The Two Bosonizations of the CKP Hierarchy: Bicharacter Construction and Vacuum Expectation Values

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Abstract We discuss the twisted vertex algebras involved in the two bosonizations of the CKP hierarchy. We show that they can be realized through the bicharacter construction of twisted vertex algebras, by endowing their Fock spaces with additional Hopf module-algebra structure and selecting appropriate bicharacters. We use the bicharacter descriptions to derive certain vacuum expectation values and identities.

1 Introduction

Vertex algebra is the mathematical concept axiomatizing the properties of some, simplest, "algebras" of vertex operators from conformal and string theory (see for instance [10, 15, 17, 19, 21]). Vertex algebras are closely associated with integrable systems through the works of Date, Jimbo, Kashiwara and Miwa (e.g., [12, 14, 22]), Igor Frenkel [16], Victor Kac [19, 20] and many others. The boson-fermion correspondence, which is a super vertex algebra isomorphism between the charged free fermions super vertex algebra and the lattice super vertex algebra of the rank one odd lattice (see e.g. [19]), is a phenomenon well known and applied in many areas of representation theory and mathematical physics. It is famously associated with the Kadomtsev–Petviashvili (KP) hierarchy and the algebraic Hirota approach, whereby the KP hierarchy is written as a single Hirota equation involving the two charged fermion fields. Bosonization, which is one of the directions of any boson-fermion correspondence, translates the fermionic fields in the Hirota equation into bosonic fields, a process necessary to translate the purely algebraic Hirota equation into a hierarchy of actual differential equations. Besides "the" boson-fermion correspondence (which we can think of as type A, as it is associated with the infinite-dimensional a_∞

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Lie algebra), there are other boson-fermion correspondences. Date, Jimbo, Kashiwara and Miwa introduced the boson-fermion correspondence of type B, which is associated to the BKP hierarchy ([14], see also [24]). They also introduced the CKP hierarchy and suggested an algebraic Hirota equation for it [13], as well as an approach to its bosonization (which though they did not complete). These, as well as other examples of bosonizations and boson-fermion correspondences, cannot be described through super vertex algebras due to the more general types of singularities in the Operator Product Expansions (OPEs) of the fields involved. Hence in [4, 5, 9] we developed the notion of a twisted vertex algebra, which accommodated these examples of boson-fermion correspondences. Twisted vertex algebras allow poles at $z = \pm w$ in the OPEs, as needed for the descriptions of the boson-fermion correspondences of type B, C and D; we recall the definition in the next Sect. 2. (Twisted vertex algebras, which perhaps should have been named "twisted chiral algebras" to avoid the potential confusion with twisted modules for vertex algebras, in general allow for poles at roots of unity in the OPEs, [5, 9], but we won't need the general case here). The case of the CKP hierarchy and its Hirota equation, as introduced in [13], is particularly interesting and was shown recently to provide a wealth of surprises [2, 8, 23]. The first surprise was, that unlike any of the other cases, there is not one, but two bosonizations attached to the Hirota equation of the CKP hierarchy: one via a twisted Heisenberg field (see (11) below), and one via an untwisted Heisenberg field (see (12)). The second surprise was that, unlike the boson-fermion correspondence of type A, the spaces of highest weight vectors for these two Heisenberg actions themselves have structures of nontrivial twisted vertex algebras. As a result, this one case of the CKP hierarchy provided at least 5 different examples of twisted vertex algebras, with more than one isomorphism between them (we will recall them briefly in Sect. 2), and we suspect that is not the end of this story. Any such twisted vertex algebra isomorphism provides a wealth of applications, for instance, in [2] we studied the several Virasoro conformal structures involved, and the resulting character identities. Such examples should be understood very well, and specifically the reason behind the existence of more than one isomorphism involving these twisted vertex algebras. In this paper we attempt to understand another aspect of these twisted vertex algebras, namely the Hopf module-algebra structure underlying these examples. The more categorical approach to vertex algebras (super, twisted or quantum) was first suggested by Borcherds in [11], through the bicharacter construction via imposing a Hopf module-algebra structure on the underlying spaces. The bicharacter construction is not well understood among the vertex algebra researchers, as there is a major difference between the bicharacter description of a vertex algebra and the operator-based description typically used. In the operator-based description the examples are presented in terms of generating fields (vertex operators) and their OPEs (or commutation relations). With the bicharacter construction one starts instead with a (supercommutative supercocommutative) Hopf algebra M and its free Leibnitz module (see Sect. 3). The vertex operators then need to be defined through the bicharacter, as well as two auxiliary maps (the projection and the exponential maps, see Sect. 3, (40)). OPEs then result from the choice of a bicharacter r on the underlying Hopf algebra M— different bicharacters r will dictate different OPEs even with the same underlying algebra M. In Sect. 3 we show what choices for the underlying algebra Hopf algebra M and a bicharacter r describe the examples of the twisted vertex algebras associated to the CKP hierarchy. Even though the bicharacter description is somewhat counterintuitive, it provides another aspect of the algebraic structure involved, as well as allows us to see some of the properties from a different point of view in a more general light. One such property is the structure of the vacuum expectation values— we show that the type of vacuum expectation values really depends only on the structure of the underlying Hopf algebra M, for any bicharacter r, and not on the exact nature of the resulting generating fields. For instance, the fact that the vacuum expectation values are expressed through Pfaffians in all three of the boson-fermion correspondences of types B, C and D follows directly from the fact that the underlying Hopf algebra M involved is generated by a single odd primitive element, even though the bicharacters for each of the three cases are different, producing in turn different generating OPEs. In this paper we show that when the underlying Hopf algebra M is generated by a single even primitive element, then the resulting vacuum expectation values are Hafnians for any choice of bicharacter (see Proposition 4.4). As an immediate Corollary 4.6, we obtain the Hafnian-Pfaffian-Product identity of [18], which was obtained also in [23] by direct calculation. There are more identities, including Borchardt's identity involving determinant and permanent, that follow from the underlying bicharacter description of the untwisted bosonization, but we will have to leave those for a lengthier paper.

2 Twisted Vertex Algebras Related to the Two Bosonizations of the CKP Hierarchy

Throughout this paper we will use common concepts and technical tools from the areas of vertex algebras and conformal field theory, such as the notions of field, locality, Operator Product Expansions (OPEs), normal ordered products, etc., for which we refer the reader to any book on the topic (such as [17, 19, 21]). We will also use the extension of these technical tools to the case of N-point locality, as introduced in [9]. In what follows we assume the underlying vector spaces are super ($\mathbb{Z}_2$ graded) vector spaces over the field of complex numbers $\mathbb{C}$. For any homogeneous element a in a super vector space we denote by $\tilde{a}$ its $\mathbb{Z}_2$ grading, called parity. In this category the flip map τ is defined by

$$\tau(a \otimes b) = (-1)^{\tilde{a}\cdot\tilde{b}}(b \otimes a) \tag{1}$$

for any homogeneous elements a, b in the super vector space, and extended by linearity.

