Math 495 Handout: February 5 2008

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Lax Operators

- Last time, someone claimed I had "lost" them. Let me try the whole thing again, but in a backwards sort of way to see if I can make it clearer.
- ***Recall the concept of eigenvectors:*** If a matrix L and a vector **v** satisfy $L\mathbf{v} = \lambda\mathbf{v}$ for some number λ, we say that **v** is an eigenvector for L with eigenvalue λ. And $n \times n$ matrix can have *at most* n eigenvalues and they are the roots of the polynomial $p(\lambda) = \det(L - \lambda I)$. This set of eigenvalues is called the "spectrum" of the matrix.

➢ For example, for the matrix and vectors

$$C = \begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix} \qquad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \mathbf{v}_4 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

you can check that

$$C\mathbf{v}_1 = \mathbf{v}_1 \qquad C\mathbf{v}_4 = 4\mathbf{v}_4.$$

So, C has eigenvalues $\lambda = 1$ and $\lambda = 4$.

- ***A Nice Fact from Linear Algebra:*** It L and G are matrices (with G invertible), then the matrix $M = GLG^{-1}$ is said to be *similar* to L. The sense in which it is similar is that it has exactly the same eigenvalues...and that we can even make eigenvectors for M from known eigenvectors of L.

> ***Theorem:*** Suppose $L\mathbf{v} = \lambda v$. Then $\mathbf{w} = G\mathbf{v}$ is an eigenvector for $M = GLG^{-1}$ with eigenvalue λ.
>
> ***Proof:***
>
> $$\begin{aligned} M\mathbf{w} &= GLG^{-1}(G\mathbf{v}) = GLG^{-1}G\mathbf{v} \\ &= GL\mathbf{v} = G\lambda\mathbf{v} = \lambda G\mathbf{v} = \lambda\mathbf{w}. \end{aligned}$$

- ***Non Isospectrality:*** As we saw last time, a matrix that depends on a variable t may have different eigenvalues for different values of t. (Equivalently, we could say that the eigenvalues also depend on t). This is actually the *usual* situation. If you randomly pick a t-dependent matrix, you'll probably get one whose eigenvalues are functions of the variable.

➢ For example, let's introduce the matrix

$$G = \begin{pmatrix} e^t & e^{2t-1} \\ 1 & e^t \end{pmatrix}.$$

Its eigenvalues are $\lambda = e^t \pm \sqrt{e^{2t} - 1}$.

- ***Cheating?:*** Well, as Julian pointed out last time, there is a simple way to *make* isospectral matrices. It is so easy, it seems like cheating! We can just take any constant matrix C and a time dependent matrix $G = G(t)$ and let $L = GCG^{-1}$ (a "dressed" version of C). Then, according to the theorem above, L always has the same eigenvalues as C. Moreover, we know we can make eigenvectors for L by "dressing" the eigenvectors of C.

➢ Continuing with our earlier examples, we can multiply it out to find that

$$L = GCG^{-1} = \begin{pmatrix} 4 - 2e^t - 3e^{2t} & -3e^t + 2e^{2t} + 3e^{3t} \\ -2 - 3e^t & 1 + 2e^t + 3e^{2t} \end{pmatrix}.$$

As predicted, this always has eigenvalues $\lambda = 1$ and $\lambda = 4$ no matter what t is. In fact, we can say more specifically that

$$\mathbf{w}_1 = G\mathbf{v}_1 = \begin{pmatrix} e^t \\ 1 \end{pmatrix} \qquad \mathbf{w}_4 = G\mathbf{v}_4 = \begin{pmatrix} -3 + 2e^t + 3e^{2t} \\ 2 + e^t \end{pmatrix}$$

are eigenvectors with these eigenvalues!

- But, I would like to relate this to *Lax Equations*. In the case of matrices, I can do this with a theorem as well:

Theorem: If $G = G(t)$ is a time dependent matrix, C is a constant matrix and $L = GCG^{-1}$ is a "dressed" matrix, then

$$\dot{L} = [M, L]$$

where $M = \dot{G}G^{-1}$.

Proof:

$$\begin{aligned} \dot{L} &= \frac{d}{dt}(GCG^{-1}) = \dot{G}CG^{-1} + GC\frac{d}{dt}G^{-1} \\ &= \dot{G}(G^{-1}G)CG^{-1} - GCG^{-1}\dot{G}G^{-1} = ML - LM = [M, L] \end{aligned}$$

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- Among the things I used in there were (a) the product rule for matrices, (b) the fact that $\dot{C} = 0$, (c) that I can introduce a factor of GG^{-1} anywhere and (d) that $\frac{d}{dt}G^{-1} = -G^{-1}\dot{G}G^{-1}$. (This last fact I got wrong in my notes from last time...left off the minus...and it can be easily derived from the product rule).

➢ It is rather instructive to check that this is true in this example. We can easily compute that

$$M = \dot{G}G^{-1} = \begin{pmatrix} -e^{2t} & e^t\left(1 - e^{2t}\right) + 2e^{3t} \\ -e^t & e^{2t} \end{pmatrix}$$

and then verify that indeed $[M,L]=ML-LM$ is *exactly* the same as

$$\dot{L}=\begin{pmatrix}-2e^t\left(1+3e^t\right) & e^t\left(-3+4e^t+9e^{2t}\right)\\ -3e^t & 2e^t\left(1+3e^t\right)\end{pmatrix}$$

- ***Back to KdV:*** Let $u(x,t)$ be any function of x and t.

$$L=\partial^2+u(x,t) \qquad \text{and} \qquad M=\partial^3+\frac{3}{2}u\partial+\frac{3}{4}u_x$$

In 1969, Peter Lax showed that the equation

$$\dot{L}=[M,L]$$

is exactly $u_t=\frac{3}{2}uu_x+\frac{1}{4}u_{xxx}$. In other words, if u is a solution to the KdV equation then L satisfies this operator equation in the same form as we saw above. (In fact, we rederived this fact at the beginning of class last time. There is nothing difficult in computing it, because you just multiply it out and everything works. But, to have *thought* to even look for this was a brilliant move by Lax!)

