Math 495 Handout: January 22 2008

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The Story

- In 1834, the ship designer John Scott Russell was riding his horse beside a canal in Scotland. Nearby, a barge pushed up a small hump of water which travelled on alone without changing its speed or shape. Russell followed it for miles on his horse. He knew that this was a new phenomenon worthy of study with potential implications for designing better ships.
- So, he built a wave tank in his yard and began studying these things he called "solitary waves". You can still read the papers he wrote about his experiments in *Report of the 14th Meeting of the British Association for the Advancement of Science*. Among the things he found were that if the wave height was taller, the wave travelled *faster*.
- Unfortunately, his work was not well received by the establishment. People as prestigious as Lord Stokes (a great mathematical physicist whose name is preserved in his discoveries such as Stokes' Theorem and Stokes Phenomenon) essentially claimed that Russell was crazy. He denied the existence of these "solitary waves".
- Essentially, Stokes' argument came down to the "fact" that the wave would have to satisfy a PDE. Either it is a linear PDE or a nonlinear one. But, it could not be a *linear* PDE. (What that I've already said above proves this?) And, Stokes believed it also could not be a nonlinear wave because the *distortion* and *dispersion* of the wave would force the shape to change.
- ➢ "Distortion" is the nonlinear phenomenon we saw in the Inviscid Burgers' Equation, leading to wave breaking. It is the effect that separates different amplitudes. "Dispersion" is the effect that causes different frequencies to separate. You can think of it as the thing that causes waves to disappear as time passes, like ripples in a pond that fade away.
- Intuitively, it seems clear that Russell's solitary wave is not being destroyed by distortion or dispersion as one would expect, but this is not a mathematical proof that it doesn't exist. Stokes *tried* writing a proof, and even published it, but the proof was later found to be flawed.
- In the end, it made no difference that no proof existed to show Russell was wrong. Everyone *believed* he was, and no research was done into these "solitary waves" beyond what he had already done.
- ➢ Side Note: Russell *did* end up designing a ship – a huge one – based on his theories. It played an important role in history as the ship that laid the first trans-Atlantic telegraph cables! (Remember this later for appropriate dramatic irony.)
- The next piece of the story takes place in 1895, after Russell and Stokes were dead. A famous Dutch mathematician, D.J. Korteweg, and his student F. de Vries, decided to model water waves on a canal using differential equations. Beginning with known equations modeling fluid flow in general, they made some simplifying assumptions (including a constant, shallow depth as one would find in a canal) and wrote:

$$u_t = \frac{3}{2}uu_x + \frac{1}{4}u_{xxx}.$$

This is the famous “KdV Equation”. (Side notes: Actually, they were not the first to write this equation. We know that Boussinesq studied it earlier. Also, they did not write it in exactly this form, but this is how I like to write it.)

- Korteweg and de Vries wrote a family of solutions to this equation which behave like Russell's solitary wave! In particular, they found that

$$u_k(x,t) = \frac{8k^2}{(e^{kx+k^3t} + e^{-kx-k^3t})^2}$$

is a solution to the KdV equation for *any* value of the constant k. It gives a solitary wave, like Russell's, that travels at speed k^2 and has height $2k^2$.

- ➢ At this point, I will stop and show you some stuff on Mathematica including that these are solutions and what their graphs look like when animated.

- Think about what is *amazing* here. Two things, really: they found an exact formula for many solutions to a nonlinear PDE, and the solution seems to be able to avoid distortion and dissipation despite Stokes' intuition to the contrary! (Loosely speaking, physicists like to say that the distortion and dissipation are balanced so they cancel out.)

- There is now a *long* pause in the story. During this time, Korteweg goes on to do great things and de Vries nearly starves to death as a teacher in rural Holland. Neither of them give much more thought to the paper they wrote together. There are interesting developments at this time in quantum physics and algebraic geometry, but nobody seems to connect them with Russell or the KdV equation.

- When we return to the story, it is the 1960s. Something strange has happened when three scientists/mathematicians tried to understand what happens in a nuclear reactor. The “Fermi-Pasta-Ulam Problem” found recurrence in a situation where they had expected everything to dissipate away to nothingness.

- To understand this, Zabusky and Kruskal returned to modeling the problem as a PDE...and they rediscovered the KdV equation. But, in the 1960's something was different. They could use *computers* to animate solutions (well, *almost* solutions) to see what happens.