In a Hopf superalgebra H we denote the coproduct and the counit by $\triangle$ and η, the antipode by S. We will write $\triangle(a) = \sum a' \otimes a''$ for the coproduct of $a \in H$ (Sweedler's notation). That means we will usually omit the indexing in $\triangle(a) =$

$\sum_k a'_k \otimes a''_k$, especially when it is clear from the context. We want to also note that by a super Hopf algebra we mean that the product on $H \otimes H$ is defined by

$$(a \otimes b)(c \otimes d) = (-1)^{\tilde{b}\cdot\tilde{c}}(ac \otimes bd) \tag{2}$$

for any a, b, c, d homogeneous elements in H, and extended by linearity.

Denote by $H_{T_{-1}}$ the Hopf algebra with a primitive generator D ($\triangle(D) = D \otimes 1 + 1 \otimes D$) and a grouplike generator T_{-1} ($\triangle(T_{-1}) = T_{-1} \otimes T_{-1}$) subject to the relations:

$$DT_{-1} = -T_{-1}D, \quad \text{and } (T_{-1})^2 = 1 \tag{3}$$

Denote by H_D the Hopf subalgebra $H_D = \mathbb{C}[D]$ of $H_{T_{-1}}$.

Denote by $\mathbf{F}^2_{\pm}(z, w)$ the space of rational functions in the variables $z, w \in \mathbb{C}$ with only poles at $z = 0,\ z = \pm w$. Note that we do not allow poles at $w = 0$, i.e., if $f(z, w) \in \mathbf{F}^2_{\pm}(z, w)$, then $f(z, 0)$ is well defined. Similarly, $\mathbf{F}^2_{\pm}(z_1, z_2, \ldots, z_l)$ is the space of rational functions in variables $z_1, z_2, \ldots, z_l$ with only poles at $z_1 = 0$, or $z_j = \pm z_k$. $\mathbf{F}^2_{\pm}(z, w)$ is a $H_{T_{-1}} \otimes H_{T_{-1}}$ Hopf algebra module by

$$D_z f(z, w) = \partial_z f(z, w), \quad (T_{-1})_z f(z, w) = f(-z, w) \tag{4}$$

$$D_w f(z, w) = \partial_w f(z, w), \quad (T_{-1})_w f(z, w) = f(z, -w) \tag{5}$$

We will denote the action of elements $h \otimes 1 \in H_{T_{-1}} \otimes H_{T_{-1}}$ on $\mathbf{F}^2_{\pm}(z, w)$ by $h_z\cdot$, and similarly $h_w\cdot$ will denote the action of elements $1 \otimes h \in H_{T_{-1}} \otimes H_{T_{-1}}$. We can now proceed to the concept of a twisted vertex algebra:

Definition 2.1 (*Twisted vertex algebra of order* 2, [4, 5, 9])
Twisted vertex algebra of order 2 is a collection of the following data (V, W, π_f, Y):

- the space of fields V: a vector super space V, which is an $H_{T_{-1}}$ module, graded as an H_D-module;
- the space of states W: a vector super space W, $W \subset V$;
- a linear surjective projection map $\pi_f : V \to W$, such that $\pi_f|_W = Id_W$
- a field-state correspondence Y: a linear map from V to the space of fields on W;
- a vacuum vector: a vector $1 = |0\rangle \in W \subset V$.

This data should satisfy the following set of axioms:

- vacuum axiom: $Y(1, z) = Id_W$;
- modified creation axiom: $Y(a, z)|0\rangle|_{z=0} = \pi_f(a)$, for any $a \in V$;
- transfer of action: $Y(ha, z) = h_z \cdot Y(a, z)$ for any $h \in H_{T_{-1}}$;
- analytic continuation: For any $a, b, c \in V$ exists $X_{z,w,0}(a \otimes b \otimes c) \in W[[z, w]] \otimes \mathbf{F}^2_{\pm}(z, w)$ such that

$$Y(a, z)Y(b, w)\pi_f(c) = i_{z,w}X_{z,w,0}(a \otimes b \otimes c) \tag{6}$$

- symmetry: $X_{z,w,0}(a \otimes b \otimes c) = X_{w,z,0}(\tau(a \otimes b) \otimes c)$;

- Completeness with respect to Operator Product Expansions (OPE's): For each $k \in \mathbb{Z}$, any $a, b, c \in V$, a, b–homogeneous w.r.to the grading by D, exist $l_k \in \mathbb{Z}$ such that

$$Res_{z=\pm w} X_{z,w,0}(a \otimes b \otimes c)(z \mp w)^k = \sum_{s}^{\text{finite}} w^{l_k^s} Y(v_k^s, w)\pi_f(c) \tag{7}$$

for some homogeneous elements $v_k^s \in V$, where $l_k^s \in \mathbb{Z}$.

Remark 1 In [5] (and [9]) we defined the more general concept of a twisted vertex algebra of order N, where the singularities in the OPEs are at roots of unity of order N. If V is an (ordinary) super vertex algebra, then the data $(V, V, \pi_f = Id_V, Y)$ is a twisted vertex algebra of order 1, and the definition above is of a twisted vertex algebra of order 2.

Examples of twisted vertex algebras of order 2 are provided by the corresponding sides of the boson-fermion correspondences of types B, C and D [2, 4–6, 8, 14, 23, 24]. In this paper we will discuss the twisted vertex algebras associated to the CKP hierarchy and its two bosonizations. We start by recalling their descriptions in terms of generating fields and Fock spaces. Note that in [9] we provided uniqueness and existence theorems for twisted vertex algebras in terms of generating fields, which allows us to claim that the various fields we describe below do indeed generate corresponding twisted vertex algebras.

The CKP hierarchy is described by a Hirota equation (see [2, 13, 23]), defined via the twisted neutral boson field $\chi(z)$

$$\chi(z) = \sum_{n\in\mathbb{Z}+1/2} \chi_n z^{-n-1/2}, \tag{8}$$

with OPE

$$\chi(z)\chi(w) \sim \frac{1}{z+w}. \tag{9}$$

In terms of commutation relations for the modes χ_n, $n \in \mathbb{Z} + 1/2$, this OPE is equivalent to

$$[\chi_m, \chi_n] = (-1)^{m-\frac{1}{2}}\delta_{m,-n}1. \tag{10}$$

The modes of the field $\chi(z)$ form a Lie algebra which we denote by L_χ. Let F_χ be the Fock module of L_χ with vacuum vector $|0\rangle$, such that $\chi_n|0\rangle = 0$ for $n > 0$.

By Proposition 2.9 of [7] (see also Theorem 7.12 of [9]) there is a two-point local twisted vertex algebra structure with a space of fields generated by $\chi(z)$ and its descendent field $\chi(-z)$, acting on the space of states $W = F_\chi$. We will denote this twisted vertex algebra for short by just F_χ.

