w)$$
$$= \frac{z^2-w^2}{z^2}V^+(z)^{-1}\beta_{\chi}(z^2)z^{-2h_0}V^+(w)V^-(z)^{-1}\gamma_{\chi}(w^2)w^{2h_0}V^-(w)$$
$$= \frac{z^2-w^2}{z^2}\cdot i_{z,w}\frac{z^2}{z^2-w^2}V^+(z)^{-1}V^+(w)\beta_{\chi}(z^2)z^{-2h_0}V^-(z)^{-1}\gamma_{\chi}(w^2)w^{2h_0}V^-(w)$$
$$= \frac{z^2-w^2}{z^2}\cdot i_{z,w}\frac{z^2}{z^2-w^2}\cdot i_{z,w}\frac{z^2}{z^2-w^2}V^+(z)^{-1}V^+(w)\beta_{\chi}(z^2)z^{-2h_0}\gamma_{\chi}(w^2)w^{2h_0}$$
$$\times\ V^-(z)^{-1}V^-(w)$$
$$= i_{z,w}\frac{z^2}{z^2-w^2}V^+(z)^{-1}V^+(w)\beta_{\chi}(z^2)z^{-2h_0}\gamma_{\chi}(w^2)w^{2h_0}V^-(z)^{-1}V^-(w).$$

Now we need to interchange $z^{-2h}{}_0$ and $\gamma_{\chi}(w^2)$. From Lemma 3.1, we have $h_0\gamma_{\chi}(w^2) = \gamma_{\chi}(w^2)(h_0+1)$ or we can see directly from

$$\gamma_{\chi}(w^2) = \frac{\chi(w)+\chi(-w)}{2} = \sum_{n\in\mathbb{Z}}\chi_{-2n-1/2}w^{2n} = \cdots + \chi_{3/2}w^{-2} + \chi_{-1/2} + \chi_{-5/2}w^2 + \cdots,$$

in addition to the fact that acting by $\chi_{-2n-1/2}$ adds charge of 1 that

$$z^{-2h_0}\gamma_{\chi}(w^2) = \frac{1}{z^2}\gamma_{\chi}(w^2)z^{-2h_0}.$$

Finally, we have from the OPE of $\beta_{\chi}(z^2)$ with $\gamma_{\chi}(w^2)$, plus the definition of a normal ordered product that

$$\beta_{\chi}(z^2)\gamma_{\chi}(w^2) =: \beta_{\chi}(z^2)\gamma_{\chi}(w^2) : + \frac{1}{z^2-w^2},$$

and so

$$H^{\beta}(z^2)H^{\gamma}(w^2) = i_{z,w}\frac{1}{z^2-w^2}V^+(z)^{-1}V^+(w)\beta_{\chi}(z^2)\gamma_{\chi}(w^2)z^{-2h_0}w^{2h_0}V^-(z)^{-1}V^-(w)$$
$$= i_{z,w}\frac{1}{z^2-w^2}V^+(z)^{-1}V^+(w)\left(:\beta_{\chi}(z^2)\gamma_{\chi}(w^2): + \frac{1}{z^2-w^2}\right)z^{-2h_0}w^{2h_0}V^-(z)^{-1}V^-(w)$$
$$= i_{z,w}\frac{1}{z^2-w^2}V^+(z)^{-1}V^+(w)\left(:\beta_{\chi}(z^2)\gamma_{\chi}(w^2):\right)z^{-2h_0}w^{2h_0}V^-(z)^{-1}V^-(w)$$
$$+\ i_{z,w}\frac{1}{(z^2-w^2)^2}V^+(z)^{-1}V^+(w)z^{-2h_0}w^{2h_0}V^-(z)^{-1}V^-(w).$$

We can similarly derive

$$
\begin{aligned}
H^{\gamma}(w^2)H^{\beta}(z^2) &= V^+(w)\gamma_\chi(w^2)w^{2h_0}V^-(w)V^+(z)^{-1}\beta_\chi(z^2)z^{-2h_0}V^-(z)^{-1}\\
&= \frac{w^2-z^2}{w^2}V^+(w)\gamma_\chi(w^2)w^{2h_0}V^+(z)^{-1}V^-(w)\beta_\chi(z^2)z^{-2h_0}V^-(z)^{-1}\\
&= \frac{w^2-z^2}{w^2}\cdot i_{w,z}\frac{w^2}{w^2-z^2}V^+(w)\gamma_\chi(w^2)w^{2h_0}V^+(z)^{-1}\beta_\chi(z^2)z^{-2h_0}V^-(w)V^-(z)^{-1}\\
&= \frac{w^2-z^2}{w^2}\cdot i_{w,z}\frac{w^2}{w^2-z^2}\cdot i_{w,z}\frac{w^2}{w^2-z^2}V^+(w)V^+(z)^{-1}\gamma_\chi(w^2)w^{2h_0}\beta_\chi(z^2)z^{-2h_0}\\
&\quad\times V^-(w)V^-(z)^{-1}\\
&= i_{w,z}\frac{1}{w^2-z^2}V^+(w)V^+(z)^{-1}\gamma_\chi(w^2)\beta_\chi(z^2)z^{-2h_0}w^{2h_0}V^-(w)V^-(z)^{-1}\\
&= i_{w,z}\frac{1}{w^2-z^2}V^+(w)V^+(z)^{-1}\left(:\gamma_\chi(w^2)\beta_\chi(z^2): - i_{w,z}\frac{1}{w^2-z^2}\right)z^{-2h_0}w^{2h_0}V^-(w)V^-(z)^{-1}\\
&= i_{w,z}\frac{1}{w^2-z^2}V^+(w)V^+(z)^{-1}\left(:\beta_\chi(z^2)\gamma_\chi(w^2):\right)z^{-2h_0}w^{2h_0}V^-(w)V^-(z)^{-1}\\
&\quad - i_{w,z}\frac{1}{(w^2-z^2)^2}V^+(w)V^+(z)^{-1}z^{-2h_0}w^{2h_0}V^-(w)V^-(z)^{-1}.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
H^{\beta}(z^2)H^{\gamma}(w^2)+H^{\gamma}(w^2)H^{\beta}(z^2) &= \delta(z^2,w^2)V^+(w)V^+(z)^{-1}\left(:\beta_\chi(z^2)\gamma_\chi(w^2):\right)z^{-2h_0}w^{2h_0}V^-(w)V^-(z)^{-1}\\
&\quad+\partial_{w^2}\delta(z^2,w^2)V^+(w)V^+(z)^{-1}z^{-2h_0}w^{2h_0}V^-(w)V^-(z)^{-1}.
\end{aligned}
$$