- ➢ Here is how we can approximate KdV dynamics with a computer. Suppose you pick an *initial profile* $u(x,0) = f(x)$ for u. Now, as time move forward, this graph will change, but how does it change? At each point, we can tell whether it will go up or down and by how much by looking at u_t (which is positive when u will increase and negative when u will decrease). But, hey, since we're assuming it is a solution to KdV, we know that this is the same as the righthand side of the KdV equation and we can compute that. So, a program such as the one I'll show you gives a good approximation of KdV dynamics.

- What Kruskal and Zabusky found is that (a) whatever initial shape you start with, as long as it is zero except for a *localized* disturbance, it eventually breaks up into a bunch of solitary waves sorted by height (b) when those solitary waves do collide, they emerge from the collisions apparently undamaged. In order to capture this “particle” like aspect of these waves, they coined the name **solitons**.

- In a later paper, Kruskal along with Gardner, Greene and Miura found a way to write the *exact* formula for the solution having any number of “humps” of any sizes! All of these *multi-soliton solutions* can be written just using exponential functions and ordinary algebra!

- (In fact, their method for finding the solutions was very involved and beyond the scope of this class. Essentially, they were doing *quantum physics*. The method, known as "inverse scattering" involves treating the solution u as the potential energy in a quantum physics problem and seeing how other waves scatter off of it. Remarkably, the solitons are characterized by the lack of reflection in scattering...but I'm not planning to say more about that. Perhaps for a project?)

- For example, here is a 2-soliton solution to the KdV equation which seems to combine the solitons with $k = 1$ and $k = 2$:

$$u_{1,2}(x,t) = \frac{24\left(e^{2(t+x)} + 6e^{6(3t+x)} + 4e^{4(4t+x)} + 4e^{20t+8x} + e^{34t+10x}\right)}{\left(1 + 3e^{2(t+x)} + e^{6(3t+x)} + 3e^{4(4t+x)}\right)^2}$$

- Note that this is *not* the sum of two one-solitons, even though it may look that way when you watch an animation. Actually, because of the nonlinearity of the KdV equation, there is no reason to think that the sum of two one-solitons would even be a solution (and it is not)!

Question 1: Check (using the computer) whether the sum of the one-solitons with $k = 1$ and with $k = 2$ is a solution.

Question 2: Visually compare the 2-soliton solution to a sum of two 1-solitons to see how they differ. (Note: It is most convenient to consider $u_1(x + .5, t)$ and $u_2(x - .25, t)$ because these align the humps before the collision.) How are they the same, how are they different?

- One important part of the difference is the so-called "phase shift" which occurs as a consequence of the collision. There are two ways to think about it (and the difference between them is philosophical rather than mathematical). Kruskal-Zabusky described it as the shorter one being shifted *backwards* and the taller one being shifted *forwards* by the interaction. I prefer to think of it as the one in *back* becoming smaller and the one in *front* becoming larger so that they have exchanged momenta like colliding billiard balls. (For more information about these two interpretations, please see my recent paper in *J. Nonlin. Sci.* written with two C of C students.)

- **Back to the story:** Notice how this discovery *amplifies* what Korteweg and de Vries found back in 1895! If it was amazing that there was a solution with a single stable hump, how much more amazing it is that there are many humps and that they are not only stable on their own but through interactions! If it was amazing that they could find the one-soliton family of solutions to a nonlinear PDE that could be written in terms of exponential functions, how much more amazing is it that we can write solutions with any number of humps using nothing more complicated!

- Russell was right back in 1834 when he realized that this was a phenomenon worth studying, but it took until the late 1960s before everyone else realized it. But, at this point, it was not clear what they had found. Was the KdV equation the only one with these sorts of solutions? If so, why? If not, what made it work and what other equations could (a) be solved and (b) support soliton-like solutions? And finally, what can be *done* with this information?

➢ Ashley asked me on the first day of class "What are solitons and what is soliton theory?" It is a good question, and perhaps I can answer it now. Solitons are this phenomenon that Russell-Korteweg-de Vries-Kruskal-Zabusky-Gardner-Greene-Miura discovered and *soliton theory* is the attempt to answer all of the questions in the preceding paragraph. At this point, since the phenomenon is still being investigated, we may not know the full answers. What I can say is that a lot of progress has been made.