To describe the bosonizations maps, we need to recall the Heisenberg fields initiating the bosonizations.

Proposition 2.2 ([7]) *I. Let*

$$h_\chi^{\mathbb{Z}+1/2}(z)=\frac{1}{2}:\chi(z)\chi(-z):.$$

We have $h_\chi^{\mathbb{Z}+1/2}(-z)=h_\chi^{\mathbb{Z}+1/2}(z)$*, and we index* $h_\chi^{\mathbb{Z}+1/2}(z)$ *as* $h_\chi^{\mathbb{Z}+1/2}(z)=\sum_{n\in\mathbb{Z}+1/2} h_n^t z^{-2n-1}$*. The field* $h_\chi^{\mathbb{Z}+1/2}(z)$ *has OPE with itself given by:*

$$h_\chi^{\mathbb{Z}+1/2}(z)h_\chi^{\mathbb{Z}+1/2}(w)\sim-\frac{z^2+w^2}{2(z^2-w^2)^2}\sim-\frac{1}{4}\frac{1}{(z-w)^2}-\frac{1}{4}\frac{1}{(z+w)^2},\tag{11}$$

and its modes, $h_n^t,\ n\in\mathbb{Z}+1/2$*, generate a* ***twisted*** *Heisenberg algebra* $\mathcal{H}_{\mathbb{Z}+1/2}$ *with relations*
$[h_m^t,h_n^t]=-m\delta_{m+n,0}1,\ \ m,n\in\mathbb{Z}+1/2$*.*
II. Let

$$h_\chi^{\mathbb{Z}}(z)=\frac{1}{4z}\left(:\chi(z)\chi(z):-:\chi(-z)\chi(-z):\right).$$

We have $h_\chi^{\mathbb{Z}}(-z)=h_\chi^{\mathbb{Z}}(z)$*, and we index* $h_\chi^{\mathbb{Z}}(z)$ *as* $h_\chi^{\mathbb{Z}}(z)=\sum_{n\in\mathbb{Z}} h_n^{\mathbb{Z}} z^{-2n-2}$*. The field* $h_\chi^{\mathbb{Z}}(z)$ *has OPE with itself given by:*

$$h_\chi^{\mathbb{Z}}(z)h_\chi^{\mathbb{Z}}(w)\sim-\frac{1}{(z^2-w^2)^2},\tag{12}$$

and its modes, $h_n^{\mathbb{Z}},\ n\in\mathbb{Z}$*, generate an* ***untwisted*** *Heisenberg algebra* $\mathcal{H}_{\mathbb{Z}}$ *with relations* $[h_m^{\mathbb{Z}},h_n^{\mathbb{Z}}]=-m\delta_{m+n,0}1,\ \ m,n\in\mathbb{Z}$*.*

Denote by F_χ^{hwv} the vector space spanned by the highest weight vectors for the untwisted Heisenberg algebra representation on F_χ, and by F_χ^{t-hwv} the vector space spanned by the highest weight vectors for the twisted Heisenberg algebra representation on F_χ. These spaces themselves have structures of twisted vertex algebras. First, using the Heisenberg fields, we can define the exponentiated boson fields as follows: Let

$$V^-(z)=\exp\Big(-\sum_{n>0}\frac{1}{n}h_n^{\mathbb{Z}}z^{-2n}\Big);\qquad V^+(z)=\exp\Big(\sum_{n>0}\frac{1}{n}h_{-n}^{\mathbb{Z}}z^{2n}\Big).\tag{13}$$

Define

$$\beta_\chi(z^2)=\frac{\chi(z)-\chi(-z)}{2z};\quad \gamma_\chi(z^2)=\frac{\chi(z)+\chi(-z)}{2}.\tag{14}$$

and

$$H^\beta(z^2)=V^+(z)^{-1}\beta_\chi(z^2)z^{-2h_0^{\mathbb{Z}}}V^-(z)^{-1},\qquad H^\gamma(z^2)=V^+(z)\gamma_\chi(z^2)z^{2h_0^{\mathbb{Z}}}V^-(z).\tag{15}$$

($V^-(z)$ and $V^+(z)$ are actually functions of z^2, so the notation is unambiguous).

Similarly denote

$$V_t^-(z) = \exp\Big(\sum_{n>0}\frac{2}{2n-1}h^t_{\frac{2n-1}{2}}z^{-2n+1}\Big); \qquad V_t^+(z) = \exp\Big(-\sum_{n>0}\frac{2}{2n-1}h^t_{-\frac{2n-1}{2}}z^{2n-1}\Big); \tag{16}$$

and

$$H^\chi(z) = V_t^+(z)^{-1}\chi(z)V_t^-(z)^{-1}. \tag{17}$$

Theorem 2.3 *I. [8] The vector space F_χ^{hwv} spanned by the highest weight vectors for the untwisted Heisenberg algebra $\mathcal{H}_\mathbb{Z}$ has a structure of a super vertex algebra, strongly generated by the fields $H^\beta(z)$ and $H^\gamma(z)$, with vacuum vector $|0\rangle$, translation operator $T = L_{-1}^{hwv}$, and vertex operator map induced by*

$$Y(\chi_{-1/2}|0\rangle, z) = H^\gamma(z), \quad Y(\chi_{-3/2}|0\rangle, z) = H^\beta(z). \tag{18}$$

This vertex algebra structure is a realization of the symplectic fermion vertex algebra, indicated by the OPEs:

$$H^\beta(z)H^\gamma(w) \sim \frac{1}{(z-w)^2}, \quad H^\gamma(z)H^\beta(w) \sim -\frac{1}{(z-w)^2}; \tag{19}$$

$$H^\beta(z)H^\beta(w) \sim 0; \quad H^\gamma(z)H^\gamma(w) \sim 0. \tag{20}$$

II. [2] The vector space F_χ^{t-hwv} spanned by the highest weight vectors for the twisted Heisenberg algebra $\mathcal{H}_{\mathbb{Z}+1/2}$ has a structure of an $N = 2$ twisted vertex algebra, generated by the field $H^\chi(z)$, with vacuum vector $|0\rangle$, and vertex operator map induced by

$$Y(\chi_{-1/2}|0\rangle, z) = H^\chi(z). \tag{21}$$

This twisted vertex algebra structure is twisted fermionic, indicated by the OPEs:

$$H^\chi(z)H^\chi(w) \sim \frac{z-w}{(z+w)^2}. \tag{22}$$

Corollary 2.4 *I. [1] Define the symplectic fermion Fock space:*

$$SF := \{H^\beta_{(m_k)}\dots H^\beta_{(m_2)}H^\beta_{(m_1)}H^\gamma_{(n_s)}\dots H^\gamma_{(n_2)}H^\gamma_{(n_1)}|0\rangle \mid$$
$$\mid m_k < \dots m_2 < m_1,\ n_s < \dots n_2 < n_1;\ \ m_i, n_j \in \mathbb{Z}_{<0},\ i = 1, 2, \dots, k;\ j = 1, 2, \dots s\}.$$

