Math 495 Handout: February 5 2008

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Lax Operators

- Last time we saw some "differential algebra". At first, it may not be clear exactly how that should be related to the KdV equation. After all, the equations represented by differential operators are all *linear* equations, but KdV is *nonlinear*.
- In fact, in his second paper on "solitons", Kruskal (with Gardner, Greene and Miura) already recognized that there was something special about the Schrödinger operator $\partial^2 + u(x,t)$ where u is a solution to the KdV equation. But, to follow what they did there requires more analysis than I want to pursue here. Shortly after, however, Peter Lax (who won the Abel prize last year) made a brilliant observation about the KdV equation which is quite easy to see. Today we will learn at least a bit about that!

Question 1: Let $L = \partial^2 + u(x)$ and $M = \partial^3 + \alpha_1(x)\partial + \alpha_0$. Half of you compute $L \circ M$ and half of you compute $M \circ L$.

Question 2: What must be true about α_1 and α_2 so that $[M, L]$ is an operator of order *zero* (i.e. so that all of the positive powers of ∂ cancel out)?

Question 3: Let α_1 and α_2 be as in your answer to the previous question. Then what is $[M, L]$?

- Look familiar? Notice that although I pulled the $3/2$ and $1/4$ coefficients in the KdV equation out of nowhere when I introduced the equation for the first time, here they appear *like magic* even though my original formulas for L and M were about as simple as you can imagine. *That* is why I use this form of the KdV equation it makes the differential algebra work out beautifully.

➢ **Definition:** Let L and M be linear operators of any kind (e.g. differential operators, matrices, etc.) which depend on an extra parameter t. (That means that for each fixed value of t we get a particular operator L and M, but that they may change as the value of t changes.) We say they are a *Lax Pair* if they satisfy the equation

$$\boxed{\dot{L} = [M, L]}$$

(where here I am using the common shorthand $\dot{L} := \frac{d}{dt}L$).

- For example, since for our choices of L and M above we see that $\dot{L} = u_t$ and $[M, L]$ is the right side of the KdV equation (assuming the free constant has the right value), we can say that L and M form a Lax pair for the KdV equation.
- Important: Not every equation *has* a Lax pair! This is already a special property that the KdV equation has.

- In a sense, the formula in the box above is the *universal* form of a soliton equation. Not only can the KdV equation be written as "the derivative of some operator equals the commutator of that operator with another operator"...but *all* soliton equations seem to have this form! Later in the course (including tonight's homework) we will meet other soliton equations, and each one has a "Lax Form". The difference between them is that the operators may not be differential operators of order 2 and 3, but may have other orders...or may even be operators of other kinds!

- And, the converse is (almost) true as well. If you can write an equation in this form, then it is quite likely to be a soliton equation (and therefore have soliton solutions and also be solvable by a variety of methods). To make this explicit we would have to learn more about Hilbert spaces than I am hoping to discuss this semester. (If the operator L is a "self-adjoint, bounded operator on a Hilbert space" then we know for sure.) But, I think for our purposes it is enough to know that the Lax form is a good clue to an equation being a soliton equation. And, in all of the cases we will see, it really does indicate "integrability".

- For homework, you will be asked to find an operator M of order 5 such that $[M, L]$ has order zero. Then $\dot{L} = [M, L]$ is a partial differential equation that we have not yet seen in this class. But, I can assure you that it is has multi-soliton solutions that we can write down just like the KdV equation. The same is true, in fact, for every odd order. This gives a *hierarchy* of soliton equations that together are called the KdV hierarchy.

- Two "hallmarks" of soliton theory are the Lax equation and the fact that soliton equations fit together into hierarchies. This is not a theorem (we don't really have a definition of "soliton equation" that would allow us to state it rigorously); it's more like a scientific discovery supported by lots of experimentation.

- **What is the significance of a Lax pair?**
  - **Isospectrality:** Because of the Lax pair, we "know" that $L\psi = \lambda\psi$ where L and ψ depend on t but λ is *constant*. This gives us a *conservation law*: the eigenvalue is conserved by the flow.
  - **Dressing:** Because of the Lax pair, we "know" that there is a *constant* operator C such that

    $$L(t) = W(t) \circ C \circ W^{-1}(t) \Leftrightarrow L(t) \circ W(t) = W(t) \circ C$$

    This gives us a method for creating a solution by "dressing" the time independent operator C by a time dependent W.

