Math 495 Handout: March 11, 2008

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The KP Equation

- The KdV equation was written to represent the motion of the surface of water. However, since the solution $u(x,t)$ depends on only *one* spatial variable (that's what we call x, while t is the temporal variable), it can only describe waves on a very narrow, straight body of water. This is alright for modeling waves on a canal, but we usually think of water as having a *two*-dimensional surface.
- The **KP Equation** is the nonlinear partial differential equation for a function $u(x,y,t)$ that can be written as

$$u_{yy} = \frac{4}{3}u_{xt} - 2u_x^2 - 2uu_{xx} - \frac{1}{3}u_{xxxx}.$$

- **We already know some solutions:** Suppose $u(x,t)$ is a solution of the KdV equation (in the form that we have studied it). Then the *same* function u is also a solution of the KP equation! This becomes obvious when you write the equation in the equivalent form

$$u_{yy} = \frac{4}{3}\frac{\partial}{\partial x}\left(u_t - \frac{3}{2}uu_x + \frac{1}{4}u_{xxx}\right).$$

Question 1: Verify that this does indeed represent the same equation as the one above, and explain why KdV solutions are always solutions of this one as well.

Question 2: Draw a Venn diagram representing the solutions to the KP equation and the solutions to the KdV equations as sets.

➢ How do we *picture* solutions to the KP equation?

- As in the KdV equation, the variable t represents *time*. So, if we fix $t = 0$ then we are looking at one instant of the wave represented by u. Then we can change t and see how the wave changes...making what is essentially a movie with frames. The *difference* is that the function $u(x,y,0)$ is a function of *two* variables and so its graph is a surface in 3-dimensional space.
- Remember how we graph a function $z = f(x,y)$. For each point (x,y) in the plane $z = 0$ you plot a point above it at height $f(x,y)$ (or below if it is negative). This makes a surface that passes a "vertical line test" because no two points on it have the same x and y coordinates.

  We'll have to learn a few new Mathematica commands to be able to see animations of these "waves", but it should mostly seem familiar. First, note that you can plot a function of two variables using `Plot3D[f[x,y],{x,...,..},{y,...,...}]`. (Here, of course, you are intended to fill in the minimum and maximum coordinate values in place of the dots.) Options that you might want to consider adding include `PlotRange->{{..,..},{..,..},{..,..}}` and `ColorFunction->"Heat"`. (I've also started using `Manipulate` instead of `Animate` because it seems to give me more control.)

- What are the "one-solitons" of the KP equation?
- For any numbers λ_1 and λ_2, the function

$$u(x,y,t) = \frac{2(\lambda_1-\lambda_2)^2 e^{(\lambda_1+\lambda_2)x+(\lambda_1^2+\lambda_2^2)y+(\lambda_1^3+\lambda_2^3)t}}{\left(e^{\lambda_1 x+\lambda_1^2 y+\lambda_1^3 t}+e^{\lambda_2 x+\lambda_2^2 y+\lambda_2^3 t}\right)^2}$$

  is a solution to KP. Its graph looks like a straight line "wave front" – a wall of water like a tsunami – travelling at constant speed across the xy-plane. The slope is determined by $\lambda_1+\lambda_2$.

- Some people might argue that this solution is not really a soliton, in the sense that it is not localized. However, since it reduces to our usual KdV soliton formula in the case $\lambda_1=-\lambda_2$ (see homework), I think it makes sense to still call it a soliton solution.