We have as vertex algebras (twisted vertex algebras of order 1)

$$F_\chi^{hwv} \cong SF; \tag{23}$$

and as vector spaces

$$F_\chi \cong F_\chi^{hwv} \otimes \mathbb{C}[x_1, x_2, \dots, x_n, \dots] \cong SF \otimes \mathbb{C}[x_1, x_2, \dots, x_n, \dots]. \quad (24)$$

II. Define the twisted symplectic fermion Fock space:

$$SF^t := \{H^\chi_{(n_s)} \dots H^\chi_{(n_2)} H^\chi_{(n_1)}|0\rangle \mid n_s < \dots n_2 < n_1;\ n_j \in \mathbb{Z} + 1/2, n_j < 0,\ j = 1, 2, \dots s\}. \quad (25)$$

We have as twisted vertex algebras

$$F_\emptyset^{t-hwv} \cong SF^t; \quad (26)$$

and as vector spaces

$$F_\chi \cong F_\chi^{t-hwv} \otimes \mathbb{C}[t_1, t_3, \dots, t_{2n-1}, \dots] \cong SF^t \otimes \mathbb{C}[t_1, t_3, \dots, t_{2n-1}, \dots]. \quad (27)$$

The Theorem above allows us to express the generating field in terms of the (correspondingly twisted or untwisted) exponentiated boson fields, plus the (twisted or untwisted) symplectic fermion fields, thereby completing the corresponding bosonization process.

Theorem 2.5 ([2, 7, 23]) *I. The twisted bosonization of the CKP hierarchy is the isomorphism σ^t between the twisted vertex algebra generated by the field $\chi(z)$ and its descendant $\chi(-z)$ on the Fock space F_χ; and the twisted vertex algebra generated by the field $\chi^{tb}(z) := \sigma^t \chi(z)(\sigma^t)^{-1} := V^+_{\sigma t}(z) H^{\sigma\chi}(z) V^-_{\sigma t}(z)$ and its descendent $\chi^{tb}(-z)$ on the space $SF^t \otimes \mathbb{C}[t_1, t_3, \dots, t_{2n-1}, \dots]$.*
II. The untwisted bosonization of the CKP hierarchy is the isomorphism σ between the twisted vertex algebra generated by the field $\chi(z)$ and its descendant $\chi(-z)$ on the Fock space F_χ; and the twisted vertex algebra generated by the fields $\sigma\beta_\chi(z^2)\sigma^{-1} := V^+(z) H^\beta(z^2) V^-(z) z^{2h_0}$ and $\sigma\gamma_\chi(z^2)\sigma^{-1} := V^+(z)^{-1} H^\gamma(z^2) V^-(z)^{-1} z^{-2h_0}$ on the space $SF \otimes \mathbb{C}[x_1, x_2, \dots, x_n, \dots]$

3 Bicharacter Construction of Twisted Vertex Algebra Examples

A completely different, but very general way of producing examples of twisted vertex algebras is provided by the super-bicharacter construction, introduced by Borcherds in [11] (the super case was introduced in [3], see also [5]). Even though the bicharacter construction does produce a variety of examples of twisted vertex algebras, it is a matter of nontrivial further examination to match an example produced by a bicharacter to an example provided in terms of generating fields, and prove that the corresponding realizations (by generating fields and by a bicharacter) describe the same vertex algebra. The goal of this paper is to identify the super-bicharacter descriptions of the various twisted vertex algebras involved in the two bosonizations of type C we described in the previous Sect. 2.

We therefore need to briefly recall the super-bicharacter construction, more details can be found in [5].

Definition 3.1 (*Super-bicharacter*) Let M be a Hopf supercommutative and super-cocommutative superalgebra. A bicharacter on M is a linear map r from $M \otimes M$ to $\mathbf{F}^2_{\pm}(z, w)$, such that

$$r_{z,w}(1 \otimes a) = \eta(a) = r_{z,w}(a \otimes 1), \tag{28}$$

$$r_{z,w}(ab \otimes c) = \sum (-1)^{\tilde{b}\tilde{c}'} r_{z,w}(a \otimes c')r_{z,w}(b \otimes c''), \tag{29}$$

$$r_{z,w}(a \otimes bc) = \sum (-1)^{\tilde{a}''\tilde{b}} r_{z,w}(a' \otimes b)r_{z,w}(a'' \otimes c). \tag{30}$$

From now on we will always work with *even* bicharacters, i.e., bicharacters r such that $r_{z,w}(a \otimes b) = 0$ whenever $\tilde{a} \neq \tilde{b}$. The transpose of a bicharacter is defined by

$$r^{\tau}_{z,w}(a \otimes b) = r_{w,z} \circ \tau(a \otimes b). \tag{31}$$

A bicharacter r is called *symmetric* if $r = r^{\tau}$.

Definition 3.2 ($H_{T_{-1}} \otimes H_{T_{-1}}$*-covariant bicharacter*) Let M be a Hopf supercommutative and supecocommutative superalgebra, r be a bicharacter on M. Suppose in addition M is an $H_{T_{-1}}$-module algebra. We call the bicharacter $H_{T_{-1}} \otimes H_{T_{-1}}$-covariant if it additionally satisfies :

$$r_{z,w}(h(a) \otimes g(b)) = h_z g_w \cdot r_{z,w}(a \otimes b), \tag{32}$$

for all $a, b \in M$, $h, g \in H_{T_{-1}}$.

A way to produce $H_{T_{-1}} \otimes H_{T_{-1}}$-covariant bicharacters is by using free Leibnitz modules and potentially imposing compatible relations on the quotient modules as well as on the bicharacters:

Lemma/Definition 3.3 (Free Leibnitz module) *Suppose M is a super commutative algebra and H is an entirely even cocommutative coalgebra. Then there is a universal supercommutative algebra $H(M)$ such that there is a map $h \otimes m \to hm := h(m)$ from $H \otimes M$ to $H(M)$ such that $H(M)$ is a left module for H and*

$$h(mn) = \sum h^{'}(m)h^{''}(n), \quad h(1) = \eta(h), \tag{33}$$

for any $m, n \in M$, $h \in H$. We will call $H(M)$, defined as above, the "free H Leibnitz module of M" (or universal H-Leibnitz module of M).

Any module-algebra over a Hopf algebra H is a quotient of a free Leibnitz module modulo some further relations. Hence we will define the concepts needed on free Leibnitz modules first, and then we will ensure that in the examples the relations we may have to impose are consistent.

