Dissipation at singularities of the Nonlinear Schrödinger Equation through limits of regularisations

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Abstract

The possibility of physically relevant singular solutions of the nonlinear Schrödinger equation (NLSE) with sustained dissipation into the singularity is considered through numerical study of a dissipative regularisation and its small dissipation limit. A new form of such dissipative solutions is conjectured for certain parameter ranges where this behaviour was previously not expected, involving a multi-focusing mechanism. A possible mechanism is discussed involving a new family of stationary singular solutions of the NLSE.

1 Introduction

The Nonlinear Schrödinger Equation (NLSE)

$$\psi_t(t,x) = i\Delta\psi(t,x) + i|\psi(t,x)|^{2\sigma}\psi(t,x),\ t \geq 0,\ x \in \mathbb{R}^d \tag{1}$$

is a generic model for the slowly varying envelope of a wave-train in conservative, dispersive, mildly nonlinear wave phenomena. Physical applications (all with the cubic nonlinearity $\sigma = 1$) include the collapse of various wave-modes in plasmas [1, 2], where $d = 3$, as well as laser self-focusing [3], where $d = 2$. This *wave collapse* (or *self-focusing*) is manifested in solutions of Equation (1) by the development of large gradients in small regions and in some cases, singularities in finite time.

The cubic nonlinearity $\sigma = 1$ will be assumed in most places. However there are occasions to consider other values, such as mathematical phenomena that arise only with a different value, and all results and conjectures generalise to $\sigma > 1/2$, so in places these generalisations will be noted in brackets.

For $d \geq 2$ [$\sigma d \geq 2$], certain initial data lead to finite time singularities as shown by Vlasov, Petritshev and Talanov [4] and Glassey [5]. Physically these will not occur due to phenomena ignored by the above equation, and for plasma physics applications, one such effect is multi-photon absorption which can be modelled by adding a nonlinear dissipation term to get the Dissipative Nonlinear Schrödinger Equation (DNLSE)

$$\psi_t = i\Delta\psi + i|\psi|^2\psi - \beta|\psi|^m\psi. \tag{2}$$

as previously studied by Talanov and Vlasov [6] and by Zakharov, Kosmatov and Shvets [7].

In this paper two possible effects of this modification are considered: the physical question of whether the combination of focusing to a near-singularity plus dissipation localized in the resulting small regions of high field intensity can lead to substantial dissipation (energy transfer from wave to plasma) before the focus disperses, even for very small β; and the related mathematical question of whether such behavior for small β can be modelled by dissipative weak solutions arising in the limit $\beta \searrow 0$. Such weak solutions, akin to shock waves, could be useful if they give behavior that is relatively insensitive to the details of the nonlinear dissipative term, since such details will not necessarily be known precisely in practice.

It will be argued from a combination of numerical simulations and analysis that such sustained dissipation does occur for a range of parameters that includes the main plasma physics application, but with the limit involving a new oscillatory pattern of inward flux and dissipation for d near 2 [σd near 2] in a way which suggests the occurence of weak solutions rather than the almost stationary form observed by the above authors for $3 < d \leq 4$ [$2+1/\sigma < d < 2 + 2/\sigma$].

Special numerical methods have been developed to achieve the fine resolution needed near the singularity or collapse centre: these will be described briefly here but are dealt with at more length elsewhere [8, 9].

As these results are presented for the general mathematical interest of understanding a "canonical" singularity formation problem as well as for the sake of the physical applications discussed above, it is convenient to assume spherical symmetry and to allow non-integer dimensions.

Restriction to spherical symmetry can be supported by the considerable evidence [10, 8, 12] that for more general initial data, solutions with a developing singularity or focus are locally asymptotic to spherically symmetric forms.

Non-integer dimensions allow all qualitatively different known behaviours to be achieved with the cubic nonlinearity $\sigma = 1$ and allow the critical phenomena that occur at each integer dimension to be studied through limits. Alternatively, analogous results can all be obtained with dimension $d = 3$ (and some with dimension $d = 2$) by choosing σ appropriately, but this is numerically and analytically more difficult and both analysis and numerical comparisons suggest that the results are analogous.

