Math 495 Handout: March 18, 2008

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Wedge Products and the Grassmann Cone

- Again, it may seem that we're taking a short break from soliton theory to do some algebraic geometry. However, just as we were eventually able to connect our discussion of elliptic curves to the KdV equation, our discussion today will turn out to be deeply related to the structure of the solutions of the KP equation.
- Recall from our discussion of elliptic curves the idea of "algebraic geometry". Although it has grown to be a very sophisticated area of mathematics, the original concept was rather simple. Associate to a polynomial p the geometric object which is the set of points in space for which that polynomial has the value zero. Then, for instance, if your idea of space is the plane $\mathbb{R}^2$ and the polynomial is $p(x,y)=x^2+y^2-1$ then the geometric object is the unit circle. And, if your idea of space is 3-dimensional Euclidean space $\mathbb{R}^3$ the the polynomial is $P(x,y,z)=x^2+y^2-z^2+1$ then the object is a *two-sheeted hyperboloid*.
- *Intersections:* One thing that can make algebraic geometry a little more interesting and useful is to consider not just a single polynomial, but a collection of polynomials. Then, looking at the set of points which make all of the polynomials zero simultaneously is the same as looking at the *intersection* of the objects associated with each one separately. For example, the polynomial $Q(x,y,z)=z-2$ is associate to the plane at height $z=2$. If we look at the points satisfying $P=Q=0$ (with P as above) then we intersect this plane with the hyperboloid to see a circle of radius $\sqrt{3}$ floating around the z-axist at height $z=2$ in $\mathbb{R}^3$.
- *Local Dimension:* Using the three previous examples, we can discuss dimension and connectedness. When I discuss the dimension of a smooth geometric object, I imagine myself being really tiny and standing on the object. If I am so small that I cannot see the curvature of the object (as we cannot see the curvature of the Earth while standing on the ground) then it will look like $\mathbb{R}^n$ for some number n. If this is true at every point, we say the object is n-dimensional. So, for instance, the circle in the plane and the circle floating in 3-space are both 1-dimensional while the hyperboloid is 2-dimensional.
- *Connectedness:* Another geometric question we can ask about these geometric objects is whether they are *connected*. This means, if you pick two points on the object, can you necessarily get from one to the other without ever leaving the object. In the case of the circles, it is clear to us that you *can* always get from one point to another, and so the circles are connected. The hyperboloid, however, is not connected. If one point is on the top sheet and the other point is on the bottom sheet, there is no way to get from one to the other without leaving the surface.
- *What if you were blind?* In thinking about both dimension and connectedness, you probably *visualized* the objects. You imagined seeing them. However, I'd like to point out that we could have determined these geometric properties without using anything like vision. For instance, I know that the kernel of a differential operator of order 4 has dimension 4...but there is no associated "picture" in my mind. Similarly, I could answer the question of whether the surface

is connected by trying to construct a continuous function $r(t) = \langle x(t), y(t), z(t)\rangle$ so that $r(0)$ is one of the points, $r(1)$ is the other point and $r(t)$ lies on the surface for every $0 \leq t \leq 1$.

Two Vector Spaces:

- Let $V = \mathbb{R}^4$ be the set of vectors with four real number components. Let us use the notation e_i for the basis vector with a 1 in the i^{th} component and zeroes elsewhere. Note that I can then write any vector $v \in V$ as

$$v = c_1e_1 + c_2e_2 + c_3e_3 + c_4e_4 = \sum_{i=1}^{4} c_ie_i$$

by choosing appropriate coordinates/cooeficients $c_1, \ldots, c_4$. (Nothing too tricky there, eh?)

- Let $\bigwedge$ be the 6-dimensional vector space with basis

$$e_{12},\ e_{13},\ e_{14},\ e_{23},\ e_{24},\ \text{and} \quad e_{34}.$$

This, maybe, is a little stranger. You may wonder, for instance, "why didn't you just name them e_1 through e_6?" You may also wonder "What's with the weird name for the vector space?" But, trust me, there is a reason for this.
In any case, the point of this is that you can write any vector $\omega \in \bigwedge$ as

$$\omega = c_{12}e_{12} + \cdots + c_{34}e_{34} = \sum_{1\leq i<j\leq 4} c_{ij}e_{ij}.$$

The Wedge Product:

- Let v, w_1 and w_2 be vectors in V. I want to define a kind of "product" which turns a pair of vectors in V into a single vector in $\bigwedge$. Let v, w_1 and w_2 be vectors in V. This "wedge product" has the following properties:

$$w_1 \wedge w_2 = -w_2 \wedge w_1$$

$$v \wedge (aw_1 + bw_2) = av \wedge w + bv \wedge w_2.$$

In other words, it is anti-commutative and distributes linearly over scalar multiplication and vector addition. In fact, the only other property I need to specify to completely identify this product is that

$$e_i \wedge e_j = e_{ij} \quad \text{if} \quad i < j.$$

- Using these properties, it should be possible to figure out what $w_1 \wedge w_2$ is for *any two* vectors $w_1, w_2 \in V$.