Now we use the standard properties of the delta function (see, e.g., Ref. 19), namely,

$$
\delta(z^2,w^2)f(z^2)=\delta(z^2,w^2)f(w^2) \quad \text{and} \quad \partial_{w^2}\delta(z^2,w^2)f(z^2)=\partial_{w^2}\delta(z^2,w^2)f(w^2)+\delta(z^2,w^2)\partial_{w^2}f(w^2).
$$

Consequently,

$$
\begin{aligned}
H^{\beta}(z^2)H^{\gamma}(w^2)+H^{\gamma}(w^2)H^{\beta}(z^2) &= \delta(z^2,w^2):\beta_\chi(w^2)\gamma_\chi(w^2):+\\
&\quad+\partial_{w^2}\delta(z^2,w^2)+\delta(z^2,w^2)\\
&\quad\times\left(-\sum_{n>0}h_{-n}w^{2n-2}-\sum_{n>0}h_nw^{-2n-2}-h_0w^{-2}\right)\\
&= \delta(z^2,w^2)h_\chi(w)+\partial_{w^2}\delta(z^2,w^2)-\delta(z^2,w^2)h_\chi(w)\\
&= \partial_{w^2}\delta(z^2,w^2).
\end{aligned}
$$

Now we prove the first trivial relation

$$
\begin{aligned}
H^{\beta}(z^2)H^{\beta}(w^2) &= V^+(z)^{-1}\beta_\chi(z^2)z^{-2h_0}V^-(z)^{-1}V^+(w)^{-1}\beta_\chi(w^2)w^{-2h_0}V^-(w)^{-1}\\
&= i_{z,w}\frac{z^2}{z^2-w^2}V^+(z)^{-1}\beta_\chi(z^2)V^+(w)^{-1}z^{-2h_0}V^-(z)^{-1}\beta_\chi(w^2)w^{-2h_0}V^-(w)^{-1}\\
&= i_{z,w}\frac{z^2}{z^2-w^2}\cdot\frac{z^2-w^2}{z^2}V^+(z)^{-1}V^+(w)^{-1}\beta_\chi(z^2)z^{-2h_0}V^-(z)^{-1}\beta_\chi(w^2)w^{-2h_0}V^-(w)^{-1}\\
&= i_{z,w}\frac{z^2}{z^2-w^2}\cdot\frac{z^2-w^2}{z^2}\cdot\frac{z^2-w^2}{z^2}V^+(z)^{-1}V^+(w)^{-1}\beta_\chi(z^2)z^{-2h_0}\beta_\chi(w^2)w^{-2h_0}\\
&\quad\times V^-(z)^{-1}V^-(w)^{-1}\\
&= (z^2-w^2)V^+(z)^{-1}V^+(w)^{-1}\beta_\chi(z^2)\beta_\chi(w^2)z^{-2h_0}w^{-2h_0}V^-(z)^{-1}V^-(w)^{-1}.
\end{aligned}
$$

Therefore,

$$
H^{\beta}(w^2)H^{\beta}(z^2)=(w^2-z^2)V^+(z)^{-1}V^+(w)^{-1}\beta_\chi(w^2)\beta_\chi(z^2)z^{-2h_0}w^{-2h_0}V^-(z)^{-1}V^-(w)^{-1}
$$

and so

$$
H^{\beta}(z^2)H^{\beta}(w^2)+H^{\beta}(w)H^{\beta}(z^2)=0.
$$

The relation

$$H^\gamma(z^2)H^\gamma(w^2)+H^\gamma(w)H^\gamma(z^2)=0$$

is proved similarly. □

We index the fields $H^\beta(z^2)$ and $H^\gamma(z^2)$ in the standard vertex algebra notation

$$H^\beta(z^2)=\sum_{n\in\mathbb{Z}}H^\beta_{(n)}z^{-2n-2},\quad H^\gamma(z^2)=\sum_{n\in\mathbb{Z}}H^\gamma_{(n)}z^{-2n-2}. \tag{3.21}$$

First, note that since the fields $H^\beta(z^2)$ and $H^\gamma(z^2)$ depend only on z^2, we can re-scale back to z as it is necessary for a super vertex algebra. Before we proceed, we want to offer two comments.