Before defining vertex operators via bicharacters, we need two auxiliary maps: a T_{-1}-projection map, and an exponential map. In what follows let M be a supercommutative supercocommutative super Hopf algebra, V be the free Leibnitz module $V = H_{T_{-1}}(M)$. To unclutter the notation, we will just write T instead of T_{-1}. Note that for any M the free Leibnitz module $W = H_D(M)$ is a sub-Hopf algebra of $V = H_{T_{-1}}(M)$, and thus the exponential map $e^{zD} = \sum_{n\geq 0} z^n \frac{D^n}{n!} : W \to W[[z]]$ is well defined on W. Further, since T is grouplike, each element in $V = H_{T_{-1}}(M)$ can be written uniquely as a linear combination of elements of the form $a = a_0 \cdot a_1$, where $a_0 \in W$, and $a_1 = T(\bar{a}_1)$, with $\bar{a}_1 \in W$. For convenience, we denote $\bar{a}_0 := a_0$, since $a_0 \in W$.

Definition 3.4 (*T-projection Map* π_T) Let M be a commutative cocommutative Hopf algebra, let V be the free Leibnitz module $V = H_{T_{-1}}(M)$, $W = H_D(M)$, and let $a \in V$ is such that $a = a_0 \cdot a_1$, where $a_i = T^i \bar{a}_i$ for $\bar{a}_i \in W$, $i = 0, 1$. Define the projection map $\pi_T : V \to W$ to be the algebra homomorphism map defined by:

$$\pi_T(a_i) = \bar{a}_i, \quad i = 0, 1, \quad \pi_T(a) = \pi_T(a_0) \cdot \pi_T(a_1) = \bar{a}_0 \cdot \bar{a}_1; \tag{34}$$

Since V is the span of such elements $a = a_0 \cdot a_1$ as above, we extend π_T to the entire V by linearity.

Definition 3.5 (*Exponential Map* $\mathcal{E}_{\mathbf{z}}$) Define the map $\mathcal{E}_z : V \to W[[z]]$ to be the algebra homomorphism map such that

$$\mathcal{E}_z(\bar{a}) = e^{zD}\bar{a}, \quad \text{for any } \bar{a} \in W; \tag{35}$$

$$\mathcal{E}_z(a) = e^{-zD}\bar{a}; \quad \text{for any } a = T(\bar{a}) \text{ with } \bar{a} \in W; \tag{36}$$

$$\mathcal{E}_z(a \cdot b) = \mathcal{E}_z(a) \cdot \mathcal{E}_z(b), \quad \text{for any } a, b \in V. \tag{37}$$

Since V is the span of elements of the form $a_0 \cdot a_1$, where $a_0 \in W$, and $a_1 = T(\bar{a_1})$ with $\bar{a_1} \in W$, we can extend $\mathcal{E}_z$ by linearity to a well-defined map on the entire V.

Now we can proceed to the construction of actual vertex operators from a bicharacter on a Leibnitz module.

Definition 3.6 (*Two-variable vertex operator from a bicharacter*) [5] Let M be a supercommutative supercocommutative super Hopf algebra, let V be the free Leibnitz module $V = H_{T_{-1}}(M)$, r a $H_{T_{-1}} \otimes H_{T_{-1}}$-covariant bicharacter on V with values in $\mathbf{F}^2_{\pm}(z, w)$, $W = H_D(M)$ be the free H_D-Leibnitz sub-module-algebra of V. Let $\mathcal{E}_z$ be the exponential map $\mathcal{E}_z : V \to W[[z]]$ defined in Definition 3.5. Define a singular multiplication map (called two-variable vertex operator)

$$X_{z,w} \colon V^{\otimes 2} \to W[[z, w]] \otimes \mathbf{F}^2_{\pm}(z, w), \tag{38}$$

by

$$X_{z,w}(a \otimes b) = \sum (-1)^{\tilde{a''}\tilde{b'}} (\mathcal{E}_z a')(\mathcal{E}_w b') r_{z,w}(a'' \otimes b''), \tag{39}$$

where a, b are homogeneous elements of the super space V. The map $X_{z,w}$ is extended by linearity to the entire V.

Definition 3.7 (*Vertex operators $Y(a,z)$ and field-state correspondence*) [5] Let V, W, $\mathcal{E}_z$ be as above, $\pi_T : V \to W$ be the projection map defined in Definition 3.4. Define the vertex operator $Y(a,z)$ associated to $a \in V$ by

$$Y(a,z)\pi_T(b) = X_{z,0}(a\otimes b) = \sum (-1)^{\tilde{a''}\tilde{b'}}(\mathcal{E}_z a')\pi_T(b')r_{z,0}(a''\otimes b''), \qquad (40)$$

for any $b \in V$.

Note that it is clear $Y(a,z)$ is actually a field on W (for the definition of a field, see e.g. [19]), and the map $Y : a \in V \to Y(a,z)$ is going to be the field-state correspondence map for the twisted vertex algebra with space of fields V and space of states W. Finally, to finish the bicharacter construction, we need to define three-variable vertex operators for the analytic continuation property of twisted vertex algebras:

Definition 3.8 *(Three-variable fields from a bicharacter)* [5] Let V, W, $\mathcal{E}_z$, $\pi_T : V \to W$ be as above. Let a, b, c be arbitrary homogeneous elements of the super space V. Define the three variable field

$$X_{z_1,z_2,z_3} \colon V^{\otimes 3} \to W[[z_1,z_2,z_3]]\otimes \mathbf{F}^2_{\pm}(z_1,z_2,z_3), \qquad (41)$$

by

$$X_{z_1,z_2,z_3}(a\otimes b\otimes c) = \sum (-1)^{f(\tilde{a},\tilde{b},\tilde{c})}\mathcal{E}_{z_1}a^{(1)}\mathcal{E}_{z_2}b^{(1)}\mathcal{E}_{z_3}c^{(1)}r_{z_1,z_2}(a^{(2)}\otimes b^{(2)})r_{z_1,z_3} \times (a^{(3)}\otimes c^{(2)})r_{z_2,z_3}(b^{(3)}\otimes c^{(3)}),$$

where $f(\tilde{a},\tilde{b},\tilde{c}) = \tilde{b^{(3)}}(\tilde{c^{(1)}}+\tilde{c^{(2)}}) + (\tilde{a^{(2)}}+\tilde{a^{(3)}})(\tilde{b^{(1)}}+\tilde{c^{(1)}}) + \tilde{a^{(3)}}\tilde{b^{(2)}} + \tilde{b^{(2)}}\tilde{c^{(1)}}$. Here as usual we denote $\Delta^2(a) = a^{(1)}\otimes a^{(2)}\otimes a^{(3)}$ for any $a \in V$. The map X_{z_1,z_2,z_3} is extended to the entire V by linearity.

