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Differential Algebra I

- This is the first lesson in this class in which we'll really "get into" the math. Until now, I've been speaking in generalities. But, let's learn something precise: the algebra of differential operators.
- In elementary school you learned the algebra of numbers. In high school you learned the algebra of functions (especially polynomials and rational functions). Then, in college you learned linear algebra (well, most of you) and abstract algebra. This last class shows you how *general* the term "algebra" can be. It applies to many different mathematical sets which have some sort of multiplication or addition defined on them. The algebra we will learn today is used in many areas of math and physics, and is *essential to understanding what makes the KdV equation special*.
- **Differential Operators as a Set:** An ordinary differential operator is a polynomial in the symbol ∂ with coefficients that are functions which can be differentiated any finite number of times. When writing it down, for reasons that will be clear later, it is necessary to put the coefficients on the left and the powers of ∂ on the right.

  A *partial* differential operator is the same thing but with functions of more variables and a different symbol corresponding to each variable. We may not ever see partial differential operators in this class. So, I will often just say "differential operator" unless there is some potential for confusion.

  So, in general, a differential operator L is an object of this form:

$$L = c_N(x)\partial^N + c_{N-1}(x)\partial^{N-1} + \cdots + c_1(x)\partial + c_0(x) = \sum_{i=0}^{N} c_i(x)\partial^i. \qquad (*)$$

  If L has this form with $c_N \not\equiv 0$ then we say that it is an operator of *order* N. If $c_N = 1$, which is the nicest case, we say that L is *monic*.
- **Addition:** Addition of differential operators is simple: you just add the coefficients as you would on any polynomial.

$$\left(\sum_{i=0}^{N} c_i(x)\partial^i\right) + \left(\sum_{j=0}^{N} q_i(x)\partial^i\right) = \sum_{i=0}^{N} \left(c_i(x) + q_i(x)\right)\partial^i.$$

- **Multiplication:** This is where things get interesting. Multiplication here is *non-commutative*. To emphasize this fact, I will denote multiplication of the operators L and M by $L \circ M$ (or by $M \circ L$ which is *different* because it is multiplication in the other order).
  Multiplying any operator L by a function from the left side ("$f(x) \circ L$") is not so interesting...you just multiply each coefficient by $f(x)$ as usual.
  Also, multiplying powers of ∂ by each other just satisfies the usual rule
  $$\partial^m \circ \partial^n = \partial^{m+n}.$$
  Things get different, however, when a power of ∂ is on the left and a function is on the right.

> As an example I will mention that when multiplying a function $f(x)$ by ∂ on the right, you must use
> $$\partial \circ f(x) = f(x)\partial + f'(x).$$

Actually, the example in the box is not only an example. It is essentially all you need to know. This is because multiplication distributes over addition and because you can always apply one ∂ at a time. Here is a more extensive example.

➢ Suppose $L = \partial^2 - e^x$ and $M = \partial^2 + x\partial - x^2$. Then

$$\begin{aligned} L \circ M &= \left(\partial^2 - e^x\right) \circ \left(\partial^2 + x\partial - x^2\right) \\ &= \partial^2 \circ \partial^2 + \partial^2 \circ x\partial - \partial^2 \circ x^2 - e^x \circ \partial^2 + xe^x\partial + x^2e^x \\ &= \partial^4 + \partial \circ (x\partial^2 + \partial) - \partial \circ (x^2\partial + 2x) - e^x\partial^2 + xe^x\partial + x^2e^x \\ &= \partial^4 + x\partial^3 + \partial^2 + \partial^2 - x^2\partial^2 - 2x\partial - 2x\partial - 2 - e^x\partial^2 + xe^x\partial + x^2e^x \\ &= \partial^4 + x\partial^3 + (2 - x^2 - e^x)\partial^2 - (4x + e^x)\partial - 2 + x^2e^x \end{aligned}$$

Question 1: Compute $M \circ L$ and compare.