Remark 3.8. The proposition above ensures that the fields $H^\beta(z)$ and $H^\gamma(z)$ satisfy the OPE relations of the symplectic fermion vertex algebra introduced by Kausch, see, e.g., Refs. 21 and 22. Observe that these fields are defined on the entire F_χ, by Definition 3.3, as we will need them to be, since the ultimate goal of this paper is to express the field $\chi(z)$ defining F_χ in terms of exponentiated boson fields. Therefore, as a necessary step, we needed to understand the properties of the fields $H^\beta(z)$ and $H^\gamma(z)$, and we proved that the symplectic fermion commutation relations between them hold **on the whole of** F_χ. But, due to Lemma 3.5, we can infer that the symplectic fermion commutation relations also hold on F_χ^{hwv}, as $H^\beta(z)$ and $H^\gamma(z)$ can be considered to also be fields restricted to F_χ^{hwv}. Normally, such a restriction should be indicated by the notation, but here we will depend on the context and not introduce a new notation for the restricted $H^\beta(z)$ and $H^\gamma(z)$ on F_χ^{hwv}.

Remark 3.9. We want to discuss an interesting phenomenon here: as we pointed above, the fields $H^\beta(z)$ and $H^\gamma(z)$ are defined on the entire F_χ. In Ref. 6, we proved that the field $\chi(z)$ and its descendant field $\chi(-z)$ generate a twisted vertex algebra on the space F_χ, which is isomorphic to the $\beta-\gamma$ system and its Fock space, but considered as a twisted vertex algebra (i.e., with singularities both at $z=w$ and $z=-w$ formally allowed). This isomorphism allows us to view the fields $H^\beta(z)$ and $H^\gamma(z)$ as defined on the (entire) Fock space of the $\beta-\gamma$ system. But, interestingly, they are not actually part of the vertex algebra structure there! The reason is that any vertex algebra structure requires a state-field correspondence (which is invertible to a field-state correspondence via the creation axiom). Thus, if the fields $H^\beta(z)$ and $H^\gamma(z)$ were vertex operators as part of the $\beta-\gamma$ vertex algebra structure, they would correspond to vertex operators assigned to some states, say v_β and v_γ from the $\beta-\gamma$ Fock space, and we would have

$$H^\beta(z)=Y(v_\beta,z),\quad H^\gamma(z)=Y(v_\gamma,z).$$

But then, if they were part of the vertex algebra structure, these vertex operators $Y(v_\beta,z)$ and $Y(v_\beta,z)$ would satisfy the creation axiom,

$$Y(v_\beta,z)|0\rangle|_{z=0}=v_\beta,\quad Y(v_\gamma,z)|0\rangle|_{z=0}=v_\gamma.$$

But as we see just below, (3.22) and (3.23), we then must have

$$v_\beta=\chi_{-3/2}|0\rangle,\quad v_\gamma=\chi_{-1/2}|0\rangle.$$

Now in the $\beta-\gamma$ Fock space, the elements $\chi_{-3/2}|0\rangle$ and $\chi_{-1/2}|0\rangle$ are identified with $\beta_{(-1)}|0\rangle$ and $\gamma_{(-1)}|0\rangle$. And these are the states assigned to the original fields $\beta(z)$ and $\gamma(z)$, i.e.,

$$\beta(z)=Y(\beta_{(-1)}|0\rangle,z),\quad \gamma(z)=Y(\gamma_{(-1)}|0\rangle,z).$$

Which means that we would have on the $\beta-\gamma$ Fock space

$$H^\beta(z)|0\rangle|_{z=0}=Y(v_\beta,z)|0\rangle|_{z=0}=\beta(z)|0\rangle|_{z=0},\quad H^\gamma(z)|0\rangle|_{z=0}=Y(v_\gamma,z)|0\rangle|_{z=0}=\gamma(z)|0\rangle|_{z=0}.$$

Which, since the state-field correspondence in a vertex algebra is an isomorphism, would mean

$$H^\beta(z)=\beta(z),\quad H^\gamma(z)=\gamma(z).$$

But that is manifestly not true. This proves that even though the fields $H^\beta(z)$ and $H^\gamma(z)$ are defined on the entire $\beta-\gamma$ Fock space, they are definitely not part of the vertex algebra structure of the $\beta-\gamma$ system, and therefore neither are they part of any vertex subalgebra of the $\beta-\gamma$ vertex algebra.

We now will show that we even have the additional more restrictive structure of a (classical) super vertex algebra on the space F_χ^{hwv}. Observe that the symplectic fermion OPE structure proved in Proposition 3.6 does not ensure that the other conditions for a super vertex algebra structure are satisfied. First, as we mentioned above, we know that the OPEs hold on the entire F_χ, but it is clear that we cannot have a state-field correspondence between the chiral structure generated by the fields $H^\beta(z^2)$ and $H^\gamma(z^2)$ and the entire F_χ. But we will show that instead the smaller space F_χ^{hwv} satisfies the other conditions for the existence of a vertex algebra structure generated by $H^\beta(z^2)$ and $H^\gamma(z^2)$. Most importantly that the state-field correspondence is satisfied, which in turn will allow a process of producing the highest weight vectors from the vacuum $|0\rangle$. To show that a vertex algebra structure is present on F_χ^{hwv}, we will show that the conditions of the existence Theorem 4.5[19] are satisfied. It is immediate to check that the creation condition is satisfied,

$$H^\beta(z^2)|0\rangle = V^+(z)^{-1}\beta_\chi(z^2)|0\rangle = \chi_{-3/2}|0\rangle + O(z^2) \tag{3.22}$$

and

$$H^\gamma(z^2)|0\rangle = V^+(z)\gamma_\chi(z^2)|0\rangle = \chi_{-1/2}|0\rangle + O(z^2). \tag{3.23}$$