- Let's see why this makes sense, first starting with *matrices*.

Question 4: Show that if M_1 and M_2 are square matrices and that they are related by the formula $M_1 = G \circ M_2 \circ G^{-1}$ for some invertible matrix G, then M_1 and M_2 have the same eigenvalues. (Hint: Let v be an eigenvector for M_1 with eigenvalue λ and make an eigenvector for M_2 out of it.)

- This is why we say that two matrices are *similar* if they are related in this way. They certainly have a lot in common if they have all of the same eigenvalues...and if you can easily turn the eigenvectors of one into the eigenvectors of the other.

Question 5: Continuing with the above, suppose G actually depends on a parameter t even though M_2 and v do not. Show that (a) M_1 satisfies a Lax equation with $A = \dot{G} \circ G^{-1}$ and (b) the eigenvector ψ for M_1 satisfies $\dot{\psi} = A\psi$. (Hint: Use the fact that $\frac{d}{dt}G^{-1} = G^{-1} \circ \dot{G} \circ G^{-1}$.)

- From these questions we can see that a Lax equation would be related to isospectrality and dressing for ordinary matrices. To see it for general operators take just a tiny "leap of faith" since we do not have all of the tools. Watch:
  Suppose $L(t)\psi(t) = \lambda\psi(t)$ with $\dot{\lambda} = 0$ and $\dot{\psi} = A\psi$. Then

$$\begin{aligned}
0 &= \frac{d}{dt}(L[\psi] - \lambda\psi) = \dot{L}[\psi] + L[\dot{\psi}] - \lambda\dot{\psi} \\
&= \dot{L}[\psi] + L[A\psi] - \lambda A[\psi] = \dot{L}[\psi] + L \circ A[\psi] - A[\lambda\psi] \\
&= \dot{L}[\psi] + L \circ A[\psi] - A[L\psi] = (\dot{L} + L \circ A - A \circ L)[\psi] \\
&= (\dot{L} - [A, L])[\psi]
\end{aligned}$$

  Now, this does not show that $\dot{L} - [A, L]$ is actually equal to zero. But, if there were *enough* ψ's like this (and what "enough" is depends on the type of operators we're considering) then this would show that the Lax equation must be satisfied.

The Moral: Linear differential equations can be written in terms of differential operators just in the form $L(u) = m$. In general, it is not clear how a nonlinear equation can be written with differential operators. However, if it can be written as $\dot{L} = [M, L]$ (where now the solution to the equation shows up in the coefficients of the operators themselves) then it is a *special* sort of nonlinear equation and quite likely a soliton equation like KdV.

Homework

1. Both of these matrices depend on t, but one of them is special because its eigenvalues *do not*. Find the eigenvalues of *both* of these matrices and identify which one is *isospectral*.

$$L_1 = \begin{pmatrix} 2t+1 & -t(2t+1) & 2e^{-2t}t \\ 2 & -2t & 2e^{-2t} \\ 0 & 0 & -1 \end{pmatrix} \qquad L_2 = \begin{pmatrix} -6+4e^t & 4-2e^t & 0 \\ -12+6e^t & 8-3e^t & 0 \\ 0 & 0 & 2t \end{pmatrix}.$$

2. Let L be your answer to the previous question (the isospectral matrix). Find a and b so that L satisfies a *Lax equation* of the form $\dot{L} = [M, L]$ where M is

$$M = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b \end{pmatrix}.$$

3. The "Toda Lattice" is a famous dynamical system that describes a collection of n particles arranged in a ring. The particles are repelled by their neighbors through a force that grows exponentially as the distance between them decreases. If we call the position of the ith particle $p_i(t)$, then the system takes the form of an equation "$\ddot{p}_i(t) =$ SOMETHING" for *each* of the values of i. (Note: $\ddot{f} = \frac{d^2}{dt^2}f$.) If the force was anything other than exponential, then we might not be able to solve this system of equations, but with an exponential force it turns out to be integrable...*and to have soliton solutions* where a localized disturbance travels through the chain of particles without changing its speed or shape!