Recall we usually omit writing the indexing in $\triangle(a) = \sum_p a'_p \otimes a''_p$, and write it just as $\triangle(a) = \sum a'\otimes a''$ to unclutter notation, but this summation is always implicitly present.

Lemma 3.9 ($n = 3$ Analytic continuation, [5]) *Let V, W, $\mathcal{E}_z$, $\pi_T : V \to W$ be as above. We have for any $a, b, c \in V$*

$$i_{z_1,z_2,z_3}X_{z_1,z_2,z_3}(a\otimes b\otimes c) = Y(a,z_1)Y(b,z_2)\mathcal{E}_{z_3}c = Y(a,z_1)Y(b,z_2)Y(c,z_3)1.$$

Let r be a $H_{T_1}\otimes H_{T_1}$-covariant bicharacter on V, with values in $\mathbf{F}^2_{\pm}(z,w)$. For any $a, b \in V$ the bicharacter $r_{z,w}(a\otimes b)$ is just a function of z and w in $\mathbf{F}^2_{\pm}(z,w)$ and thus can be expanded as a Laurent series around $z = \pm w$: $r_{z,w}(a\otimes b) = \sum_{l=0}^{M_{a,b}-1} \frac{f^{i,l}_{a,b}}{(z-(-1)^i w)^{l+1}} + reg.$, $i = 0, 1$. We denote by $M_{a,b}$ the order of the pole at $z = (-1)^i w$ and note that $f^{i,l}_{a,b} = f^{i,l}_{a,b}(w)$ is a function only of w.

Proposition 3.10 ([5]) *Let V, W, r be as above, let again $M_{pq} = M_{a''_p, b''_q}$. For any $a,\ b \in V$ we have*

$$Y(a,z)Y(b,w) = i_{z,w} \sum_{p,q} \sum_{k=0}^{M_{pq}-1} \frac{\sum_{l=M_{p,q}-1-k}^{M_{p,q}-1} (-1)^{\tilde{a''}\tilde{b'}} f^{i,l}_{a'',b''} Y\big((T^i_{-1} D^{(l-k)} a').b', w\big)}{(z-(-1)^i w)^{k+1}} + Reg^{(-1)^i}_{(z,w)}(a \otimes b).$$

The term $Reg^{(-1)^i}_{(z,w)}(a \otimes b)$ denotes the regular part in the Laurent expansion above, it depends on $a, b \in V$, z and w, and $i = 0, 1\}$.

Corollary 3.11 (Bicharacter formula for OPEs for simple poles) *[5] Let V, W, r be as above, and let $a,\ b \in V$ are such that the bicharacters $r_{z,w}(a'' \otimes b'')$ have at most simple poles at each a'', b''. Then*

$$Y(a,z)Y(b,w) = i_{z,w} \sum_{p,q} \sum_{i=0,1} (-1)^{\tilde{a''}\tilde{b'}} f^{i,0}_{a'',b''} \frac{Y((T^i a').b', w)}{z-(-1)^i w} + :a(z)b(w):. \tag{42}$$

Finally, the following theorem summarizes the super-bicharacter construction of twisted vertex algebras:

Theorem 3.12 ([5]) *Let M be a commutative cocommutative Hopf algebra, let V be the free Leibnitz module $V = H_{T_{-1}}(M)$, r be a shift-restricted $H_{T_{-1}} \otimes H_{T_{-1}}$-covariant symmetric bicharacter on V with values in $\mathbf{F}^2_{\pm}(z,w)$, $W = H_D(M)$ be the free H_D-Leibnitz sub-module-algebra of V. Let $\pi_T : V \to W$ be the projection map as in Definition 3.4 and Y be the field-state correspondence defined by (40), via (39). The set of data (V, W, π_T, Y) constructed as above satisfies the definition of a twisted vertex algebra for **any** (shift-restricted) supercommutative $H_{T_{-1}} \otimes H_{T_{-1}}$-covariant bicharacter on V.*

4 Bicharacter Realizations of the Twisted Vertex Algebras Arising from the CKP Hierarchy

In most of the examples in the literature vertex operators are presented in terms of generating fields and commutation relations. This is the case with the various twisted vertex algebras arising in the two bosonizations of the CKP hierarchy as we saw in Sect. 2. With the bicharacter construction as we recalled in the previous Sect. 3, one starts instead with the supercommutative supercocommutative Hopf algebra M and its free Leibnitz module $H_{T_{-1}}(M)$; and then the OPEs and thus the commutation relations are dictated by the choice of the bicharacter r. Most importantly, for each such Hopf algebra M there are many choices of a symmetric bicharacter r, and so each such pair (M, r) will give rise to a different twisted vertex algebra (V, W, π_T, Y); even if M, and therefore the spaces V and W are the same as Hopf algebras—it is the field-state correspondence Y that changes with the choice of a bicharacter. Thus, among other advantages, the bicharacter construction provides additional information about the underlying similarities among some seemingly unrelated twisted vertex algebras.

To describe the examples of twisted vertex algebras produced via the bicharacter construction we need to just select the appropriate Hopf algebra M, then choose a bicharacter, and then impose (potentially) relations on the free Leibnitz module $H_{T_{-1}}(M)$. Then we need to prove that the vertex operators (fields) produced by the bicharacter via (40) indeed are identical to the generating fields we need, by calculating their OPEs. We start with the twisted vertex algebra F_χ, which is by far the easiest in terms of both generating fields and bicharacter description:

Proposition 4.1 *The twisted vertex algebra generated by the field $\chi(z)$ and its descendant $\chi(-z)$ on the Fock space can be realized by the bicharacter construction from the pair (M_χ, r^χ), where M_χ is the Hopf algebra $M_\chi = \mathbb{C}[\chi]$ generated by a single **even** primitive element χ, and the bicharacter r^χ is defined by*

$$r^\chi_{z,w}(\chi \otimes \chi) = \frac{1}{z+w}, \tag{43}$$

and extended by covariance. The Fock space F_χ can be identified (as a Hopf algebra) with the free Leibnitz module $H_D(M_\chi)$, and the resulting twisted vertex algebra constructed via Theorem 3.12 is equivalent to the twisted vertex algebra generated by the field $\chi(z)$ and its descendant $\chi(-z)$ on the Fock space F_χ.

Proof The free H_D-Leibnitz module over the algebra $\mathbb{C}[\chi]$ is isomorphic to the polynomial algebra in infinitely many variables $\mathbb{C}[p_1, p_2, \dots, p_n, \dots]$, since we can identify $p_1 = \chi$ and $D^n\chi = n! \cdot p_{n+1}$; and there are no other relations due to the universality of the free Leibnitz module. Hence we can identify F_χ with the free Leibnitz module $H_D(M_\chi)$.