In order to show that the operators $H^\beta_{(n)}$ and $H^\gamma_{(n)}$ generate the vector space F_χ^{hwv} by a successive action on the vacuum $|0\rangle$, we observe that the vector

$$H_{(n_1)}H_{(n_2)}\dots H_{(n_k)}|0\rangle,$$

where $H_{(ns)}$ is either $H^\beta_{(n_s)}$ or $H^\gamma_{(n_s)}$, will appear as a coefficient in the multivariable expression

$$H(z_1^2)H(z_2^2)\dots H(z_k^2)|0\rangle,$$

where again $H(z_s^2)$ is either $H^\beta(z_s^2)$ or $H^\gamma(z_s^2)$. We first observe that as a consequence of Lemma 3.5, these coefficients are themselves highest weight vectors for the Heisenberg action.

By extending the calculation in the proof of the previous proposition, we can observe that

$$H(z_1^2)H(z_2^2)\dots H(z_k^2) = \prod_{s>l} i_{z_s,z_l}(z_s^2 - z_l^2)^{\pm} \prod_{s=1}^{k}(V^+(z_s)^{\pm}) \prod_{s=1}^{k}(\beta - \text{or} - \gamma)_\chi(z_s^2) \prod_{s=1}^{k}\left(z_s^{\pm 2h_0}V^-(z_s)^{\pm}\right),$$

where $\pm$ depends on whether the $H(z_s^2)$ is $H^\beta(z_s^2)$ or $H^\gamma(z_s^2)$. Therefore, we have

$$H(z_1^2)H(z_2^2)\dots H(z_k^2)|0\rangle = \prod_{s>l} i_{z_s,z_l}(z_s^2 - z_l^2)^{\pm} \prod_{s=1}^{k}(V^+(z_s)^{\pm}) \prod_{s=1}^{k}(\beta - \text{or} - \gamma)_\chi(z_s^2)|0\rangle.$$

Now the nonzero coefficients in the above multivariate expression will be precisely those for which the coefficients in $\prod_{s=1}^{k}(\beta - \text{or} - \gamma)_\chi(z_s^2)|0\rangle$ cannot be canceled by an action of the operators from $\prod_{s=1}^{k}(V^+(z_s)^{\pm})$. The coefficients in $\prod_{s=1}^{k}(\beta - \text{or} - \gamma)_\chi(z_s^2)|0\rangle$ are the elements $\left(\chi_{-j_k}\right)^{m_k}\dots\left(\chi_{-j_2}\right)^{m_2}\left(\chi_{-j_1}\right)^{m_1}|0\rangle$, and they span F_χ. Thus the nonzero coefficients will correspond precisely to monomials $\left(\chi_{-j_k}\right)^{m_k}\dots\left(\chi_{-j_2}\right)^{m_2}\left(\chi_{-j_1}\right)^{m_1}|0\rangle$ that cannot be obtained by acting with the Heisenberg algebra on combinations of similar monomials but of lower degree. Due to the fact that the representation of the Heisenberg algebra on F_χ is completely reducible, those correspond precisely to the highest weight vectors for the Heisenberg action. Thus we see that successive action by the operators $H^\beta_{(n)}$ and $H^\gamma_{(n)}$ will generate the space F_χ^{hwv} of the highest weight vectors for the Heisenberg action. In fact, we can see directly that this is a strong generation, i.e., the only indexes appearing in the generating elements $H_{(n_1)}H_{(n_2)}\dots H_{(n_k)}|0\rangle$ are negative, $n_s < 0, s = 1, 2, \dots, k$.

Finally, to apply the existence Theorem 4.5 of Ref. 19, we need a Virasoro element, which will define the translation operator. As is well known, from the start, the symplectic vertex algebra was of interest due to the properties of its Virasoro field and its (logarithmic) modules. Namely, it is immediate to calculate that the field (observe that on the space F_χ^{hwv}, this normal ordered product is well defined)

$$L^{hwv}(z^2) := :H^\gamma(z^2)H^\beta(z^2): = \sum_{n\in\mathbb{Z}} L_n^{hwv} z^{-2n-4} \tag{3.24}$$

is a Virasoro field with central charge $c = -2$, namely,

$$L^{hwv}(z^2)L^{hwv}(w^2) \sim \frac{2L^{hwv}(w^2)}{(z^2-w^2)^2} + \frac{\partial_{w^2}L^{hwv}(w^2)}{z^2-w^2} - \frac{1}{(z^2-w^2)^4}.$$

This can easily be proved by Wick's Theorem using the OPEs derived in Proposition 3.6, so we omit it. Thus we can take L^{hwv}_{-1} as a translation operator T on F^{hwv}_χ. We can then immediately calculate that

$$T|0\rangle = 0, \quad [T, H^\beta(z^2)] = \partial_{z^2}H^\beta(z^2) \quad \text{and} \quad [T, H^\gamma(z^2)] = \partial_{z^2}H^\gamma(z^2),$$

which completes the requirements of the existence Theorem 4.5 of Ref. 19. Thus, after observing that we can re-scale from z^2 to z [as all relevant fields, namely, $H^\beta(z^2)$, $H^\gamma(z^2)$, and $L^{hwv}(z^2)$, depend only on z^2], we arrive at the following.