In this question, I will give you matrices which form the Lax pair for the Toda lattice with $n = 3$ particles. From this, you will be able to derive the equations of the Toda lattice yourself. We will use:

$$L = \begin{pmatrix} -\frac{1}{2}\dot{p}_1 & \frac{1}{2}e^{\frac{1}{2}(p_1-p_2)} & \frac{1}{2}e^{\frac{1}{2}(p_3-p_1)} \\ \frac{1}{2}e^{\frac{1}{2}(p_1-p_2)} & -\frac{1}{2}\dot{p}_2 & \frac{1}{2}e^{\frac{1}{2}(p_2-p_3)} \\ \frac{1}{2}e^{\frac{1}{2}(p_3-p_1)} & \frac{1}{2}e^{\frac{1}{2}(p_2-p_3)} & -\frac{1}{2}\dot{p}_3 \end{pmatrix}$$

$$M = \begin{pmatrix} 0 & \frac{1}{2}e^{\frac{1}{2}(p_1-p_2)} & -\frac{1}{2}e^{\frac{1}{2}(p_3-p_1)} \\ -\frac{1}{2}e^{\frac{1}{2}(p_1-p_2)} & 0 & \frac{1}{2}e^{\frac{1}{2}(p_2-p_3)} \\ \frac{1}{2}e^{\frac{1}{2}(p_3-p_1)} & -\frac{1}{2}e^{\frac{1}{2}(p_2-p_3)} & 0 \end{pmatrix}$$

(a) Compute the derivative of the entry in the first column and the second row of L. (Remember, when I write p_i I really mean $p_i(t)$ so its derivative is $\dot{p}_i = \frac{d}{dt}p_i(t) = p_i'(t)$.) Compute also the entry in the first column and the second row of $[M, L]$. (Hint: They should turn out to be exactly the same.)

➢ Please trust me when I tell you that the *off-diagonal* entries in $\dot{L}$ and $[M, L]$ are all exactly the same (as they were above). Then we can ignore these in looking at the Lax equation.

(b) Now compute the *diagonal* entries of $\dot{L}$ and $[M, L]$. If we equate these, then we get the equations of the Toda lattice. So, fill in the blanks to tell me what the Toda lattice equations are:

$$\ddot{p}_1 = \_\_\_\_\_\_\_\_ \qquad \ddot{p}_2 = \_\_\_\_\_\_\_\_ \qquad \ddot{p}_3 = \_\_\_\_\_\_\_\_$$

4. Let $L = \partial^2 + u(x)$ as before and let

$$M = \partial^5 + \alpha(x)\partial^3 + \beta(x)\partial^2 + \gamma(x)\partial + \delta(x)$$

be a monic fifth order differential operator (with no ∂^4 term).

(a) Compute $[M, L]$. Big Hint:

$$\begin{aligned} M \circ L = \partial^7 &+ (\alpha(x) + u(x))\partial^5 + (\beta(x) + 5u'(x))\,\partial^4 \\ &+ (\gamma(x) + \alpha(x)u(x) + 10u''(x))\,\partial^3 \\ &+ \left(\delta(x) + \beta(x)u(x) + 3\alpha(x)u'(x) + 10u^{(3)}(x)\right)\partial^2 \\ &+ \left(\gamma(x)u(x) + 2\beta(x)u'(x) + 3\alpha(x)u''(x) + 5u^{(4)}(x)\right)\partial \\ &+ \delta(x)u(x) + \gamma(x)u'(x) + \beta(x)u''(x) + \alpha(x)u^{(3)}(x) + u^{(5)}(x) \end{aligned}$$

and

$$\begin{aligned} L \circ M = \partial^7 &+ (\alpha(x) + u(x))\partial^5 + (\beta(x) + 2\alpha'(x))\,\partial^4 \\ &+ (\gamma(x) + \alpha(x)u(x) + 2\beta'(x) + \alpha''(x))\,\partial^3 \end{aligned}$$

$$+ (\delta(x) + \beta(x)u(x) + 2\gamma'(x) + \beta''(x))\,\partial^2$$
$$+ (\gamma(x)u(x) + 2\delta'(x) + \gamma''(x))\,\partial$$
$$+\delta(x)u(x) + \delta''(x)$$

(b) Find what $\alpha(x)$ must be so that the ∂^4 term in the commutator vanishes.

