Math 495 Handout: April 10, 2008

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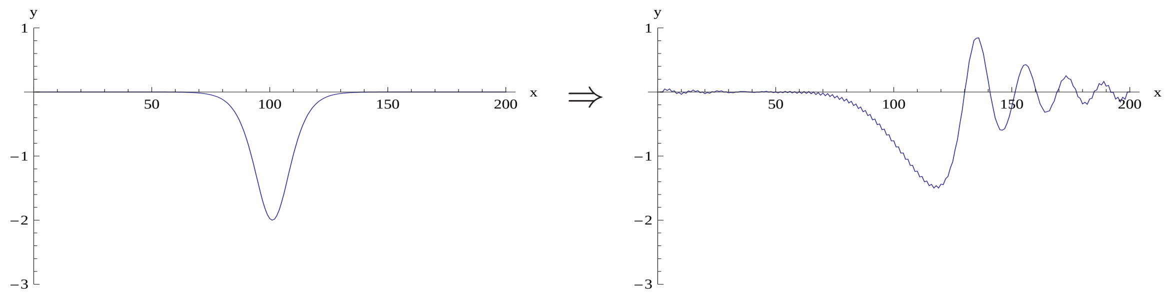
Loose Ends

- Of course, as I said on the first day of class, there is no way I could really teach you even *half* of "Soliton Theory" in this course. We just saw a few **glimpses** of this cutting edge area of mathematical research.
- Still, there are a few more things I'd like to mention before we go on to hear from each of you in the form of your project presentations.

> The new material below will not be tested on the final exam nor is there any homework being assigned today.

- ***Upside-down Soliton Profile?:*** Near the beginning of the class, someone asked me what happens under KdV dynamics to an initial profile that looks like a nice KdV 1-soliton, but inverted so that it is a dip instead of a rise. This is a reasonable question, since any initial shape has a uniquely defined evolution specified by the KdV equation, and because this seems like a reasonable shape to consider as a start. (In fact, there are some other soliton equations for which this initial profile does "translate" like a soliton and is known as a "dark soliton"...but I knew that this does not happen for KdV.) Still, I did not know an answer and when I asked various people I was met with different (and uncertain) answers. One thing that became clear was that even the small difference of changing the sign on the initial profile changed the problem so that it becomes *very* difficult to give an exact answer while the true 1-soliton profile is super-easy. (From a scattering theory point of view, the problem becomes complicated by the fact that the spectral parameter appears in a rational form and so there are singularities. From an algebro-geometric point of view, one has to consider the spectral parameter as living on some Riemann surface instead of just being a number as usual These situations are tricky to actually work with.)
- ***A Partial Answer:*** But, we still wanted to have an idea of what happens. Even though soliton theory was not able to provide us with a nice answer, I have been able to resort to "old-fashioned" techniques to just get a good estimate of what would happen. These techniques predate soliton theory. In fact, I'm using the exact same algorithm that Kruskal and Zabusky used in the 1960's to study KdV dynamics when the discovered solitons! In this sequence of pictures we can see that a bunch of ripples develop heading off to the right, and the initial wave seems to be developing a shock as well. This is as far as I was able to take it before numerical errors began popping up.



- ***Higher Times:*** Another idea I wanted to discuss now, partly because you may have thought of it on your own and partly because it relates to several of the projects, is the idea of "higher time flows". Let's build up to it in two stages.

- ***Remember Spongebob:*** Remember back when we learned about Lax equations, we saw that there was another KdV like equation that came from using $L = \partial^2 + u(x)$ and a *fifth* order operator M. This is in contrast to the KdV equation in which we have a *third* order M. In both cases, we wrote "t" for the time variable because, well, because we were thinking of it as time. But, suppose we wrote t_5 for the Spongebob time and t_3 for the KdV time. Then we could write a *single* function $u(x, t_3, t_5)$ which solves the KdV equation in terms of t_3 and solves the Spongebob equation in terms of t_5.

➢ This is actually connected to the whole "constants of motion" idea. You know that there are lots of constants of motion for the KdV equation...well we can name one of them t_5. The amazing thing is that we can treat that constant as a time variable and then t_3 (the "real" time) becomes a constant of motion for that other evolution equation.

- ***Nicely Weighted Functions:*** Remember also that we said a function was "nicely weighted" if $f_{xx} = f_y$ and $f_{xxx} = f_t$. In other words, if y has weight two and t has weight three. But, why stop there? Why not have a weight four variable and a weight five variable and so on? In general, we could have a variable called t_n and say that the function f is nicely weighted if

$$\frac{\partial^n}{\partial x^n} f = \frac{\partial}{\partial t_n}.$$

➢ Under this notation, we'd say $x = t_1$, $y = t_2$ and $t = t_3$. Then, an example of a nicely weighted function could be written as

$$e^{\sum t_i \lambda^i}.$$

- In fact, this is the mathematically natural thing to do at this point. Intead of limiting ourselves to x and t (the variables that came from the KdV equation describing the evolution of a wave on a canal) or even x, y and t (the extension to two spatial dimensions in the KP equation) we should think of infinitely many variables t_i and each one of those can be thought of as a time in an appropriate context. We're used to $t = t_3$ being time, as it is in both KP and KdV, but for the Spongebob equation, t_5 is time...and for the Boussinesq equation that Julian is learning about in his project, t_2 is time!

- Putting them together this way opens up the possibility of studying lots of different equations of mathematical physics. We can find many of them here by finding the right combination of variables among the t_i's. More importantly, the mathematical theory gets *nicer* when all of the variables are there. As Sai may mention in his project, it is when all of these variables are used simultaneously that the whole theory fits together into a nice, algebraic equation that seems to encapsulate the whole theory most naturally.