Theorem 3.10. *The vector space F^{hwv}_χ spanned by the highest weight vectors has a structure of a super vertex algebra, strongly generated by the fields $H^\beta(z)$ and $H^\gamma(z)$, with vacuum vector $|0\rangle$, translation operator $T = L^{hwv}_{-1}$, and vertex operator map induced by*

$$Y(\chi_{-1/2}|0\rangle, z) = H^\gamma(z), \quad Y(\chi_{-3/2}|0\rangle, z) = H^\beta(z). \tag{3.25}$$

This vertex algebra structure is a realization of the symplectic fermion vertex algebra, indicated by the OPEs,

$$H^\beta(z)H^\gamma(w) \sim \frac{1}{(z-w)^2}, \quad H^\gamma(z)H^\beta(w) \sim -\frac{1}{(z-w)^2}, \tag{3.26}$$

$$H^\beta(z)H^\beta(w) \sim 0, \quad H^\gamma(z)H^\gamma(w) \sim 0. \tag{3.27}$$

Remark 3.11. We were asked to comment on the connection between the theorem above and a certain vertex algebra coset structure. Specifically, the vector space F^{hwv}_χ can be identified with the vector space of "what physicists would generally regard as the coset symmetry algebra" of the $\beta-\gamma$ system by the Heisenberg field $h(z)$, see Ref. 26, the quote is from Ref. 8. We will call the space F^{hwv}_χ for short the "physicists' coset" space (apologies for the lack of a better name) of the $\beta-\gamma$ vertex algebra by the Heisenberg field $h(z)$ [the coset of the $\beta-\gamma$ chiral algebra by $\widehat{u}(1)$ in the notation of Ref. 26]. But we want to start by noting that a coset, be it mathematician's or "physicists' coset," is of course not just a vector space, but a vector space with additional structure on it—a vertex algebra structure in the case of the mathematicians' concept of a coset [although often for convenience, one speaks of the vertex algebra structure $(V, Y, |0\rangle, T)$ and the vector space V on which it is defined interchangeably]. The mathematics definition of a coset as a commutant vertex subalgebra (see, e.g., Ref. 19, Corollary 4.6, and Remark 4.6b, and for more details, see Ref. 23, Sec. 3.11) is the vertex algebra equivalent of the original coset construction of Goddard-Kent-Olive (GKO), Refs. 16 and 15; a theorem of Frenkel-Zhu[13] generalizes the GKO construction to the case when "the coset energy-momentum tensor is not the same as that of its parent" (quote from Ref. 8 Sec. 4.3); see also Ref. 23, Theorem 3.11.12. As the authors point out in Ref. 8, Sec. 4.3, the mathematician's coset by $h(z)$ can be quickly proved to be the space spanned of the charge 0 highest weight vectors, so in this particular case, it is the space $F^{hwv}_\chi \cap F^{(0)}_\chi$, which is strictly smaller than F^{hwv}_χ. The mathematician's coset $F^{hwv}_\chi \cap F^{(0)}_\chi$ is studied among other places in Ref. 28, in connection with the triplet vertex algebra.

In general, the "physicists' coset" can be larger than the "mathematics' coset" and is typically a vertex algebra extension of the "mathematics' coset," as it is in this case. (Here we would like to thank Thomas Creutzig for the clarification of the concept of a "physicists' coset" and for the very helpful explanations and discussion.) In this particular case (see Refs. 26 and 8), the "physicists' coset" has as underlying vector space the space spanned by all the Heisenberg highest weight vectors, i.e., F^{hwv}_χ. But, we want to note that the fields $H^\beta(z)$ and $H^\gamma(z)$ we used to prove the symplectic vertex algebra structure on F^{hwv}_χ are defined on the whole of F_χ and satisfy the symplectic fermion OPEs on the whole of F_χ, as we comment in Remark 3.8. Hence it seems that in this they differ from the chiral fields described in Ref. 26 purely on the "physicists' coset" F^{hwv}_χ (although the two structures should coincide on the smaller space F_{χ}^{hwv}). Thus we could not have used the chiral structure and the OPEs of Ref. 26 for our purposes without further calculations, as *a priori,* they are only defined in Ref. 26 on the smaller space F_{χ}^{hwv}. Nevertheless, Theorem 3.10 does give another mathematical proof that the vector space F_{χ}^{hwv} carries the symplectic fermion vertex algebra structure, through $H^{\beta}(z)$ and $H^{\gamma}(z)$, or, in the language of Ref. 26, "carries the same chiral algebra as the theory of symplectic fermions."

Again, we want to stress that the ultimate goal of this paper is to express the defining field $\chi(z)$ in terms of exponentiated bosons, and as a necessary part of this process we needed information on the properties of the fields $H^{\beta}(z)$ and $H^{\gamma}(z)$, as they were specifically defined by Definition 3.3, and the space F_{χ}^{hwv} they generate, since they are the intermediaries in the conversion to exponentiated bosons.