2 Background: Known and Previously Conjectured Singular Solutions

The Nonlinear Schrödinger Equation (1) has two important conserved quantities, the "power"

$$P = P(\psi) := \int |\psi(t,x)|^2 \, dx \tag{3}$$

and the "energy"

$$H = H(\psi) := \int \left(|\nabla \psi(t,x)|^2 - \frac{1}{2}|\psi(t,x)|^4 \right) \, dx. \tag{4}$$

It was shown by Vlasov, Petritshev and Talanov [4] and Glassey [5] that solutions of the NLSE become singular at a finite *collapse time* T if $d \geq 2$ [$\sigma d \geq 2$] and the energy H is negative.

In general all that is known about the structure of the singularity is that the supremum and H^1 norms go to infinity. In the *super-critical* case $d > 2$ [$\sigma d > 2$] explicit spherically symmetric self-similar solutions are known that develop point singularities: up to a phase shift and translation these are

$$\psi(t,x) = \frac{1}{\sqrt{2\kappa(T-t)}} Q\left(\frac{|x|}{\sqrt{2\kappa(T-t)}} \right) \exp\left(\frac{i}{2\kappa \ln[1/(T-t)]} \right) \tag{5}$$

where $\kappa > 0$, $Q(0) = q_0 > 0$, $Q'(0) = 0$ and $Q(r)$ is a solution of the ODE

$$-\left(Q'' + \frac{d-1}{r} Q' \right) + Q - |Q|^2 Q - i\kappa(Q + rQ') = 0. \tag{6}$$

Numerical evidence for a large variety of initial data and values of d [and of σ] suggests that in this super-critical case, singular solutions are asymptotic to translations of the above solutions as one approaches the time and place of the singularity, with values of κ and q_0 that depend only on d [and σ], not the initial data [13].

For the critical case a related but more complicated picture has been developed [14, 15, 16, 17]. In particular, the structure near the place and time of a singularity again appears to be asymptotic to a universal, spherically symmetric form. The most important differences are that the singular solutions are now only "almost self-similar", and that all singular solutions that are known, heuristically described or numerically observed have a fixed quantum of "power" concentrating into the singularity locus as $t \to T$. This quantum is in fact the total power in the known explicit singular solutions, for which all power concentrates into the singularity (see [14] for example.)

Of course, in the physical situations being modelled by the NLSE there will not be a true singularity, and one wants to know what happens after the collapse time T. Fortunately there is hope for weak solutions continuing beyond the collapse time, due to numerical evidence that solutions generically have a smooth pointwise limit as $t \nearrow T$ for every point except the collapse locus [8]. Since the same numerical results indicate (contrary to some earlier speculations) that the singularity at $t = T$ is not "removable", the point singularity should be expected to persist at later times.

One then might try to relate such singular weak solutions of the NLSE to solutions of some more physically accurate and singularity-free modification of the NLSE, such as the DNLSE of Equation (2).

Previous authors, starting with Zakharov and Kuznetsov [18], have conjectured several different scenarios for different values of d [and σ], based on the presence or absence of two features. The first is instantaneous power loss into the singularity due to a wave collapse in which a fixed positive amount of the power density is contained in a ball whose radius shrinks to zero as the time of collapse is approached, which occurs only in the critical case. The second feature is sustained power loss (dissipation) due to power flux into a sustained singularity, motivated by the existence of stationary solutions with power flux into a singular in various cases.

In the critical case $d = 2$ [$\sigma d = 2$], relevant to laser self-focusing, only the the first of these is known to occur, so the natural conjecture is *strong collapse*: weak solutions with an instantaneous power loss at the collapse time of the power quantum that concentrates into the singularity, and possible further instantaneous power dissipation events at discrete time intervals due to repeated collapses into the singularity.

For $2 < d < 3$ [$2/\sigma < d < 2 + 1/\sigma$], with no strong collapse and no known stationary singular solutions, these authors predicted *weak collapse* with no dissipation. This was supported by numerical solutions of the DNLSE with fixed initial data and different β, which suggested that the solutions go through a cycle of wave collapse leading to rapid nonlinear dissipation followed by dispersal which essentially ends the dissipation, with the total power dissipated in this cycle going to zero in the limit $\beta \searrow 0$. This pattern is sugggested if one considers only the early parts of the graphs in Figures 4 and 5 below. Talanov and Vlasov [6] also suggested that for parameters close to the critical case there might be a succession of brief events each dissipating approximately the same quantum of power as in the previous case.