Consider the field $Y(\chi, z)$ produced via (40). We can use Corollary 3.11 to calculate the OPE of $Y(\chi, z)Y(\chi, w)$. Since χ is primitive, the only singular bicharacter among any of the coproducts of χ'' is $r^\chi_{z,w}(\chi \otimes \chi)$. Thus

$$Y(\chi, z)Y(\chi, w) \sim i_{z,w} \sum f^{1,0}_{\chi'',\chi''} \frac{Y\big((T\chi').\chi', w\big)}{z+w} \sim i_{z,w} \frac{Y\big((T1).1, w\big)}{z+w} \sim \frac{1}{z+w}$$

Hence we see that we can identify the field $Y(\chi, z)$ with the generating filed $\chi(z)$ as defined by (9).

We now proceed to the bicharacter description of the twisted symplectic fermion vertex algebra on the space of highest weight vectors F_χ^{t-hwv}.

Proposition 4.2 *Let $M_{\phi-\chi}$ be the Grassmann super Hopf algebra $M_{\phi-\chi} = \mathbb{C}\{\phi^\chi\}$ generated by a single **odd** primitive element ϕ^χ, and let the symmetric bicharacter $r^{\phi-\chi}$ be defined by*

$$r^{\phi-\chi}_{z,w}(\phi^\chi \otimes \phi^\chi) = \frac{z-w}{(z+w)^2}, \tag{44}$$

and extended by covariance. By Theorem 3.12 the pair $(M_{\phi-\chi}, r^{\phi-\chi})$ gives rise to a twisted vertex algebra. The space $F_{\chi}^{t-hwv} \cong SF^t$ can be identified (as a Hopf algebra) with the free Leibnitz module $H_D(M_{\phi-\chi})$, and the resulting twisted vertex algebra constructed via Theorem 3.12 is equivalent to the twisted vertex algebra generated by the field $H^{\chi}(z)$ and its descendant $H^{\chi}(-z)$ on the space $F_{\chi}^{t-hwv} \cong SF^t$.

The proof of this proposition is similar to the proof of the Lemma above, with the minor difference that one has to account for the minuses that arise from using an odd primitive element.

To account for the entire twisted vertex algebra on the space of states $SF^t \otimes \mathbb{C}[t_1, t_3, \ldots, t_{2n-1}, \ldots]$ produced by the twisted bosonization isomorphism σ^t we need to introduce a lattice part. Let $L_1 = \mathbb{C}[\mathbb{Z}\alpha]$ be the group algebra of the rank-one free abelian group $\mathbb{Z}\alpha$. The group algebra is generated by $e^{m\alpha},\ m \in \mathbb{Z}$, with relations $e^{m\alpha}e^{n\alpha} = e^{(m+n)\alpha},\ e^0 = 1$. Note that as an algebra $L_1 = \mathbb{C}[e^{\alpha}, e^{-\alpha}]$, modulo the relation $R:\ e^{\alpha}e^{-\alpha} = 1$. The Hopf algebra structure is determined by e^{α} and $e^{-\alpha}$ being grouplike. Consider the free H_D Leibnitz module $H_D(L_1)$. Since $H_D(L_1)$ is universal, it contains an element $h = (De^{\alpha})e^{-\alpha}$. It follows that h is primitive, and one can show that $H_D(L_1)$ is isomorphic as an algebra to $L_1 \otimes H_D(\mathbb{C}[h])$ (for a proof, see [5]). We have already seen that the free Leibnitz module $H_D(\mathbb{C}[h])$ can be identified with the polynomial algebra on infinitely many variables.

Now consider the algebra $M_{\chi-tb} = M_{\phi-\chi} \otimes L_1$. It follows that the free Leibnitz module $H_D(M_{\chi-tb})$ is equivalent to

$$H_D(M_{\chi-tb}) \cong L_1 \otimes H_D(\mathbb{C}[h]) \otimes SF^t \cong L_1 \otimes SF^t \otimes \mathbb{C}[t_1, t_3, \ldots, t_{2n-1}, \ldots]. \tag{45}$$

Now consider the larger free Leibnitz module $\tilde{V} = H_{T_{-1}}(M_{\chi-tb})$, and its sub-Hopf algebra $\tilde{W} = H_D(L_1)$. We now define the following quotient space, by imposing a relation:

$$V^{\chi-tb} = \tilde{V}/\{Te^{\alpha} = e^{-\alpha}\}, \tag{46}$$

Denote the quotient relations generated from $\{Te^{\alpha} = e^{-\alpha}\}$ by $\mathcal{R}^{\chi-tb}$. The space $V^{\chi-tb}$ will be the space of fields for our twisted vertex algebra. In order for the construction to work, we need to use a bicharacter compatible with the relation $\mathcal{R}^{\chi-tb}$, in particular, such a bicharacter would have to satisfy

$$r_{-z,w}(e^{\alpha} \otimes e^{\alpha}) = r_{z,-w}(e^{\alpha} \otimes e^{\alpha}) = \frac{1}{r_{z,w}(e^{\alpha} \otimes e^{\alpha})}. \tag{47}$$

We choose the following bicharacter $r^{\chi-tb}$:

$$r_{z,w}^{\chi-tb}(e^{\alpha} \otimes e^{\alpha}) = \frac{z+w}{z-w}. \tag{48}$$

It satisfies the required relations, and thus we can extend by covariance and the bicharacter properties to the whole of $V^{\chi-tb}$. Note that the relation $\mathcal{R}^{\chi-tb}$, together with how we defined the projection map (specifically its product property), implies additionally that on $\pi_T\left(V^{\chi-tb}\right)$ we will have to have $\pi_T(e^{\alpha}) = -1$. Hence, finally, we have the space of states $W^{\chi-tb}$ for our twisted vertex algebra:

$$W^{\chi-tb} = \pi_T\left(V^{\chi-tb}\right) \cong SF^t \otimes \mathbb{C}[t_1, t_3, \ldots, t_{2n-1}, \ldots]. \tag{49}$$

Thus, from Theorem 3.12, we have the following proposition:

Proposition 4.3 *The twisted vertex algebra with space of fields* $V^{\chi-tb} = H_{T_{-1}}(M_{\chi-tb})/\mathcal{R}^{\chi-tb}$ *and space of states* $W^{\chi-tb} = \pi_T\left(V^{\chi-tb}\right)$*, generated via the covariant bicharacter* $r^{\chi-tb}_{z,w}$*, is isomorphic to the twisted vertex algebra generated by the field* $\chi^{tb}(z) = V^{+}_{\sigma t}(z)H^{\sigma\chi}(z)V^{-}_{\sigma t}(z)$ *on the space* $SF^t \otimes \mathbb{C}[t_1, t_3, \ldots, t_{2n-1}, \ldots]$.