Question 1: What is $e_i \wedge e_j$ if $j < i$?

Question 2: What is $e_i \wedge e_j$ if $j = i$?

Question 3: What is $(e_1 + 2e_3) \wedge (e_2 + 3e_3 - e_4)$?

Is it decomposable?:

- Now, given a vector $\omega \in \bigwedge$, we can ask whether we can *decompose* (i.e. "factor") it as a product of two vectors in V. For example, $\omega_1 = e_{12} - e_{13} - e_{23} = (e_1 + e_3) \wedge (e_1 + e_2)$. But, there is *no* way to decompose $\omega_2 = e_{12} + e_{34}$ as $w_1 \wedge w_2$!
  (I like to think of this as being analogous to the question of whether an integer is *prime*. We can factor 8 as 2×4, but we cannot factor 7.)
- Let's use the symbol Γ to denote the subset of $\bigwedge$ of ω's that *are* decomposable. Thus $\omega_1 \in \Gamma$ but $\omega_2 \notin \Gamma$.

Question 4: Can you see how we can use ω_2 to quickly see that Γ is not a *subspace* of $\bigwedge$?

➤ So, Γ does not have the sort of structure that would make it amenable to study though linear algebra, but it does have the sort of structure that we study with algebraic geometry!

- There is actually a way to check whether ω is in Γ by just plugging its coefficients into a polynomial. The polynomial equation

$$\Pi(c_{12}, \ldots, c_{34}) = c_{12}c_{34} - c_{13}c_{24} + c_{14}c_{23} = 0$$

  is called the Plücker relation. If ω has the coefficients $c_{12}, \ldots, c_{34}$ I will "abuse the notation" and write $\Pi(\omega)$ for the number you get when you plug those coefficients into Π.

Decomposability Test: ω is decomposable if and only if $\Pi(\omega) = 0$.

- In particular, we can characterize Γ as an algebro-geometric object:

$$\Gamma = \left\{ \omega = \sum_{1 \le i < j \le 4} c_{ij} e_{ij} \,|\, \Pi(\omega) = 0 \right\}.$$

  When we think of it geometrically like this, we call it the *Grassmann Cone*.
- (I'm calling it a "cone" because it has the property that if ω is in it then so is $\lambda\omega$ for any number λ. If you think about it, a circular cone through the origin has this property too. And, I'm naming it after Hermann Günther Grassmann, the 19th century German mathematician who played an essential role in creating vectors, wedge products, and much of the mathematics that underlies modern supersymmetric particle physics.)
- In what sense is this "geometric"? Well, it doesn't seem any different than the circles or hyperboloid we used to start the lesson. It is the set of points in a vector space whose coordinates satisfy a particular polynomial equation. We can note, for example, that it is *connected*...there is a path from any point on it to any other point on it that doesn't leave the object.
- There is a beautiful generalization of what we did above that works when V has any dimension n and $\bigwedge$ is made by taking the wedge product of k vectors. It works almost exactly the same way for any $2 \le k < n$, but the Plücker relations get longer and messier and you need more than one of them. Perhaps one of you could learn about it for your project. (Also, in a paper just published this semester in Proceedings of the American Mathematical Society, a colleage and I along with two C of C students showed that there is a previously undiscovered alternative to these longer Plücker relations which is somewhat nicer.)
- Now, what is not at all clear at this point is what this has to do with soliton theory...but we'll get to that next time!

Homework

1. Compute the requested wedge products, writing the result out as a linear combination of the e_{ij}'s:

   (a) $(e_1 + e_2 + e_3) \wedge (e_2 + e_3 + e_4)$ (b) $e_2 \wedge (9e_1 + 0.5e_2)$

   (c) $e_3 \wedge (9e_1 + 0.5e_2)$

2. For each of the following elements of $\bigwedge$, use the Plücker relation to determine whether it is decomposable. If it is, then find a "factorization" as a wedge product of two vectors from V. (This may involve some guesswork and experimentation on your part.)

   (a) $\pi e_{13} - \pi e_{14} + 9e_{23} - 9e_{24}$ (b) $e_{13} - e_{24} + e_{34}$

   (c) $e_{12} + e_{14} - e_{23} + e_{34}$

3. Prove that $ae_{12} + be_{23}$ is decomposable for *any* choice of constants a and b. Find a "factorization".

4. Suppose I tell you that $ae_{12} + be_{34} \in \Gamma$. What can you tell me about the constants a and b?