We want to note that besides its own beauty, and being necessary for Sec. IV, Theorem 3.10, specifically the strong generation, actually gives us a constructive way to produce the highest weight vectors in F_{χ}^{hwv}. The decomposition of Proposition 2.3 gives us a count of the highest weight vectors of given degree, but not a way to actually construct them besides solving the defining equations of being a highest weight vector by brute force. Among other things, we can now officially identify the space of the highest weight vectors F_{χ}^{hwv} and the Fock space of the symplectic fermions which was used as a starting point in Refs. 1 and 21 (denoted by *SF* in Ref. 1), by using the state-field correspondence and the creation axiom.

Corollary 3.12. Define (Ref. 1)

$$SF := \Big\{H^{\beta}_{(n_s)}\dots H^{\beta}_{(n_2)}\ \ H^{\beta}_{(n_1)}H^{\gamma}_{(m_k)}\dots H^{\gamma}_{(m_2)}H^{\gamma}_{(m_1)}|0\rangle \mid \\ \mid n_s<\cdots n_2<n_1;\ m_k<\cdots m_2<m_1,\ m_i,n_j\in\mathbb{Z}_{<0},\ i=1,2,\dots,k; j=1,2,\dots,s\Big\}.$$

We have as vertex algebras

$$F_{\chi}^{hwv}\cong SF \tag{3.28}$$

and as vector spaces

$$F_{\chi}\cong F_{\chi}^{hwv}\otimes\mathbb{C}[x_1,x_2,\dots,x_n,\dots]\cong SF\otimes\mathbb{C}[x_1,x_2,\dots,x_n,\dots]. \tag{3.29}$$

For instance, we have the following example:

Example 3.13. For the two special families of highest weight vectors, $\chi^n_{-\frac{1}{2}}|0\rangle$ and $\chi^n_{-\frac{3}{2}}|0\rangle$, one can easily check that

$$\chi^n_{-\frac{1}{2}}|0\rangle = H^{\gamma}_{(-n)}\dots H^{\gamma}_{(-2)}H^{\gamma}_{(-1)}|0\rangle, \tag{3.30}$$

$$\chi^n_{-\frac{3}{2}}|0\rangle = H^{\beta}_{(-n)}\dots H^{\beta}_{(-2)}H^{\beta}_{(-1)}|0\rangle. \tag{3.31}$$

The charge 0° 4 vector $v_{4;0}$ from (2.13) that we used as a counter example in Sec. II can be obtained as

$$v_{4;0}=2H^{\beta}_{(-1)}H^{\gamma}_{(-1)}|0\rangle=(h_{-1}^2-h_{-2})|0\rangle-2h_{-1}\chi_{-3/2}\chi_{-1/2}|0\rangle+2\chi_{-\frac{7}{2}}\chi_{-1/2}|0\rangle. \tag{3.32}$$

IV. COMPLETE BOSONIZATION

From Definition 3.3, the fields $\beta_{\chi}(z^2)$ and $\gamma_{\chi}(z^2)$ needed to express the generating field

$$\chi(z)=\gamma_{\chi}(z^2)+z\beta_{\chi}(z^2)$$

can be written as

$$\beta_{\chi}(z^2)=V^+(z)H^{\beta}(z^2)V^-(z)z^{2h_0},\qquad \gamma_{\chi}(z^2)=V^+(z)^{-1}H^{\gamma}(z^2)V^-(z)^{-1}z^{-2h_0}. \tag{4.1}$$

Due to Corollary 3.12, we can write

$$F_{\chi}\cong SF\otimes\mathbb{C}[x_1,x_2,\dots,x_n,\dots]. \tag{4.2}$$

The fields $V^+(z)$ and $V^-(z)$ [consequently $V^+(z)^{-1}$ and $V^-(z)^{-1}$] are bosonic, via the action

$$h_n\mapsto i\partial_{x_n},\quad h_{-n}\mapsto inx_n\cdot,\quad \text{for any } n>0. \tag{4.3}$$

Remark 4.1. As we mentioned before, we can use an arbitrary re-scaling $h_n \to s_n h_n$, $s_n \neq 0$, $s_n \in \mathbb{C}$, for $n > 0$, so that we could have used instead the identification

$$h_n \mapsto -\partial_{x_n}, \quad h_{-n} \mapsto n x_n \cdot, \quad \text{for any } \ n \in \mathbb{N}.$$

The identification we use here underlines the potential complexification, as seen in (4.10) and (4.11).

But the fields $H^\beta(z^2)$ and $H^\gamma(z^2)$ required to complete the description of the generating field $\chi(z)$ are fermionic. We can, as was done in Ref. 27 for the twisted bosonization, introduce super-variables and derivatives with respect to those super-variables to describe the fields $H^\beta(z)$ and $H^\gamma(w)$ and their action on the space of the highest weight vectors F_χ^{hwv}. But in this case, for this second bosonization, we can do better, as it is known that the symplectic fermions can be embedded into a lattice vertex algebra. Namely, as in the Friedan-Martinec-Shenker (FMS) bosonization,[14] and following Refs. 21 and 22, we can view the fields $H^\beta(z)$ and $H^\gamma(w)$ as

$$H^\beta(z) \mapsto \psi^-(z), \quad H^\gamma(z) \mapsto \partial_z \psi^+(z), \tag{4.4}$$