For $3 \leq d \leq 4$ $[2 + 2/\sigma \leq d \leq 2 + 4/\sigma]$, relevant to plasma wave collapse, Talanov and Vlasov and also Malkin [19] and Zakharov, Kosmatov and Shvets [7] discovered the previously mentioned stationary singular dissipative solutions of the NLSE and conjectured what will here be called *super-strong collapse*: weak solutions of the DNLSE having sustained dissipation through power loss into a persistent collapse region of high intensity and hence intense nonlinear dissipation. These solutions would have a "quasi-stationary" state which approximates a stationary singular solution of the NLSE, though for physically relevant solutions which must have finite power, the approximation is of necessity only local in both space and time.

These stationary singular solutions are just solutions $\psi(r) = \psi(|x|)$ of the Stationary NLSE

$$\psi'' + \frac{d-1}{r}\psi' + |\psi|^2\psi = 0 \tag{7}$$

with a singulrity at the origin. The simplest case is for $d = 4$ $[d = 2 + 2/\sigma]$, where Equation (7) and hence the NLSE have the explicit singular solutions

$$\psi(r) = B\exp(i\log r\sqrt{B^2-1})/r, \quad r = |x|, \text{ any } B > 1. \tag{8}$$

The "power density" $|\psi|^2$ flows into the singularity at a rate proportional to

$$P := \lim_{r\to 0} |\psi|^2 r^{d-1}\frac{d}{dr}\arg\psi = B^2\sqrt{B^2-1}. \tag{9}$$

For $3 < d < 4$ $[2 + 1/\sigma < d < 2 + 2/\sigma]$ the existence of such solutions is suggested by formal asymptotic expansions with leading order term again $O(1/r)$, while for $d = 3$ $[d = 2 + 1/\sigma]$ there is a logarithmic modification in the expansions, and this family of solutions dissappears for $d < 3$ $[d < 2 + 1/\sigma]$, which is why weak collapse continued to be conjectured in that case.

The above-cited authors conjectured that for some interval of time after T, the limit $\beta \searrow 0$ of solutions to the DNLSE with fixed initial data gives local convergence to stationary singular solutions of the NLSE, and in particular the limit of the dissipation rates is the rate for this limit solution. Thus for small β, the rate of power dissipation and the intensity profile of the solution should not depend much on the form of the dissipative term. Of course for solutions which must have finite power, this can only be true locally in space and time.

3 More Stationary Singular Solutions and New Conjectures

In the work discussed above it was conjectured that collapse would be "weak" for $2 < d < 3$ $[2/\sigma < d < 2+1/\sigma]$ due to the nonexistence of the above $O(1/r)$ solutions to Equation (7). However other formal asymptotic for solutions exist in exactly this case, with some existence theorems. The simplest are real valued solutions of the form

$$\psi(r) = \frac{B}{r^\alpha}(1 + b_1 r^\delta + b_2 r^{2\delta} + \cdots) \tag{10}$$

with $\alpha = d - 2$, $\delta = 2(3 - d)$ and B the sole free parameter. The only obstructions to these formal power series expansions, occurring for a discrete set of d values, can be overcome by adding the usual "log-power" terms and any such terms needed are bounded at the origin. The existence of singular solutions with the above leading order asymptotics has been proven by P.L. Lions [20]. (There, only integer dimensions are considered, so the results are proven for $d = 2$ and any σ: extension to the current cases should be routine.)

Such real valued solutions have no flux, but one can get solutions $\psi = \phi e^{i\theta}$ with arbitrary inward flux P by combining a real solution of

$$\phi'' + \frac{d-1}{r}\phi' + |\phi|^2\phi - P^2/(r^{2d-2}\phi^3) = 0$$

with the phase given by $\theta' = Pr^{1-d}/\phi^2$. The extra term is bounded at the singularity, so formal asymptotic expansions exist with the same leading order terms as above, B/r^α for any B, and the same comments apply about obstructions.

One conjecture arising from this observation is that solutions of the NLSE develop persistent singularities that are locally asymptotic to these special singular solutions, leading to steady sustained dissipation into the singularities. The next question is how this is manifested in solutions of the DNLSE. The above singular solutions do not apply in the *dissipative core* of the wave collapse where $|\psi| = 0(\beta^{-1/(m-2)})$ so that the dissipative term is significant.

However they are good approximations a bit further away, in the *focusing shoulder* where $1 << |\psi| << \beta^{-1/(m-2)}$.

It can then be conjectured that in this shoulder region, solutions approximate the singularities described above, producing a fairly steady flux into the core, wherein a possibly more complicated pattern of dissipation occurs.

