Math 495 Handout: January 17, 2008

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Nonlinear Differential Equations

- *Linear* differential equations are pretty nice. We proved last time that the solution set to a homogeneous linear differential equation is always a vector space. That gives us a way to understand the structure of the solutions as well as a way to make new solutions out of known ones. If the equation is not homogeneous, then much of this still works. (The solution set is then a vector space *plus* some particular solution.)

- **In contrast**, *nonlinear* differential equations are a real mess. One illustration of that fact is that I will not attempt to prove any general statements about them. There simply is nothing universal that can be said. Instead, we will look at a few key examples to develop some intuition about what can go wrong with nonlinear equations and also try to understand why the idea of solitons was initially met with skepticism.

- In general, it can be very difficult to find any solutions to a nonlinear differential equation. I have intentionally "cooked" up this one so that I can find at least some solutions. Consider

$$(f + f_{yy})(f_x + f_y - 1) = 0.$$

I can see right away that $f(x,y) = \sin(y)$ and $f(x,y) = x$ are both solutions. But, can we take linear combinations of them to make more?

Question 1: For what values of the parameters λ and μ is

$$f(x,y) = \lambda \sin(y) + \mu x$$

also a solution?

- **Waves that Shock!** One of the elementary examples of a nonlinear wave equation that people always see is the "Inviscid Burgers' Equation":

$$u_t + uu_x = 0.$$

I can write solutions whose graphs are straight lines (like $u(x,t) = 0$). Otherwise, I am not certain that I can write down a *formula* for *any* solutions of this equation. So, that right away is an interesting phenomenon that you have not seen before. I have been carefully picking equations for which solutions could be written in terms of functions you know, but that is an *unusual* situation for nonlinear equations. More are like this.

But, this equation is not so bad in other ways. In fact, even though I cannot write a formula for its nontrivial solutions, I can tell you exactly what they would do! You can pick any "initial profile" $u(x,0) = f(x)$ and I can tell you how it would *evolve* in time according to this equation.

- The “Method of Characteristics” is useful for figuring out the behavior of solutions to some differential equations. The basic idea is that you track the behavior along a curve (or “characteristic”) $x = c(t)$ in the xt-plane. By an appropriate choice of the curve, things can work out nicely. Perhaps you’ve seen it in another class. If not, here is just an example. Rather than looking at all values of x and all values of t, let’s focus at just one x value for each time. Then the function becomes $u(c(t), t)$, which is just a function of one variable. What is its derivative?
$$\frac{d}{dt}u(c(t), t) = u_x(c(t), t)c'(t) + u_t(c(t), t).$$
Now, let’s pick a point where we want the curve to start at time $t = 0$: (x_0, t). We know the height of the wave there is $f(x_0)$. This is where we have to be clever in picking the function $c(t)$. What happens if I just define the curve to be $x = c(t) = f(x_0)t + x_0$? Then the derivative becomes
$$u_x(f(x_0)t + x_0, t)f(x_0) + u_t(f(x_0)t + x_0, t).$$
Plugging in $t = 0$ gives
$$u_x(x_0, 0)f(x_0) + u_t(x_0, 0) = u_x(x_0, 0)u(x_0, 0) + u_t(x_0, 0) = 0$$
where the last equality is true precisely because $u(x, t)$ is assumed to be a solution of the equation.
What does it mean to say that the derivative is equal to zero? That means that (infinitessimally) the value of u is neither going up nor down as we move along the characteristic. In other words, the height stays the same.
This is often the case with characteristics: If you were on a boat riding along a wavy curve and you stay at the point $x = c(t)$ at time t then you stay at the same height.
Now, we can think of a bunch of different people doing this at different points along the wave. Each one moves along a straight line in the xt-plane, but does not go up or down. The slopes of the lines are determined by their initial height. In fact, they move to the right at a speed exactly equal to their initial height (or move left if they begin at a negative height.)
Using Mathematica, we can make a movie of what these solutions look like, with each point along the curve moving to the right at a speed determined by the height. However, there are some unpleasant things that happen:
    - Shock waves form.
    - The solutions become “multi-valued”.
    - The values of the function $u(x, t)$ cannot be written as an algebraic formula of x and t with any of the functions we know.
- Note: This is actually not an unrealistic situation either! This equation is a simple model of waves as they approach the beach, and so this “breaking” phenomenon is one you should recognize. The “inviscid” part refers to the fact that the liquid is not very viscous. If there is viscosity (e.g. honey) then things are quite different. A term involving u_{xx} is added in and the behavior changes drastically. (In fact, I could get us into solitons by going in that direction, but instead I would like to proceed as history progressed by going to the KdV Equation next.)

Question 2: So, what will the dynamics look like if we start with a profile that looks like a “bell-curve”?

Question 3: What will happen to an initial profile that looks like a sine-wave?

Question 4: Can you think of a function $f(x)$ (whose graph is not a straight line) that we can write a formula for and so that the initial profile stays nice (not distorted, described by a function we know) for all time?

- Note that changing the coefficients on the equation makes no significant difference. If I wanted to solve the equation

$$U_t + 2UU_x = 0$$

  instead, I could just take any solution $u(x,t)$ of the original equation and then $U(x,t) = u(x,t/2)$ solves this new one.

Homework

1. Actually, it is possible to find solutions to this equation in the form

$$u(x,t) = c_1(t)x + c_2(t).$$

   Note that for each fixed value of t in the domain of c_1 and c_2, the graph is a straight line. Can you find an example of nonconstant functions c_1 and c_2 for which this gives an exact solution of the Inviscid Burgers' equation? Can you find the *general* formula for the solution? (Show your work and explain your reasoning.)
2. If the initial profile $u(x,0) = f(x)$ of a solution of Inviscid Burgers' equation looks like a single hump, we saw that it will lean to the right and "break" (just like an ocean wave). Suppose I started instead with an initial profile that was *twice as tall*: $u(x,0) = 2f(x)$...how would that solution be different? (Compare and contrast this with the situation we saw earlier in the case of a linear equation.)
3. Suppose I have a solution $u(x,t)$ of the inviscid Burgers' equation and I want to produce a solution of the equation

$$2U_t + 9UU_x = 0$$

   by choosing constants λ and γ so that $U(x,t) = u(\lambda x, \gamma t)$ is a solution of this new equation. What choices of λ and γ will be sure to work? (Note: It will be good if you give me any one choice of constants that work...but I would like your answer to be general and list all possible choices that will be sure to work!)
4. **Reading Asignment** Download the PDF document at
   http://www.claymath.org/millennium/Navier-Stokes_Equations/navierstokes.pdf
   It is a description of one of the Clay Mathematics Institutes "million dollar questions". Answer it and you can win a huge cash prize! (Good luck with that ;). Anyway, I mostly want you to read the first page. It describes a differential equation called the Navier-Stokes equation and poses the question of whether its solutions develop shocks or other similar problems (blow-ups, multivaluedness, etc.) Apparently, this is a very difficult problem, and also an important one. I'm just asking you to answer the following (open-ended) essay questions:
   (a) How is the Navier-Stokes equation different from the Inviscid Burgers' Equation above?
   (b) How is the Navier-Stokes equation the same? (In particular, what choices can we make for things like n, p and f_i so that they are *exactly* the same?)
   (c) Take a look through some of the rest of the document beyond the first page. Find (and briefly describe) two things you *do* understand and two things you *don't* understand.