- **Action as Operators:** As well as being able to be added and multiplied by each other (making them an algebra), these differential operators have another job "on the side". They also *act* on functions, turning one function into another. Specifically, if L is the general ordinary differential operator from $(*)$ above
  $$L(f) = \sum_{i=0}^{N} c_i(x)\frac{d^i f}{dx^i}.$$
  For instance, using the M from the previous worked example we compute that
  $$M(\sin(x)) = x\cos(x) - (1 + x^2)\sin(x)$$
  (I will try to remember to use parentheses as in these definitions for the application of an operator to a function, but I must warn you that the usual way to write this in most of the mathematical literature is just Lf.)

- **Meaning of the Multiplication Rules:** Now, we can see why differential algebra has the rules that it does. *Multiplication* of differential operators is the same as their *composition* as operators. That is, for any L, M and $f(x)$ we have

$$\boxed{(L \circ M)(f) = L(M(f)).}$$

For example, we already know that $M(\sin(x)) = x\cos(x) - (1+x^2)\sin(x)$. Now we can apply L to this new function to get

$$L(x\cos(x) - (1+x^2)\sin(x)) = \left(x^2 + e^x\left(x^2+1\right) - 3\right)\sin(x) - (5+e^x)\,x\cos(x).$$

But, we could have done that in one step by taking the formula for $L \circ M$ and applying it from the start.

- **Differential Operators Act Linearly:** An important fact about differential operators is that they are *linear* operators. This means that if $f(x)$ and $g(x)$ are functions and λ and μ are numbers, then

$$L(\lambda f + \mu g) = \lambda L(f) + \mu L(g).$$

Thus, everything you know about operators from linear algebra apply to these operators...except that in general you cannot write these as matrices since they are acting on an infinite dimensional space.
For instance, using the terminology of linear algebra, we say the *kernel* of an ODO L is

$$\ker L = \{f(x) | L(f) = 0\}$$

(the set of all functions it annihilates). An important fact is that the kernel of an ordinary differential operator of order N is N-dimensional.

Question 2: Does this look at all familiar to you? Where have we been using differential operators in the course without even knowing the name? What words did we use to describe $\ker L$ earlier in the course?

Question 3: Suppose $L(f) = m(x)$ and $L(g) = m(x)$. Can you make another function $h(x)$ out of f and g so that $L(h) = m(x)$ too?

- (There, I finally said it! I've been dying to tell you that since we first discussed differential equations!)

- **Quantum Physics:** This class is no place for me to really teach you about quantum physics. But, I hope I can motivate your interest in differential operators by at least mentioning that one way to view the QM revolution is to say that they play the role that numbers did in classical physics. That is, whereas quantities like speed and position were given by numbers in Newton's description of the universe, we now can show that differential operators do a *far* better job of describing what we can now demonstrate in some impressive experiments.

- **Commutators:** Whenever you have an algebra that is not commutative, it is interesting to consider the *difference* between multiplying one way and the other. This happens so often, we have come up with a name and a notation for this idea: the *commutator* of L and M is

$$[L, M] = L \circ M - M \circ L.$$

Question 4: Compute the commutator $[\partial, x]$.

- (If you found the right answer, then what you wrote above is one of the *key* results of mathematical physics: Heisenberg's Uncertainty Principle, written in its simplest mathematical form. It says that you cannot really talk about measuring the speed ("∂") and position ("x") of an object.)

- **Flashback:** Finally, our last definition of the day again harks back to linear algebra. Recall that an *eigenvector* **v** for a matrix M is a vector such that $M\mathbf{v} = \lambda\mathbf{v}$ for some number λ, which we call the *eigenvalue*. (For instance since

$$M\mathbf{v} = \begin{pmatrix} 5 & -6 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

we know that **v** is an eigenvector of M with eigenvalue $\lambda = 2$.

