The Chain Rule

The **Chain Rule** comes into play when we have to deal with a function that is a **composition** of other functions. As a review, if $y = f(u)$ and $u = g(x)$, the composite function $f(g(x))$ is the function of *x* formed by substituting $u = g(x)$ in the formula for $f(u)$.

The **Chain Rule**: If $y = f(u)$ is a differentiable function of *u* and $u = g(x)$ is in turn a differentiable function of *x*, then the composite function $y = f(g(x))$ is a differentiable function whose derivative is given by the product:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \text{ or } \frac{dy}{dx} = f'(g(x))g'(x)$$

Example: Consider $y = (x^3 + 5)^2$.

a) Differentiate this function as $y = (x^3 + 5)(x^3 + 5)$ using the product rule.

b) Differentiate this function using the chain rule.

The **General Power Rule**: For any real number *n* and differentiable function *h*,

$$\frac{d}{dx}[h(x)]^n = n[h(x)]^{n-1}\frac{d}{dx}[h(x)]$$

Example: Differentiate using the chain rule.

a) $m(x) = (x^3 + 5)^{1/3}$ b) $r(x) = \sqrt{5x - 2}$ c) $f(x) = \dfrac{1}{(x^3 + 7)^{11}}$

d) $h(x) = \dfrac{1}{\sqrt[3]{2x^4 + 3x}}$ e) $y = \dfrac{5}{6x^2 + 7x - 6}$ f) $f(x) = \dfrac{2}{(x^4 - 3x^2)^5}$

Example: Differentiate combining Product/Quotient Rule with Chain Rule.

a) $f(x)=x^3\left(x^4+3\right)^2$ b) $y=\dfrac{5x^2}{(6x-7)^3}$ c) $h(x)=2x^2\sqrt{8-x^3}$

Example: Find the equation of the line that is tangent to $f(x)=\left(x^4-15\right)^5$ when $x=2$.

Example: Find the second derivative of $f(s)=\dfrac{2}{5s+1}$.

Example: A manufacturer's total cost (in dollars) function of producing *x* items is

$$C(x)=500+\sqrt{27+x^2}$$

a) Find the exact cost of producing the third item.

b) Find the **marginal cost function.**

c) Find $C'(2)$ and interpret.

Example: A small, polluted lake was treated with a chemical to destroy bacteria. It is estimated that after *t* days of treatment, the number of bacteria in one CC of water should be $N(t)$ where $N(t) = 500(8-t)^2$.

a) Find $N'(t)$.

b) Find $N'(6)$ and interpret.

Example: A metal sphere is heated so that *t* seconds after the heat is applied, the radius $r(t)$ is given by $r(t) = 3 + .01t$ centimeters. Find the rate at which volume is changing after 100 seconds. (Volume of a sphere as a function of radius is given $V(r) = \frac{4}{3}\pi r^3$.)