- As it turns out, we *can* understand what is special about the KdV equation in several ways. This is the "algebraic and geometric structure" underlying the equation that I am going to be talking about for much of the rest of the semester. Also, as it turns out, the KdV equation is *not* the only one with these properties. Many other equations that had already been studied (such as the nonlinear Schrödinger equation and the sine-Gordon equation) are now understood to also be "integrable" and have solitons. Moreover, many, many *more* equations have now been found that are like this as well. (In the MSC classification scheme for math, one category is "KdV like equations".) We will soon move from KdV to the KP equation (just a little more complicated, but much better for seeing the algebro-geometric aspects)...but I'm hoping to show you still other soliton equation and how you can find new ones.
- Mathematically, this connection between algebra, geometry and calculus has been very useful in all of the directions. We understand nonlinear PDEs better now that we can apply algebra and geometry to it. We also have answered open questions in algebra and geometry through the use of soliton theory.
- The "soliton revolution" has had impacts in the applied sciences. Some phenomena in the real world are now understood (or conjectured) to be examples of solitons (tsunamis, the red spot on Jupiter, ball lightning, energy transfer by enzymes in biological systems, etc.) The most fascinating application (for me) is the idea that particles and solitons are not just *similar*, but in fact the same. That is, we already know that particles are not the simple point masses that we once thought they were. The 20th century demonstrated that particles are mysterious objects. I would like to think that in the 21st century, the use of solitons will help us to resolve these mysteries and better understand what particles actually are. Certainly, some progress in this direction has already been made. (For example, the solitons of the Sine-Gordon equation model electrons in a thin wire, and Skyrmions are stunningly similar to atomic nuclei.) But, only a small minority of researchers share my optimism on this topic, and we certainly have work to do before the goal is achieved.

➢ **Epilogue 1:** In 1995, one hundred years after Korteweg and de Vries published their paper, I went to the library at MIT to look at the original. The old journals, covered in dust, filled the shelves of a back room. Only one volume was not dusty...and when I took it off the shelf it fell right open to the KdV paper. Although it was not appreciated at the time (even by its authors), this paper has achieved "stardom" today.

➢ **Epilogue 2:** The greatest application of solitons so far has been their use in communication. When signals (like phone calls or data transmission) are sent through a wire, the signal fades over long distances and must be amplified. But, a soliton of light travelling through glass fibers can in theory continue without any loss of information and in practice can still travel very far. So, now, digital information is sent using optical solitons. In the late 20th century, optical fibers were laid across the Atlantic, establishing a new form of communication to replace the old wires. The old wires were laid by Russell's boat, and the new ones use his *solitary wave* concept, redeeming the vision of this man ridiculed in his own lifetime.

Homework

1. For what values of the constants c_1 and c_2 is the function

$$u(x,t) = \frac{c_1}{(x+c_2)^2}$$

a solution to the KdV equation?

2. Suppose that I have a function $u(x,t)$ that is a solution to the equation $u_t = (u_x)^2 - 2u_{xx}$ and such that its initial profile looks like $u(x,0) = x^2$. The point $(0,0)$ is on the graph at time $t = 0$...will this point initially move *up* or *down* under the evolution determined by the equation. (Show or explain how you know.) What about the point $(1,1)$ which is also on the initial profile?

3. For what value(s) of the constant c is the function

$$u(t,x) = \frac{cx}{t}$$

a solution to the KdV equation? Describe the dynamics: "The graph of this function at any fixed time looks like .... and as time passes ..."

4. If $u(x,t)$ is a multi-soliton solution of KdV, then it looks like a bunch of humps of different sizes travelling to the left at different speeds. In a short essay (one or two paragraphs), explain why they are eventually in order. Include answers to these questions: What order are they in? Are they always ordered like that?

5. Show that if $u(x,t)$ is a solution to the KdV equation then so is $u(-x,-t)$. Explain, show or describe how this would affect the graph and the dynamics of the solution.

6. **Using Mathematica:** We are going to make a matrix and take its determinant. Type the following

```
tau[x_,t_,k_,g_]:=Exp[k x+ k^3 t]+ g Exp[-k x - k^3 t]
a=tau[x,t,k1,g1]
b=tau[x,t,k2,g2]
c=D[a,x]
d=D[b,x]
M={{a,b},{c,d}}
```

Now `M` is a matrix whose second row is the derivative of its first row. You can see it better by typing

```
MatrixForm[M]
```

Now, we'll take its determinant, and then let `u` be two times the second derivative of its logarithm!

```
newtau=Det[M]
u=2 D[Log[newtau],{x,2}]
```

Show that `u` *is* a solution to the KdV equation. Now, try different values of the constants `k1`,...`g2` and animate the results. What do you see? When is it a 2-soliton solution? What can go wrong?

7. **Using Mathematica:** Try the same thing as above but using a 3×3 matrix, so that there is a third choice of "tau" in the top right corner and the third row contains the *second* derivatives of the functions in the first row. Again, show that the result is *always* a solution of KdV and try looking at some specific choices of the constants in the form of animations. For an appropriate choice, you should see a 3-soliton!