Math 495 Handout: January 31 2008

Alex Kasman
Department of Mathematics
College of Charleston

An Interlude

- I had originally planned today to show you how we can see what is *special* about the KdV equation using differential algebra ("Lax pairs"). However, I decided that maybe I was moving too quickly for you and decided instead to spend a bit more time on KdV and differential algebra separately before moving on. So, if you already downloaded the handouts or started on the homework from my earlier post, please look at how it has changed. (Essentially, I pushed everything back by one class.)

➢ More Motivation for Differential Algebra

- Recall that there is a simple way to see the relationship between *ordinary differential operators* on the one hand and *linear ordinary differential equations* on the other. In particular, if I have the operator

$$L = \partial^2 - \frac{2}{x^2}$$

and I am looking for functions in its kernel, then this is exactly the same as saying that I am looking for a function $f(x)$ satisfying the differential equation

$$f''(x) - \frac{2}{x^2}f(x) = 0.$$

- It could be that this equation is simple enough that you can just guess what the kernel would be. But, I would like to ask you *not* to even try at the moment. That is because I want to show an example of how you could find a solution to this equation using differential algebra. (In fact, this is very similar to the first few problems in your homework from last time...and in that case the formula is so complicated that I don't think you could just guess it!)

- Consider also the operator $M = \partial^2$. That one, certainly, is even *simpler* than L! But, the point is that M and L are closely related in a certain way. (And, it is actually important that the number in the numerator of the fraction in L is -2 rather than just some arbitrary constant! I am trying to subtley convince you that $-2/x^2$ is rather special.)

Question 1: Compute the products:

$$\left(\partial - \frac{1}{x}\right) \circ \left(\partial + \frac{1}{x}\right) \qquad \left(\partial + \frac{1}{x}\right) \circ \left(\partial - \frac{1}{x}\right).$$

- So, you see that the differential equations

$$f''(x) = 0 \qquad \text{and} \qquad f''(x) = \frac{2}{x^2}f(x)$$

are closely related in a way you might not have imagined in an ODO class: one can "*factor* the equation", exchange the orders of the factors and multiply out again to get the other!

- Now, homework question 3 from last time gives a method for turning solutions of one of these equations into solutions of the other. Let's try to apply it. First, we will find all of the solutions of the *easier* equation.

Question 2: Find a *basis* for the solutions of $f''(x) = 0$. (It has order two so you need a 2-dimensional space of solutions.)

- Now, according to problem 3, we can produce a solution to the other equation from these by applying the operator which is on the right in the easier equation to these.

Question 3: Take the right factor from our factorization of ∂^2 and apply it to your basis. Is each of these in the kernelof $\partial^2 - 2/x^2$?

- If this works out, then I was right about that. But, I *did* make an error when writing your homework. I mistakenly thought that we would be able to get a basis for the solutions of the harder problem. Why, in this case, does that not work?

➢ KdV Solitons

- As we know, $u_t = \frac{3}{2}uu_x + \frac{1}{4}u_{xxx}$ is the KdV equation and I am trying to convince you that it is special. One way I did this was by allowing you to look at some particular solutions with Mathematica as part of your homework assignment.
- Specifically, for $n = 2$ and $n = 3$, we looked at solutions of the form

  $$u(x,t) = 2\frac{\partial^2}{\partial x^2}\log(\det M)$$

  where M is an $n \times n$ matrix with the function

  $$k_i^{j-1}e^{k_i x + k_i^3 t} + (-k_i)^{j-1}g_i e^{-k_i x - k_i^3 t}$$

  in the (i,j) position. (Note that this makes the jth row equal to the jth derivative of the first row...this is a Wronskian!)
- In fact, except in the case where $\det M = 0$ (which happens when the columns are linearly dependent) this is *always* a solution of the KdV equation for any choice of n and any choice of the parameters.
- Among these solutions are the ones I would like to call the real multi-soliton solutions. Those solutions are characterized by the fact that the solution $u(x,t)$ is defined and positive for every x and t, that $\lim_{x\to\pm\infty} u = 0$, and that u is a fraction whose numerator and denominator are both linear combinations of functions of the form e^{ax+bt}. If for t sufficiently large the graph of the solution looks like n isolated humps travelling at different speeds, then we call it an n-soliton solution. It is possible to get all n-soliton solutions from the construction above with an $n \times n$ matrix!
- (As you saw, for some choices of the parameters, the u you get is *not* positive and *not* defined everywhere. In that case, even though it is a solution, I would not call it a real soliton solution. There is, however, a sense in which it is a "complex" soliton solution.)
- The k's control the speeds and the g's the positions at time $t = 0$ of the "disturbances" in the solution u.
- **Unrealistic 1?** It is interesting (and disturbing and reminiscent of something in quantum physics called the "Pauli exclusion principle" that there is simply no way to make a multi-soliton solution of KdV in which there are two solitons travelling at exactly the same speeds.

- **Unrealistic 2?** Even if there are n "humps" interacting in a multi-soliton solution to KdV, they interact in a pairwise fashion. This is *not* the case with interactions of particles in classical physics where a 3-particle collision can do more unusual things.
- **What it looks like at the time of collision:** Peter Lax, who figured out the clever thing relating KdV to differential algebra that we'll see later, also carefully described the way two "humps" look at the moment of "collision" under KdV dynamics. There are three cases:
  - If the velocities are sufficiently different, you briefly only see a single peak.
  - If the velocities are almost the same, you always see two peaks.
  - In between, there is supposed to be a region where you can see *three* peaks.
- (I believe this all supports my philosophical interpretation of the behavior of KdV solitons.)

Question 4: Can we produce examples in class that illustrate each of these?

Homework

No new homework assigned today. HW due Tuesday was assigned way back on the 24th.