Numerical evidence discussed below suggests that for d values from 2 up to about 3 $[2/\sigma$ to roughly $2+1/\sigma]$ dissipation in the core is in bursts due to multi-focusing: the intensity is repeatedly reduced by dissipation and dispersion enough to stop dissipation, but the inwards flux from the shoulder repeatedly rebuilds the core and restarts dissipation until substantial total dissipation and dispersion prevent further focusing. Further, as β gets smaller the dissipation events become briefer, faster and closer together, and so give a descending staircase pattern for power as a function of time with steps of descreasing size and width, and as $\beta \to 0$, the apparent limit is power smoothly decreasing at a dissipation rate that is roughly constant in time until a substantial proportion of the initial power has been dissipated and then gradually flattens out. This limiting behavior resembles that of the super-strong collapse scenario rather than either the strong or weak collapse scenarious previously conjectured for this parameter range.

For the rest of the range of d values up to 4 $[2 + 2/\sigma]$ the numerical evidence supports the previous conjecture of more or less steady, ultra-strong collapse, but it is not clear that the transition between the two forms is exactly at $d = 3$, since there is some evidence of multi-focusing above this threshold. In any case, the behavior for sufficiently small β is quite similar in the two cases until a significant amount of dissipation has occurred.

4 Overview of Numerical Difficulties and Methods

One difficulty in numerical study of these ideas is the cost associated with higher spatial dimensions. The super-strong collapse regime has $d > 2$ for any choice of σ, so physically one needs $d \geq 3$. Increasing σ also increases spatial gradients at a given intensity, making spatial resolution harder to achieve, another reason for using mostly $\sigma = 1$: then the mathematically interesting range of dimensions extends up to $d = 4$ [corresponding to $d = 3$ for $\sigma = 2$].

Both to reduce the computational costs of high spatial dimensions and to make fractional dimensions meaningful, spherical symmetry is assumed in most studies including this one. A few other studies [10, 8] have been done without this restriction, but generally show an asymptotically spherically symmetric behaviour near the focus center, giving some justification for this simplification.

The greater numerical problem comes from the fact that the behaviour near these conjectured point singularities only becomes clear when the amplitude has grown by very large factors; early numerical results led to wrong conjectures even with amplitude growth by factors of several hundred. This means that very fine space and time resolutions is needed near the focus center, requiring a high degree of time dependent grid refinement. Here this is addressed by a refinement of the methods of *dynamic adaptive spatial discretisation* developed by this author and collaborators [21, 22, 13] and independently by Zakharov and Shvets [23], based on conjectured qualitative properties of the singularities. The main modification described here is to make the grid point movement more local to the region of the focus center, while holding boundary points stationary: this better fits the behaviour far from the singularity and allows for the imposition of boundary conditions other than decay to zero at infinity.

The approach to be discussed here uses a physical radial variable r that is a time dependent function of a computational variable ρ, which is locally of dilation form near the origin but respects boundary conditions. A fixed ρ discretisation will be used, and the coordinate transformation will be given analytically as a function of a single dynamically determined parameter $l(t)$, allowing the flexibility in the choice of the ρ discretisation and easy vector/parallel implementation.

The spatial coordinate will have the same domain $[0, r_{max}]$ and have homogeneous Neumann or Dirichlet boundary conditions at r_{max} and homogeneous Neumann boundary conditions at 0 due to the symmetries.

The physical coordinates are given in terms of fixed computational coordinates through the transformation

$$r = f(\rho, l(t)), \ \ \rho \in [0, 1], \tag{11}$$

so the transformed equation is

$$\psi_t = i\Delta\psi + i|\psi|^{2\sigma}\psi - |\psi|^m\psi + \psi_r r_l l_t. \tag{12}$$

with the same boundary conditions. Note that all derivatives of ψ including those in the Laplacian are still with respect to the physical coordinate r, not ρ.

The transformation function should be odd, increasing, achieve the inner scaling by having $f_\rho(\rho, l)|_{\rho=0} = l$, and fix the outer boundary by having $f(1, l) = r_{max}$.

The choice used in this paper is

$$f(\rho, l) = l \sinh(k(l)\rho) \tag{13}$$

where $k(l)$ is determined by the condition $f(1, l) = r_{max}$ to fix the outer boundary. This exponential stretching further concentrates the mesh points in the collapse region in a way that gives good stability behaviour in the presence of the numerical advection seen in the transformed equation.