where $\psi^+(z)$ and $\psi^-(z)$ are the charged free fermion fields used in the bosonization of the KP hierarchy, via the boson-fermion correspondence (see, e.g., Refs. 19 and 24). Specifically, $\psi^+(z)$ and $\psi^-(z)$ have OPEs,

$$\psi^+(z)\psi^-(w) \sim \frac{1}{z-w}, \quad \psi^-(z)\psi^+(w) \sim \frac{1}{z-w}, \quad \psi^+(z)\psi^+(w) \sim 0, \quad \psi^-(z)\psi^-(w) \sim 0,$$

and are the generating fields of the charged free fermion super vertex algebra (see, e.g., Ref. 19). We can use the bosonization of the charged free fermion super vertex algebra via the lattice fields

$$\psi^+(z) \to e_y^\alpha(z), \quad \psi^+(z) \to e_y^{-\alpha}(z), \tag{4.5}$$

where the lattice fields $e_y^\alpha(z)$ and $e_y^{-\alpha}(z)$ act on the bosonic vector space $\mathbb{C}[e^\alpha, e^{-\alpha}] \otimes \mathbb{C}[y_1, y_2, \ldots, y_n \ldots]$ by

$$e_y^\alpha(z) = \exp\Big(\sum_{n\ge 1} y_n z^n\Big) \exp\Big(-\sum_{n\ge 1} \frac{\partial}{n\partial y_n} z^{-n}\Big) e^\alpha z^{\partial_\alpha},$$

$$e_y^{-\alpha}(z) = \exp\Big(-\sum_{n\ge 1} y_n z^n\Big) \exp\Big(\sum_{n\ge 1} \frac{\partial}{n\partial y_n} z^{-n}\Big) e^{-\alpha} z^{-\partial_\alpha},$$

as is standard in the theory of the KP hierarchy. We use the index y to indicate that these are the exponentiated boson fields acting on the variables $y_1, y_2, \ldots, y_n \ldots$. We introduce similarly the Heisenberg field $h^y(z)$,

$$h^y(z) = \sum_{n\ge 1} \frac{\partial}{\partial y_n} z^{-n-1} + h_0^y z^{-1} + \sum_{n\ge 1} n y_n z^{n-1}, \tag{4.6}$$

where h_0^y acts on $\mathbb{C}[e^\alpha, e^{-\alpha}] \otimes \mathbb{C}[y_1, y_2, \ldots, y_n \ldots]$ by $h_0^y e^{m\alpha} P(y_1, y_2, \ldots, y_n \ldots) = m e^{m\alpha} P(y_1, y_2, \ldots, y_n \ldots)$. Thus, combining the two maps, we map F_χ^{hwv} onto a subspace of $\mathbb{C}[e^\alpha, e^{-\alpha}] \otimes \mathbb{C}[y_1, y_2, \ldots, y_n \ldots]$, and

$$H^\beta(z) \to e_y^{-\alpha}(z) = \exp\Big(-\sum_{n\ge 1} y_n z^n\Big) \exp\Big(\sum_{n\ge 1} \frac{\partial}{n\partial y_n} z^{-n}\Big) e^{-\alpha} z^{-h_0^y}, \tag{4.7}$$

$$H^\gamma(z) \to \partial_z e_y^\alpha(z) =: h^y(z) \exp\Big(\sum_{n\ge 1} y_n z^n\Big) \exp\Big(-\sum_{n\ge 1} \frac{\partial}{n\partial y_n} z^{-n}\Big) e^\alpha z^{h_0^y} : . \tag{4.8}$$

Now we can combine the actions of the two Heisenberg fields: $h^y(z)$ and the original $h_\chi(z)$. Through the above map, the Fock space F_χ will be mapped onto a subspace of $\mathbb{C}[e^\alpha, e^{-\alpha}] \otimes \mathbb{C}[x_1, x_2, \ldots, x_n, \ldots; y_1, y_2, \ldots, y_n \ldots]$. The modes h_n (for clarity, we shall write h_n^x) of the field $h_\chi(z)$ will act as in (4.3), with

$$h_0^x e^{m\alpha} P(x_1, x_2, \ldots, x_n, \ldots; y_1, y_2, \ldots, y_n \ldots) = m e^{m\alpha} P(x_1, x_2, \ldots, x_n, \ldots; y_1, y_2, \ldots, y_n \ldots). \tag{4.9}$$

The action of h_0^x stems from the identifications (4.7) and (4.8) which determine the charges of the elements of $\mathbb{C}[e^\alpha, e^{-\alpha}] \otimes \mathbb{C}[y_1, y_2, \ldots, y_n \ldots]$. Thus implies that the actions $z^{-h_0^x}$ in (3.11) and $z^{h_0^y}$ in (4.8) will cancel each other. And so finally we arrive at the complete second bosonization of the CKP hierarchy.