The bicharacter descriptions of the vertex algebras involved in the bosonic side of the second, untwisted bosonization of the CKP, can be done similarly, which we will leave to a lengthier article. We want to leave the reader with one of the direct applications of the bicharacter construction, which is the derivation of certain vacuum expectation values and identities. In [5] we derived the following general vacuum expectation value formula that is valid for any twisted vertex algebra based on a pair $(\mathbb{C}\{\phi\}, r)$ with ϕ being an odd primitive variable, and **any** bicharacter r (see Proposition V.4 of [5]):

$$\langle 0 \mid \phi(z_1)\phi(z_2)\ldots\phi(z_{2n})|0\rangle = i_z Pf\left(r_{z_i,z_j}(\phi\otimes\phi)\right)_{i,j=1}^{2n}. \tag{50}$$

Here Pf denotes the Pfaffian of an antisymmetric matrix and i_z stands for the power series expansion in the region $|z_1| > |z_2| > \cdots > |z_{2n}|$.

Note that such is the case of $M_{\phi-\chi} = \mathbb{C}\{\phi^{\chi}\}$, hence we immediately have

$$\langle 0 \mid H^{\chi}(z_1)H^{\chi}(z_2)\ldots H^{\chi}(z_{2n})|0\rangle = i_z Pf\left(r^{\phi-\chi}_{z_i,z_j}(\phi^{\chi}\otimes\phi^{\chi})\right)_{i,j=1}^{2n} = i_z Pf\left(\frac{z_i - z_j}{(z_i+z_j)^2}\right)_{i,j=1}^{2n}. \tag{51}$$

Similarly, we derived a general vacuum expectation value formula that is valid for any twisted vertex algebra based on $L_1 = \mathbb{C}[\mathbb{Z}\alpha]$ and any choice of a bicharacter r. Namely, Proposition V.6 of [5] asserts that

$$\langle 0 \mid e^{m_1\alpha}(z_1)e^{m_2\alpha}(z_2)\ldots e^{m_n\alpha}(z_n)|0\rangle = i_z\delta_{m_1+m_2+\cdots+m_n,0}\prod_{i<j,\ i,j=1}^{n} r_{z_i,z_j}(e^{m_i\alpha}\otimes e^{m_j\alpha}).$$

Hence, since our twisted vertex algebra is based on the tensor product $M_{\chi-tb} = M_{\phi-\chi}\otimes L_1$, we immediately obtain that

$$\langle 0 \mid \chi^{tb}(z_1)\chi^{tb}(z_2)\ldots\chi^{tb}(z_{2n})|0\rangle = i_z Pf\left(\frac{z_i - z_j}{(z_i+z_j)^2}\right)_{i,j=1}^{2n}\cdot\prod_{i<j}^{2n}\frac{z_i+z_j}{z_i-z_j}. \tag{52}$$

To obtain vacuum expectation values for the original twisted vertex algebra generated by $\chi(z)$ on F_{χ}, we need to modify the Proposition V.4 of [5] to accommodate a pair

$(\mathbb{C}\{\chi\}, r)$ with χ being an even primitive variable (which in fact will make it easier, as there will be no minus signs to account):

Proposition 4.4 *Let V be a twisted vertex algebra based on $M = \mathbb{C}\{\chi\}$, where χ is an **even** primitive variable, and **any** supersymmetric bicharacter r (in particular, $V = H_{T_{-1}}(M) = H_D(\mathbb{C}\{\chi, T\chi\})$ and $W = H_D(\mathbb{C}\{\chi\})$). Denote by $\chi(z)$ the field $Y(\chi, z)$ produced by definition* (40)*, via* (39). *The following formula for the vacuum expectation values holds:*

$$\langle 0 \mid \chi(z_1)\chi(z_2)\ldots\chi(z_{2n})|0\rangle = i_z Hf\big(r_{z_i,z_j}(\chi\otimes\chi)\big)_{i,j=1}^{2n}. \tag{53}$$

Here Hf denotes the Hafnian of a symmetric matrix of even size: for a symmetric matrix A of size $2n$ by $2n$

$$Hf(A) = \sum_P A_{i_1,i_2}A_{i_3,i_4}\ldots A_{i_{2n-1},i_{2n}}.$$

The sum is over all permutations P, $P(k) = i_k$, of $\{1, 2, \ldots, 2n\}$, such that $i_1 < i_2,\ i_3 < i_4, \ldots, i_{2n-1} < i_{2n},\ i_1 < i_3 < \cdots < i_{2n-1}$.

Proof The proof is very similar to the proof of V.4 of [5], but we want to explain why the bicharacter of the primitive element determines that the summation is over these particular permutations only, thus resulting in a Hafnian. Since χ is a primitive element we have $r_{z,w}(\chi\otimes 1) = r_{z,w}(1\otimes\chi) = 0$ for any bicharacter. To calculate the vacuum expectation value, we need the analytic continuation $X_{z_1,z_2,\ldots,z_{2n}}(\chi\otimes\chi\ldots\chi)$, which has explicit formula in terms of the $(2n)$-character $r_{z_1,z_2,\ldots,z_{2n}}(\chi\otimes\chi\otimes\ldots\otimes\chi)$ (see [5]). The only contributions in the $(2n)$-character $r_{z_1,z_2,\ldots,z_{2n}}(\chi\otimes\chi\otimes\ldots\otimes\chi)$ will come from the following situation: a nonzero summand in the $(2n)$-character $r_{z_1,z_2,\ldots,z_{2n}}(\chi\otimes\chi\otimes\ldots\otimes\chi)$ will be a product of nonzero bicharacter factors, and that happens only when we have a sequence of either $(1, 1)$ pairs (non-contributing, but non-zero, as $r_{z,w}(1\otimes 1) = 1$), or (χ, χ) pairs (nontrivial contributions). If there is a mixed pair $(1, \chi)$ or $(\chi, 1)$ appearing in a factor in a summand, that summand will be 0. So a nonzero summand will have exactly n such nontrivial contributing pairs (χ, χ), and each pair forms one bicharacter $r_{z_{i_{2k-1}},z_{i_{2k}}}(\chi\otimes\chi)$.

Corollary 4.5 (of Propositions 4.1 and 4.4) *For the generating field $\chi(z)$ of the twisted vertex algebra F_χ we have*

$$\langle 0 \mid \chi(z_1)\chi(z_2)\ldots\chi(z_{2n})|0\rangle = i_z Hf\Big(\frac{1}{z_i + z_j}\Big)_{i,j=1}^{2n}. \tag{54}$$

We finish with the following identity (proved in [18], and re-proved in [23] by using direct calculations and Wick't Theorem), which now follows directly from the twisted bosonization isomorphism and the bicharacter description of both sides of the bosonization:

Corollary 4.6

$$Hf\Big(\frac{1}{z_i+z_j}\Big)_{i,j=1}^{2n} = Pf\Big(\frac{z_i-z_j}{(z_i+z_j)^2}\Big)_{i,j=1}^{2n} \cdot \prod_{i<j}^{2n}\frac{z_i+z_j}{z_i-z_j}. \tag{55}$$

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