- **Eigenfunctions:** Similarly, we say that f is an eigenfunction for a differential operator L with eigenvalue λ if $L(f) = \lambda f$.

Question 5: Check that $\psi(x) = \left(1 - \frac{1}{3x}\right) e^{3x}$ is an eigenfunction of $L = \partial^2 - \frac{2}{x^2}$. What is the eigenvalue?

Question 6: If a function is an eigenfunction of L, can it also be in the kernel of L?

Question 7: If a function f is an eigenfunction of L with eigenvalue λ, then I can make an operator out of L and λ that has f in its kernel...what operator is that?

- **Quantum Physics Again:** We have already seen how Heisenberg's uncertainty principle is just a fact about ODOs. The other big name in QM is Schrödinger. What he said is that the number we think of as the result of a measurement is really an *eigenvalue* of the corresponding operator. The *Schrödinger Equation* is an equation of the form

$$\left(\partial^2 + u(x)\right)(\psi(x)) = \lambda\psi(x)$$

(although usually it involves *partial* differential operators and functions of more variables). For Schrödinger, the important thing is ψ, which is the wave description of a particle. But, for *us*, it will be $u(x)$ where we see the KdV solitons.

- Next time we will see how one can recognize the KdV equation as a soliton equation using differential algebra...and also how we can already make infinitely many truly different soliton equations in the same way. Then, I am hoping to return to differential algebra again someday in the future to teach you how you can factor differential operators and why that is a useful thing to do in soliton theory. Finally, if you do like playing with differential algebra, let me know because we could also go into the interesting topic of *bispectrality* (either together or for a project) which would only be slightly off-subject.

➤ **Warning:** Remember that there is a difference between multiplication and action as operators. Don't get them confused and use the wrong one in the wrong place.

Homework

1. Let $\tau(x) = e^x + e^{-x}$ and define the two differential operators L_1 and L_2 by

$$L_1 = \partial - \frac{\tau'(x)}{\tau(x)} \qquad L_2 = \partial + \frac{\tau'(x)}{\tau(x)}.$$

   Compute the products $P = L_1 \circ L_2$ and $Q = L_2 \circ L_1$.
2. Continuing the previous problem, come up with a basis for the kernel of Q. I think you can do this with just a little bit of guesswork even if you don't know an official procedure for answering it. However, I bet you would have trouble guessing even one (non-zero) function in the kernel of P, right? (See the next question.)
3. Show that if f is in the kernel of $A \circ B$ then $B(f)$ is in the kernel of $B \circ A$. (This is actually a rather general statement that would apply to any linear operators).

   Then, use this fact to find a nonzero function in the kernel of P from the first question. (Why can you not use this same idea to find a *basis* for the kernel of P?)
4. Find the commutators:

$$[x\partial, x^2\partial^3] \qquad [\partial^3 + x, \partial^2 + e^{2x}] \qquad \left[\partial^2 - \frac{2}{x^2}, \partial^3 - \frac{3}{x^2}\partial + \frac{3}{x^3}\right]$$

5. For what value(s) of the constant n is the function $f(x) = x^n$ an eigenfunction for the operator $L = x\partial$? What is the corresponding eigenvalue?
6. Let L be an ordinary differential operator with *constant* coefficients. (That is, each function $c_i(x)$ in $(*)$ is just a number.) Then we can write L as $p(\partial)$ where $p(x)$ is an ordinary polynomial. The function $e^{\lambda x}$ is an eigenfunction for L. What is the eigenvalue? (If you're having trouble with this, try an example like $L = \partial^3 + 2\partial$ and $\lambda = 4$ first to see what happens.)
7. What is the relationship between the commutators $[L, M]$ and $[M, L]$?
8. If L is an operator of order N_1 and M is an operator of N_2 then their products $L \circ M$ and $M \circ L$ both have order $N_1 + N_2$. Show, however, that the commutator $[L, M]$ always has order *less* than $N_1 + N_2$.