The length scale $l(t)$ is based on the functional

$$l^*(\psi(t,.)) = C\frac{\int |\psi||\nabla\psi|^d}{\int |\nabla\psi|^{d+1}}. \tag{14}$$

This functional is designed to be convergent in the presence of the behaviour $|\psi| \approx |x|^{-1/\sigma}$ that develops as the singularity is approached, and to be numerically stable (which simpler measurements at the origin only are not).

Using $l(t) = l^*(\psi(t,.))$ directly would give badly non-linear and non-local terms through the presence of l_t in the equation, highly undesirable when implicit time discretisations are used. Therefore the evolution of $l(t)$ is decoupled from the main evolution equation, determining its values through a time step before that step is started with an extrapolation that is sufficiently smooth and keeps the value close to that of l^*. Specifically, $l(t)$ is determined from continuity and requiring that $\dfrac{dl}{dt} = \dfrac{l_n^* - l_{n-1}^*}{t_n - t_{n-1}} + \dfrac{l_n^* - l_n}{t_n - t_{n-1}}$ within each time step $[t_n, t_{n+1}]$, where $l_n^* = l^*(t_n)$ etc.

Finally, the time discretisation is done by a partially implicit PC method so as to to avoid solving nonlinear equations with implicit time differencing: the Laplacian term is handled by the Crank-Nicholson method (really the trapezoid rule) and the other terms by the two stages of the modified Euler method.

5 Accuracy Checks

The NLSE has several conserved quantities that can be used to check the accuracy of solutions: the "power" $\|\psi\|_2^2$ and the "energy" H introduced in Equation (4) above. However the latter comes to be the difference of two very large quantities during wave collapse, so suffers substantial errors even when the solutions is quite accurate overall. Thus the power is the main one used. In the dissipative case power is not conserved but has a simple evolution equation:

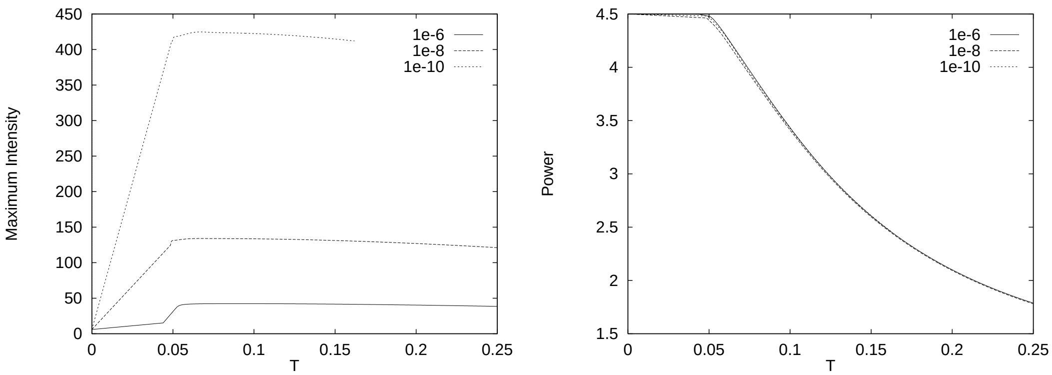

Fig. 1. (a) Solution maximum amplitude and (b) solution power, both vs. time for $d = 4$, $\psi_0 = he^{-|x|^2}$, $h = 6$, $\beta = 10^{-6}, 10^{-8}$ and 10^{-10}.

this can be integrated and the values checked against the actual power. Other functionals can be treated similarly but it is best to keep to ones that do not involve spatial derivatives: thus the $L^{2(\sigma+1)}$ norm (which forms part of H) is also checked, and data is discarded when either error exceeds an appropriate tolerance.

This has worked well in practice: the stage of a run where this test fails corresponds well to the start of significant divergence of the solution from results of computations with more refined discretisations.

6 Numerical Results

In addition to fixing $\sigma = 1$, the numerical results presented here are for initial data $\psi_0 = he^{-|x|^2}$, and have $h = 6$ with $m = 6$ except where noted. Figures 3 and 6 show that varying m and β while maintaining roughly equal maximum amplitude, length and time scales gives qualitatively very similar behaviour: this and similar unpublished results support the restriction to $m = 6$ in the following. Note also that power and other spatial integrals are computed using the unnormalised volume measure $r^{d-1}dr$ to simplify the handling of fractional dimensions.