Theorem 4.2. *The generating field $\chi(z)$ of the CKP hierarchy can be written as*

$$\chi(z) = \gamma_\chi(z^2) + z\beta_\chi(z^2),$$

where the fields $\beta_\chi(z)$ and $\gamma_\chi(z)$ can be bosonized as follows:

$$\beta_\chi(z) \to \exp\Big(i\sum_{n>0}(x_n + iy_n)z^n\Big)\exp\Big(-i\sum_{n>0}\frac{1}{n}\Big(\frac{\partial}{\partial x_n} + i\frac{\partial}{\partial y_n}\Big)z^{-n}\Big)e^{-\alpha}, \tag{4.10}$$

$$\gamma_\chi(z) \to\ :\exp\Big(-i\sum_{n>0}(x_n + iy_n)z^n\Big)h^y(z)\exp\Big(i\sum_{n>0}\frac{1}{n}\Big(\frac{\partial}{\partial x_n} + i\frac{\partial}{\partial y_n}\Big)z^{-n}\Big)e^{\alpha}:. \tag{4.11}$$

The Fock space F_χ is mapped onto a subspace of the bosonic space $\mathbb{C}[e^\alpha, e^{-\alpha}] \otimes \mathbb{C}[x_1, x_2, \ldots, x_n, \ldots; y_1, y_2, \ldots, y_n \ldots]$, with $|0\rangle \mapsto 1$. The Hirota equation (2.5) is equivalent to

$$Res_z\Big(\beta_\chi(z) \otimes \gamma_\chi(z) - \gamma_\chi(z) \otimes \beta_\chi(z)\Big) = 0. \tag{4.12}$$

V. OUTLOOK

In this paper we completed the second bosonization of the Hirota equation for the CKP hierarchy. As with any bosonization, notably the best known case—the bosonization of the KP hierarchy, one has several avenues of further work, for which the bosonization is the necessary foundation. First, the bosonization itself, Theorem 4.2, allows one to write the purely algebraic Hirota equation as an infinite hierarchy of actual differential equations. One proceeds similarly to the exposition in Ref. 18, Chap. 7, by employing the Hirota derivative technique. In the CKP case, some quirks are by now expected, and one of the difficulties is tied to the fact that there are no actual elements of F_χ, besides the vacuum vector $|0\rangle$, which solve the algebraic Hirota equation:

Lemma 5.1. If $v \in F_\chi$ solves the Hirota equation $Res_z(\chi(z) \otimes \chi(-z))(v \otimes v) = 0$, then $v = |0\rangle$.

This shows that there are no finite-sum solutions, in contrast to the KP case where every monomial in the charged free fermion Fock space is actually a solution to the corresponding KP Hirota equation. Thus one has to immediately go to a completion of F_χ, where one considers series of monomials instead of finite sums (luckily F_χ, which can be considered as a polynomial algebra for all purposes, is dense in such a completion). In the series completion we will have solutions, as the Hirota equation was suggested because the Hirota operator $S = Res_z(\chi(z) \otimes \chi(-z))$ commutes with the action of the rather large c_∞ algebra for which the Fock space F_χ is a module (Refs. 20, 29, and 7). This was part of the reason for the name CKP and for the choice of this specific algebraic Hirota equation for the CKP hierarchy. We expect that the solutions will belong to the orbit of a certain Weil representation of Sp_∞, and of course, a proof will need to be provided. The fact that we have to go to a series completion of the Fock space F_χ is incidentally completely expected for a Weil representation of the symplectic group. Investigating the solutions will require some length and so will take a separate paper to detail, as by now we have quite a bad experience with long papers that encompass more than a single topic.

Usually, besides the relation to the CKP hierarchy, for which this bosonization was designed, there are several direct applications of any bosonization. For instance, by calculating the character (graded dimension) of both the fermionic and the bosonic sides of the correspondence, one can obtain identities relating certain product formulas to certain sum formulas. For example, one can directly obtain the Jacobi triple product identity—it was done in Ref. 19 for the classical boson-fermion correspondence of type A, and in Ref. 5, for the bosonization of type D-A. Such a sum-vs-product identity perfectly illustrates the equality between the fermionic side (the product formulas) and the bosonic side (the sum formulas). Here, as is typical for the CKP quirks, this identity is complicated by the fact that the degree operator L_0 that we had to use for the decomposition does not act uniformly on the symplectic fermion side with which the highest weight vector space F_χ^{hwv} identifies. Hence the character identity is much more complicated than the Jacobi identity, its proof requires some length, and is already promised as a contribution to the conference proceeding of the AMS Special Session on Representations of Lie Algebras, Quantum Groups and Related Topics, edited by Kailash Misra and Naihung Jing.

Another direct application is by equating the vacuum expectation values on both sides of the correspondence. For instance, in the case of the bosonization of the BKP, a direct comparison of the vacuum expectation values on both sides of the correspondence produces the Schur Pfaffian identity (Ref. 4), where the Pfaffian represents the twisted neutral fermion side, and the product represents the bosonic side. As we prove in Ref. 4 via the bicharacter construction, the symplectic fermion component (the space F_{χ}^{hwv}) will produce determinant vacuum expectation values. We need to extend the bicharacter construction to the twisted boson side, but it is already clear that the vacuum expectation values there will be Hafnians (as proved by brute force in certain cases for the CKP in Ref. 27). Therefore, especially considering that there are now two bosonizations for the CKP case, one would obtain identities between certain types of Pfaffians, Hafnians (see Ref. 27), and determinants.

Most importantly, the consequences of the existence of the two bosonizations (the one described here as well as the bosonization studied in Ref. 27) need to be addressed, as well as the comparison between the Hirota equation and the original reduction approach to the CKP hierarchy. Each of these topics is worth a separate discussion, which we will commence in a consequent paper.

ACKNOWLEDGMENTS

Finally, we would like to acknowledge the helpful comments and suggestions of the referee, which facilitated a deeper understanding of the subject and made this paper better; thank you.

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