(c) Assuming α is chosen as above, find what $\beta(x)$ must be so that the ∂^3 term in the commutator vanishes.

(d) Assuming α and β are chosen as above, find what $\gamma(x)$ must be so that the ∂^2 term in the commutator vanishes.

(e) Now, assuming all of the choices already specified, you might want to find $\delta(x)$ so that the ∂ term vanishes...but you can't quite do that since there are some functions appearing there that you don't know how to anti-differentiate. Instead, just tell me what $\delta'(x)$ must be so that the ∂ term vanishes.

(f) Finally, tell me what $[M, L]$ is if all of the coefficients in M are chosen as specified above. (It should turn out to be just some expression in u and its x derivatives.

(g) Write

$$u_t = \boxed{\text{your answer to the previous question}}.$$

Congratulations, you've just discovered a soliton equation. This is an equation like the KdV equation having multi-soliton solutions that we can write exactly...but they behave a little differently. Should we name it after you?

5. I would like you to see that it is possible to make solutions to the KdV equation using an idea like the "dressing" above where some constant operator is *conjugated* by something time dependent. Since we don't know how to find the inverse of a differential operator, we'll use a formula like $L(t) \circ W(t) = W(t) \circ L_0$ where L_0 is a constant operator.

In this problem, we will prove the following statement: *Suppose we know that $\phi(x,t)$ is a function satisfying the equation $\phi_{xx}(x,t) = \lambda\phi(x,t)$ for some number λ. Suppose also that $\phi_t(x,t) = \phi_{xxx}(x,t)$. Then if $W = \partial - \frac{\phi_x}{\phi}$ there is a unique function $u(x,t)$ defined by the equation*

$$W \circ \partial^2 = \left(\partial^2 + u(x,t)\right) \circ W$$

and *it is a solution to the KdV equation.*

(a) Compute $W \circ \partial^2$. (This is super-easy, but you just need to write it in the standard form.)

(b) Compute the first and second derivative of ϕ_x/ϕ with respect to x and write them in the simplest form. (This will save you trouble in the next step.)

(c) Now compute $(\partial^2 + u(x,t)) \circ W$. (This will be tricky! Be careful and use your answer to the previous problem.)

(d) What must u be equal to in order for the two sides of the equation to be equal? (Hint: There is one definition you can make for $u(x,t)$ in terms of ϕ and λ that makes all of the coefficients equal).

(e) Using the definition for $u(x,t)$ from the previous part, compute u_t, uu_x and u_{xxx}. Show that the KdV equation is satisfied.

(f) Here are two examples of functions $\phi(x,t)$ that satisfy the requirements. What are the corresponding KdV solutions?

$$\phi_1(x,t) = e^{2x+8t} + e^{-2x-8t} \qquad \phi_2(x,t) = x.$$

6. Unlike a matrix which can only have a finite number of eigenvalues, a differential operator can have an infinite number. (I guess you saw this in your homework last time.) Here is an example from soliton theory that illustrates the isospectrality. The operator

$$L = \partial^2 + \frac{8}{\left(e^{-x-t} + e^{x+t}\right)^2}$$

is a Lax operator for the KdV solution with a single soliton travelling at speed 1. This is an eigenfunction for that operator:

$$\psi(x,t,z) = e^{tz^3+xz}\left(1 - \frac{-e^{-x-t}z^2 - e^{x+t}z^2 + e^{-x-t}z + e^{x+t}z + e^{-x-t} - e^{x+t}}{\left(e^{-x-t} + e^{x+t}\right)z^2}\right).$$

The question is: what is the *eigenvalue*? You can compute this by hand or use Mathematica, but be sure to either show or explain how you got your answer.