The case $d = 4$ has the most clear-cut behaviour and the numerical results here add to other numerical evidence [7, 24, 25] for the "super-strong collapse" scenario. However, the present calculations explore dissipation parameters β that give far larger maximum amplitudes and higher degrees of collapse, and study the effect of varying the dissipative nonlinearity power m and the amount of power in the initial data, to better document the universality of the behaviour.

Figure 1 for initial amplitude $h = 6$ and various β values indicates the quasi-

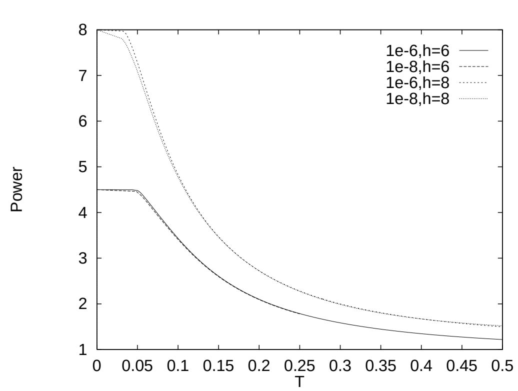

Fig. 2. Varying the initial data: height $h = 6$ as above and $h = 8$.

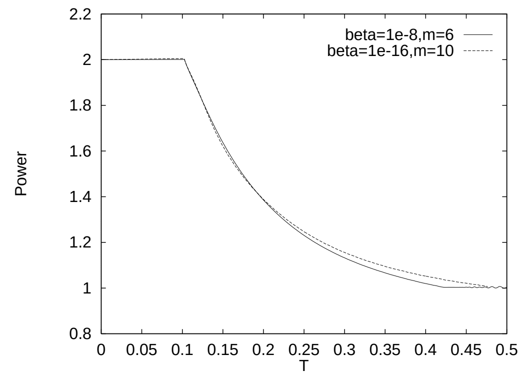

Fig. 3. Varying m and β with similar maximum amplitude: $m = 6, \beta = 10^{-8}$ as above and $m = 10, \beta = 10^{-16}$.

steady collapse state and shows that the pattern of dissipation in time is essentially independent of the parameter β. The results for different initial data in Figure 2 (see also [24]) show that the dissipation is limited only by the total available power and its distribution, with the total power dissipated increasing with the initial power: indeed it appears that the final amount of power left is approximately 1 in each case.

Further, as already shown in [25], spatial cross-sections of amplitude in the shoulder region (where the amplitude is enough for the focusing nonlinearity to be significant but small enough for the dissipation to be negligible) vary little with time and are approximately B/r. As the power is exhausted, the dissipation rate must slow down, which is reflected in the slow decrease of B and hence also the dissipation rate P as indicated by Equation (9).

At the other extreme, $d = 2$, the previously conjectured "strong collapse" (with a fixed quantum of energy being dissipated), is not seen. Instead one sees "multifocusing", a cycle of focusing and defocusing, with each cycle having a brief interval of rapid dissipation followed by defocusing and then a far longer interval with essentially no dissipation. As shown by Figure 4, for small β the dissipation curve has an almost stair-case form, and the scaling of time and amount of dissipation is such that theses curves appear to converge to a smoothly decreasing limit curve for power as a function of time. Further, increasing the power of the initial data increases the total dissipation: there again appears to be no limiting quantum of power dissipated as in the strong collapse scenario, and the total dissipation in the case show here is more than three times the predicted quantum.

Thus the behaviour in the limit as $\beta \to 0$ appears to be "super-strong collapse" again, though approached by a different and more complicated path.

For $d = 2.5$, very much the same pattern is seen (Figure 5): there appears to be a succession of weak dissipation events converging step-wise to a smooth