- And multi-solitons too...
- Then, of course, there are also solutions that look like combinations of these walls. Just like the KdV case, they almost seem like sums of the one-solitons...but if you look closely you will note that they do "interact" in a nonlinear way. More interestingly, the KP multi-soliton solutions are more diverse than the KdV ones. For instance, rather than just having a pair of wave fronts moving in different directions, they can form weblike patterns of walls. These patterns are a subject of research even in the last few years.
- We'll see how to make our own multi-soliton solutions in the next lecture, but here's an example you can consider now:

$$\frac{4\left(6e^{12(6t+x)}+75e^{5(7t+x+y)}+16e^{2(t+x+5y)}+103e^{37t+7x+5y}+50e^{39t+9x+5y}\right)}{\left(3e^{5(7t+x)}+e^{37t+7x}+2e^{5y}+4e^{2t+2x+5y}\right)^2}$$

- The initial data $u(x,y,0)$ is not enough to predict future:
- One big difference between KdV and KP is the number of spatial dimensions. But there is another difference. Since KP is not in the form $u_t =$"stuff with no t derivatives" even if we know $u(x,y,0)$ (the initial shape of the wave at time $t=0$, this is *not* enough to say what will happen next. There are many different solutions to KP which look exactly the same at a given time but behave differently when time is allowed to flow.

- The history and applications of the KP equation:
- The KP equation is a natural generalization of the KdV equation to two spatial dimensions. It was devised in 1970 by Russian Physicists B.B. Kadomtsev and V.I Petviashvili (which is obviously where it gets its name). Their motivations seemed to be purely physical. In particular, I don't think they worried about whether their new equation had a Lax equation or integrals of motion or solutions that could be written exactly. In other words, I don't think they were trying to make a "soliton equation" as we've been considering them. But, remarkably, it *does* have all of these properties.
- The KP equation does a good job of modeling surface waves on the ocean. Like KdV, it is not completely realistic. For instance, it does not treat the x and y-directions equivalently. (Oscillations in the y-direction tend to be smoother.) However, according to the Website of Bernard Deconinck in the applied math department at the University of Washington:

  > *The KP equation admits a large family of exact quasiperiodic solutions. Each such solution has N independent phases. Recent comparisons with experiments show that the family of two-phase solutions of the KP equation describes waves in shallow water with surprising accuracy. This success suggests that more complicated KP solutions might provide accurate physical models of more complex wave phenomena.*

  It has other physical significances as well (it shows up in string theory, random matrix models of matter, Bose-Einstein condensates, super conductivity, etc.)...

- ...BUT, the amazing thing about it is how much it unifies the mathematics of soliton theory. Looking at the KdV equation alone, it would be nearly impossible to understand the real algebro-geometric structure of soliton equations. But, the generalization to the KP equation represents a quantum leap to a higher level of understanding. For instance, we move from talking about hyper-elliptic curves to talking about *all* curves. More importantly, the structure of the solution set itself takes on a geometric form. That will be the topic of discussion over the next week or so.

Homework

1. Show that if you choose $\lambda_1 = k$, $\lambda_2 = -k$ and then the formula above for a 1-soliton solution to KP is actually the same as the 1-soliton of KdV that we saw back in January. (Explain how or where the dependence on y disappears from the solution when the parameters are chosen this way.)
2. There is a non-zero number c such that

$$u(x,y,t) = c\frac{x^2+2y}{(x^2-2y)^2}$$

   is a solution of the KP equation.

   (a) Determine what value of c it is. (Show me either your computations by hand or your Mathematica notebook.)

   (b) Can you describe what the solution looks like? (Hint: It is undefined where the denominator is equal to zero. Basically, you can describe it by telling me whether it goes to plus or minus infinity there and describing the shape of this region where it is undefined.)

   (c) Notice that there is no t in the solution. What does that tell you about the dynamics?
3. In this question I want you to use Mathematica to investigate the dynamics of the KP solution

$$u(x,y,t) = \frac{2e^{2(t+x)}\left(4+e^{3(3t+x+y)}+9e^{7t+x+3y}\right)}{\left(1+e^{2(t+x)}+e^{3(3t+x+y)}\right)^2}.$$

   After you type it into Mathematica, you can check that you got it right by verifying that it satisfies the KP equation. Then, choose an appropriate range of x, y and z to plot and a range of values for t so that when you animate it you can see what is happening. Send me the Mathematica file with your “movie” in it, and also try to describe in words what you see. How is it different than the multi-soliton solution we saw in class?
4. Read through the list of projects and select your favorites.