- **So,** it becomes reasonable to ask "Well, what does this tell us about L? Does it mean that the eigenvalues of L are constant? Does it tell us that L is just a constant operator "dressed up" with some time dependent one in the obvious way?"

- For KdV, the answer is *yes*...but I must be vague about what form it actually takes:
  - The ring of differential operators fits into a *larger* ring where there are multiplicative inverses (just as the integers fit into the rational numbers). Within these *pseudo-differential operators*, we can say that if $u(x,t)$ is a solution to KdV, then there is a time dependent pseudo-differential operator G so that $G\circ\partial^2\circ G^{-1}=\partial^2+u(x,t)$. (You just have to "dress up" the trivial solution $u=0$!)
  - If $u(x,t)$ is a solution of the KdV equation then for any positive number λ there is a two dimensional space of real valued eigenfunctions $\psi(x,t)$ satisfying $L\psi=\lambda\psi$.

- But, more importantly, this gives us a way to generalize what we've seen in the KdV equation to *other* equations. The "Lax Equation" is a cornerstone of soliton theory. Every equation we would want to call a "soliton equation" (a nonlinear equation we can solve explicitly with solutions that look like localized interacting disturbances) has a Lax form. Moreover, we can *almost* say the converse ("if it has a Lax form then it is a soliton equation")....but I would need to be able to refer to a few analytic details that are beyond the scope of this course. So, let's just say that having a Lax form would make an equation a good candiate for the approaches of soliton theory.

- In particular, it is true that there is a soliton equation of the form $\dot{L}=[M,L]$ with operators M of every odd order and $L=\partial^2+u$. The KdV equation is the one where the order of M is 3. For homework you will find the equation coming from order 5. These equations obviously fit together into a structure...they are not just random equations found here and there but part of a *hierarchy*. (We call them "The KdV Hierarchy".) That soliton equations fit together into hierarchies is another hallmark of soliton theory. Of course, there are *other* hierarchies, that come from looking at different sorts of operators!

Update on homework from last time:

I am not assigning any new homework, but I will provide a hint to help you answer one of the questions from last time:

4. Let $L = \partial^2 + u(x)$ as before and let

$$M = \partial^5 + \alpha(x)\partial^3 + \beta(x)\partial^2 + \gamma(x)\partial + \delta(x)$$

be a monic fifth order differential operator (with no ∂^4 term).

(a) Compute $[M, L]$. Big Hint:

$$\begin{aligned} M \circ L = &\partial^7 + (\alpha(x) + u(x))\partial^5 + (\beta(x) + 5u'(x))\,\partial^4 \\ &+ (\gamma(x) + \alpha(x)u(x) + 10u''(x))\,\partial^3 \\ &+ \left(\delta(x) + \beta(x)u(x) + 3\alpha(x)u'(x) + 10u^{(3)}(x)\right)\partial^2 \\ &+ \left(\gamma(x)u(x) + 2\beta(x)u'(x) + 3\alpha(x)u''(x) + 5u^{(4)}(x)\right)\partial \\ &+\delta(x)u(x) + \gamma(x)u'(x) + \beta(x)u''(x) + \alpha(x)u^{(3)}(x) + u^{(5)}(x) \end{aligned}$$

and

$$\begin{aligned} L \circ M = &\partial^7 + (\alpha(x) + u(x))\partial^5 + (\beta(x) + 2\alpha'(x))\,\partial^4 \\ &+ (\gamma(x) + \alpha(x)u(x) + 2\beta'(x) + \alpha''(x))\,\partial^3 \\ &+ (\delta(x) + \beta(x)u(x) + 2\gamma'(x) + \beta''(x))\,\partial^2 \\ &+ (\gamma(x)u(x) + 2\delta'(x) + \gamma''(x))\,\partial \\ &+\delta(x)u(x) + \delta''(x) \end{aligned}$$

(b) Find what $\alpha(x)$ must be so that the ∂^4 term in the commutator vanishes.

(c) Assuming α is chosen as above, find what $\beta(x)$ must be so that the ∂^3 term in the commutator vanishes.

(d) Assuming α and β are chosen as above, find what $\gamma(x)$ must be so that the ∂^2 term in the commutator vanishes.

(e) Now, assuming all of the choices already specified, you might want to find $\delta(x)$ so that the ∂ term vanishes...but you can't quite do that since there are some functions appearing there that you don't know how to anti-differentiate. Instead, just tell me what $\delta'(x)$ must be so that the ∂ term vanishes.

(f) Finally, tell me what $[M, L]$ is if all of the coefficients in M are chosen as specified above. (It should turn out to be just some expression in u and its x derivatives.

(g) Write

$$u_t = \boxed{\text{your answer to the previous question}}.$$

Congratulations, you've just discovered a soliton equation. This is an equation like the KdV equation having multi-soliton solutions that we can write exactly...but they behave a little differently. Should we name it after you?