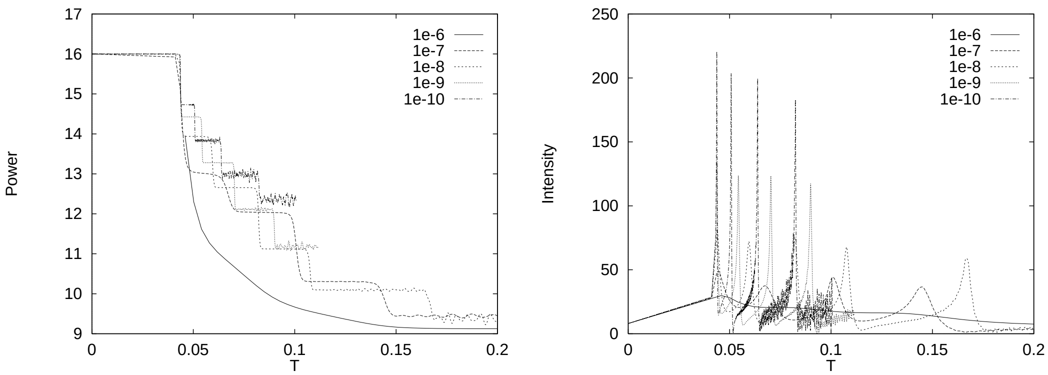

Fig. 4. $d = 2$, $h = 8$ (a) power (L^2 norm squared) vs. time (b) maximum amplitude.

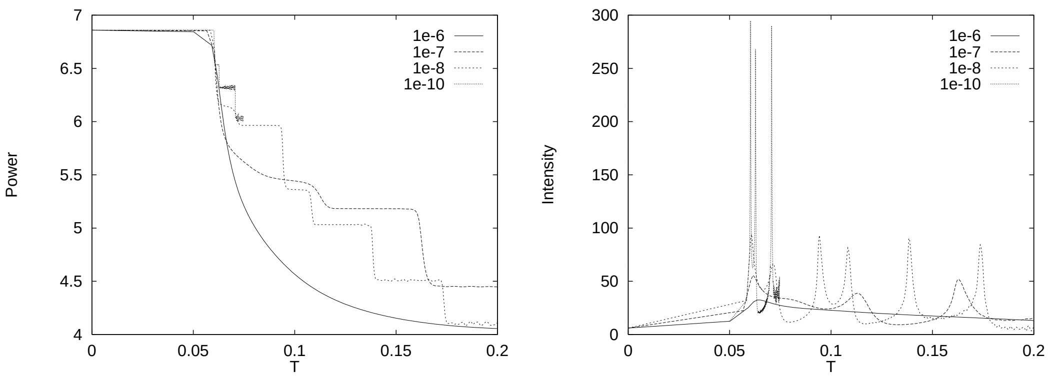

Fig. 5. $d = 2.5$, $h = 6$, otherwise as above.

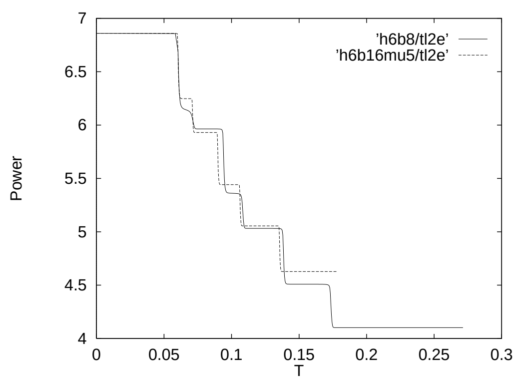

Fig. 6. $d = 2.5$, $h = 6$ varying m and β as in Figure 3.

sustained, "super-strong collapse".

Note that if one only considered behaviour up until the collapse of the initial focus, one might predict a weak collapse with the total energy dissipation converging to zero as $\beta \to 0$ in each of the last two cases, as previously suggested for $d = 2.5$ and corroborated by less extensive numerical studies in which it was not possible to solve accurately far beyond the initial collapse event.

For the physically important borderline case $d = 3$ the numerical results are

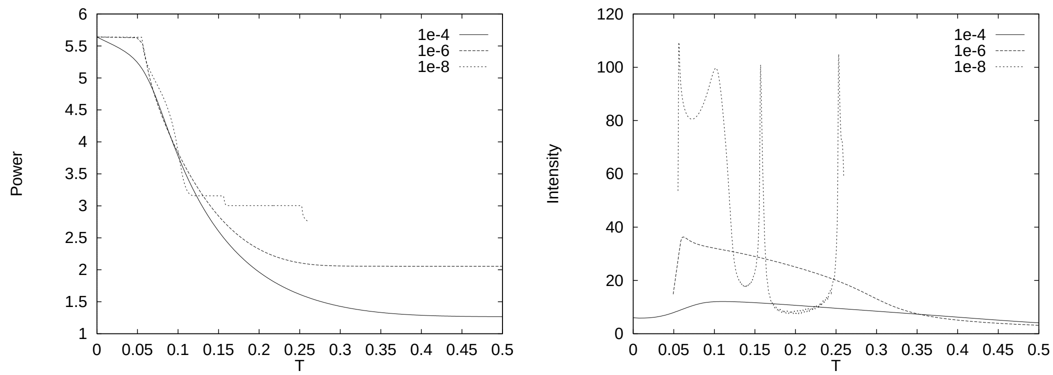

Fig. 7. $d = 3$, $h = 6$, showing a touch of multi-focusing.

