# Math 495 Handout: February 19, 2008

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## Review: A Zoo of Differential Equations

- ➢ This seems like a good opportunity to look back at what we've done.

- First, recall that a *linear* differential equation is one that can be written as a sum of derivatives of the unknown function multiplied by coefficient functions being set equal to some function. In other words, they are of the form $L(f) = g$ for a differential operator $L$! There is a lot of structure to the solution space of such an equation...no matter *what* operator $L$ might be. For instance, if $f_1$ and $f_2$ are solutions to this equation, then so is $f_1 + \lambda(f_1 - f_2)$ for any $\lambda$. In a deeper sense, we can say that the kernel of $L$ (the solutions of $L(f) = 0$) form a vector space and that the solution set to $L(f) = g$ takes the form of any one solution of this equation plus the kernel. (For example, why is the solution of $f'(x) = g(x)$ of the form $F(x) + C$? Because $C$ is the kernel!)

- Recall next that nonlinear differential equations seem to lose this niceness. For $u_t = uu_x$ (the inviscid Burger's equation), knowing two solutions $u_1$ and $u_2$ doesn't help you make a new solution $u_3$ in any way. Moreover, it has the unpleasant symptoms that we really can't write down any interesting solutions anyway...and that the solutions we can describe implicitly do not even stay solutions for long as they develop shocks and stop being functions. Its close cousin, the Navier-Stokes equation, has similar problems...but it is so hard that we cannot yet even say for sure whether it has solutions which stay nice without developing shocks or other problems. This is one of the $\$$1M questions at the Clay Math Institute.

- In contrast, we found that the KdV equation $u_t = \frac{3}{2}uu_x + \frac{1}{4}u_{xxx}$ is surprisingly nice for a nonlinear PDE. Not only does it have 1-soliton solutions that stay nice without developing shocks or other problems, but it has solutions that seem (in some sense) to be combinations of these. (We will see later how this really generalizes the linear case in which the solution set involves the "geometry" of vector spaces.) Moreover, we have seen that it has the additional structure of being able to be represented in Lax form and is somehow deeply connected with algebraic geometry.

- Understanding this has allowed us to "discover" some other special equations that are in some ways like KdV. If we were looking at a zoo of differential equations, these would be set off in a separate part of the zoo (like the bird domes at some zoos) because they are really quite different. These are the soliton equations or "integrable nonlinear systems". The "Sponge Bob" equation looks similar to KdV when we write it down, and it is not terribly different than KdV in many ways. It solutions look rather similar, in fact. The Toda Lattice ($\ddot{p}_i = e^{p_{i+1}-p_i} - e^{p_i - p_{i-1}}$) seems quite different since it describes moving particles rather than a fluid. However, it has a Lax equation (as we saw in the homework) and solitons that seem somewhat similar to the solitons of KdV.

- But, I don't want to mislead you into thinking that this "bird dome" of integrable systems contains only a few similar looking species. In fact, some of them are quite exotic and different. Consider, for instance, the *Sine Gordon Equation* $u_{xx} - u_{tt} = \sin(u)$. On the right side there, we are taking the *SINE* of the function $u$ and this has to be equal to the difference between the homogeneous second derivatives of $u$ without any sine. Does that sound possible? In fact,

  $$u(x,t) = 4\arctan\left(\exp\left[\frac{x-vt}{\sqrt{1-v^2}}\right]\right)$$

  is a solution to this equation for every choice of $v$! It is a very different sort of soliton. For one thing, it can move in either direction. Moreover, it doesn't *look* like our KdV solitons. In what sense is it the same, then? It is a localized solution which translates in time and for which there are solutions that seem to combine an arbitrary number of them together. Moreover, the SG equation can be written in Lax form and can be solved using functions from algebraic geometry.
- There are many *more* species to look at in this part of the zoo...but we'll only have time to see a few more. The next I intend to discuss (after the break) is the KP equation, which is a cousin to the KdV equation having three variables instead of just two.