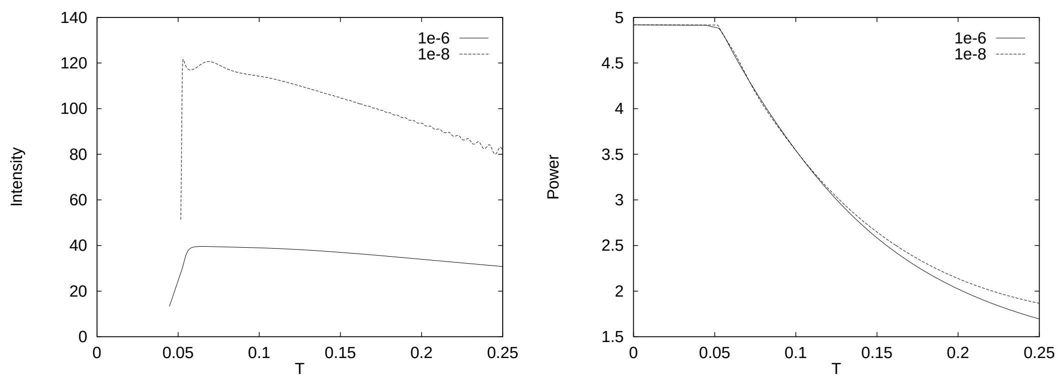

Fig. 8. $d = 3.5$, $h = 6$: a sustained focus or long period multi-focusing?

somewhat ambiguous (Figure 7): as for lower dimensions, the initial collapse event is "weak" with its power dissipation decreasing as β does, and there are subsequent focusing and dissipation intervals. However the initial focus is considerably more sustained, reflecting perhaps greater stability of the different stationary solution forms for $d \geq 3$. Also, it is not yet clear that the subsequent dissipation events are sufficiently strong or frequent to give a non-zero total dissipation in the limit as $\beta \to 0$. Further investigation of this case is suggested.

Finally, for $3 < d < 4$ ($d = 3.5$ in Figure 8) the pattern is similar to the archetypical sustained super-strong collapse seen for $d = 4$, but with some degree of slow oscillation and a hint that the total dissipation (in what might be only the first of a succession of dissipation intervals) is decreasing with β.

7 Conclusions

The main new observations here are

- For $2/\sigma \leq d \leq 2 + 1/\sigma$ at least, solutions of the DNLSE can exhibit a succession of dissipative intervals which produce substantial total power dissipation.
- This differs from both the instantaneous "strong" dissipation conjectured for the critical case $d = 2/\sigma$ and from the weak dissipation (with total dissipation going to zero as $\beta \to 0$) previously conjectured for $2/\sigma \leq d \leq 2 + 1/\sigma$. More generally, the intermittent pattern of dissipation appears not to have been described previously.
- For sufficiently small β, the total dissipation and its overall time scale is largely independent of the details of the dissipative term including the power m, and the dissipation as a function of time appears to have a continuous limiting form in the limit as $\beta \to 0$ depending only on the initial data.
- This apparently continuous limiting behaviour is possibly explained by a flux towards the dissipative region in a shoulder region where solutions take on a nearly stationary form which approximates a complex singular solutions of the stationary NLSE (7), related to the real-valued solutions introduced in [20].

In addition, the earlier conjecture of super-strong collapse is numerically tested and corroborated (at least for $d = 4$) more thoroughly than in previous papers, by virtue of the refined adaptive numerical method introduced herein.

The greatest doubts as to the long-term behaviour remain in the most important case $d = 3$, $\sigma = 1$ $[d = 2+1/\sigma]$ at the threshold between two qualitatively different regimes. Here the way forward is probably analysis of such things as the stability of known special solutions, rather than greater numerical resolution. However with both regimes now shown to be cabable of giving substantial dissipation for arbitrarily small nonlinear dissipation terms, one can conjecture that the same holds in this case too.

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