## A New Goal: Reading Mathematics

- One goal of this class is to make certain that you know how to read and write mathematics. This actually is a difficult thing to do, and definitely requires both training and practice. What we will do here is read section 5.1.1 from the book "Solitons: an introduction" by Drazin and Johnson.
- *Notational Differences:* One annoying thing that you have to get used to if you read a lot of math is that people use different notations for the same thing. In this case, for instance, the authors chose to write the KdV equation as

  $$u_t - 6uu_x + u_{xxx} = 0 \qquad (5.4)$$

  Is this really different? No, you can turn ours into this one with a simple change of variables: $u(x,t) \mapsto u(ax,bt)$. (At least, I think you can. We'll work it out in class to see.)
- Another thing is that you have to get used to someone's style. As you read further, it often gets easier because you know how that particular author writes. So, we will be reading this together in class up to right before the KdV equation is introduced. Then, you will try to read the rest for homework.
- Here is general advice on reading math:
- ➢ Read slowly. Math is dense. Read a single sentence and stop to see if you understood it. If not, read it again. Read it as many as three times to try to understand it. If after three times you still don't get it, try reading ahead a bit and see if that helps.
- ➢ Read with a piece of paper (or an open window running Mathematica) handy on which you can compute some examples and check the author's claims. For instance, in this paper, he claims that he can turn equation (1) into the simpler formula below it by using polar coordinates. Before you go on, make sure you can do it. Perhaps he's mistaken, or perhaps you don't understand well enough...either way it is worthwhile catching such problems as they arise.

- Read with a friend. Perhaps someone else will understand a line you don't get. Read through it together and you'll get farther faster.

## Homework

1. Carefully read through the rest of the section, from wherever we left off until the start of 5.1.2. Keep a list of the things you understand and the things you don't. For instance, you might write:
    - At the bottom of page 90, I understand what they mean by $u \longrightarrow 0$ as $x \longrightarrow \pm\infty$. They are asying that in sections 4.5 and 4.6 they only looked at solution whose graphs always approach the $x$-axis for large values of $|x|$...solutions like the multi-solitons we studied.
    - I can't figure out where equation $(5.8)$ comes from. It seems as if they pull it out of the blue without any explanation.
2. What do they mean "which is already in conservation form..." between (5.4) and (5.5)? Actually do the computation and verify that (5.5) does what it is supposed to do.
3. He says that if $u(x,t)$ is a solution to KdV and it is *integrable* (which here just means that for each value of $t$ $\int_{-\infty}^{\infty} u\,dx$ is a well-defined number) then this number is constant.
    (a) Can you come up with an example of a function $u(x,t)$ for which $\int_{-\infty}^{\infty} u\,dx$ is a well-defined number for each $t$ but is *not* constant? (Hint: $u$ can't be a solution of KdV...but you can pick just about any other kind of function you want as long as you know how to integrate it.)
    (b) What else that we did earlier in this class does this remind you of? (Hint: think of something that seems like it could depend on $t$ but in the context of soliton equations turns out not to.)
    (c) What does $(5.6)$ tell you about *areas* in the animations we have watched of KdV 2-soliton solution? Does this seem possible? Can you somehow check with Mathematica that it really is true?
4. At the bottom of page 90 they say "Note, however, that equation (5.6) does not hold for all solutions of the KdV equation". What is the point of this remark? Is it a serious problem? Does it contradict something that they or I said earlier?
5. At the top of 91 they work out "another conservation law". Where did this formula come from? How is it related to (5.4)? Most importantly, can you see why $(5.7)$ follows from it as a logical conclusion?
6. What are the authors saying in the paragraph that begins "That both $u$ and $u^2$ are consderved densities..." Can you summarize the entire paragraph in one or two sentences? Do you agree with the authors?
7. What are the authors saying in the paragraph that begins "At this stage, one might be forgiven for believing..."? Can you summarize the entire paragraph in one or two sentences? How does it set you up for a sequel?
8. Do the conserved densities have a natural order, or are they just listed in the order we happened to have found them? (Or, to put the question another way, why is $T_5$ called "$T_5$"? If we had discovered it before $T_4$ would it make sense to exchange their names?)