

# Math 323 Solutions

## APRIL 4 ASSIGNMENT

p. 340 #2: The eigenvalues of the matrix

$$A = \begin{pmatrix} -2 & 1 \\ -4 & 3 \end{pmatrix}$$

are determined by  $0 = \det(A - \lambda I) = (-2 - \lambda)(3 - \lambda) + 4 = \lambda^2 - \lambda - 2$ . The solutions are  $\lambda_1 = 2$  and  $\lambda_2 = -1$ . The corresponding eigenvectors are in the nullspace of

$$A - 2I = \begin{pmatrix} -4 & 1 \\ -4 & 1 \end{pmatrix}, \quad A - (-1)I = \begin{pmatrix} -1 & 1 \\ -4 & 4 \end{pmatrix},$$

respectively. Thus, a choice of eigenvectors is

$$\mathbf{v}^1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad \mathbf{v}^2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The corresponding solutions are

$$\mathbf{x}^1(t) = e^{2t} \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad \mathbf{x}^2(t) = e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The general solution is

$$\mathbf{x}(t) = c_1 \mathbf{x}^1(t) + c_2 \mathbf{x}^2(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 4 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

#8: The eigenvalues of

$$A = \begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix}$$

are determined by  $0 = \det(A - \lambda I) = \lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1)$ . We have

$$A - 4I = \begin{pmatrix} -3 & -3 \\ -2 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad A - (-1)I = \begin{pmatrix} 2 & -3 \\ -2 & 3 \end{pmatrix},$$

For  $\lambda = 4$ ,  $\lambda = -1$  respectively, a choice of eigenvectors is

$$\mathbf{v}^1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}^2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

The general solution is

$$\mathbf{x}(t) = c_1 e^{4t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Imposing the initial condition  $x_1(0) = 0$ ,  $x_2(0) = 5$  gives the linear equations

$$-c_1 + 3c_2 = 0, \quad c_1 + 2c_2 = 5,$$

which have solution  $c_1 = 3$  and  $c_2 = 1$ . The solution to the initial value problem is

$$\mathbf{x}(t) = 3e^{4t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + e^{-t} \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

#12: The eigenvalues of

$$A = \begin{pmatrix} 3 & 1 & -2 \\ -1 & 2 & 1 \\ 4 & 1 & -3 \end{pmatrix}$$

are determined by

$$0 = \det(A - \lambda I) = -\lambda^3 + 2\lambda^2 + \lambda - 2 = -(\lambda - 1)(\lambda + 1)(\lambda - 2).$$

For  $\lambda = 1$ ,  $\lambda = 2$ ,  $\lambda = -1$  respectively, eigenvectors are

$$\mathbf{v}^1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}^2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}^3 = \begin{bmatrix} -7 \\ 2 \\ -13 \end{bmatrix}.$$

The general solution is

$$\mathbf{x}(t) = c_1 e^t \mathbf{v}_1 + c_2 e^{2t} \mathbf{v}_2 + c_3 e^{-t} \mathbf{v}_3.$$

Imposing the initial conditions gives a linear system

$$\begin{pmatrix} 1 & 1 & -7 \\ 0 & 1 & 2 \\ 1 & 1 & -13 \end{pmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix},$$

which has solution  $c_1 = 9$ ,  $c_2 = 4/3$ ,  $c_3 = 4/3$ . The solution of the initial value problem is

$$\mathbf{x}(t) = 9e^t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{4}{3}e^{2t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{4}{3}e^{-t} \begin{bmatrix} -7 \\ 2 \\ -13 \end{bmatrix}.$$

p.344, #3: The characteristic polynomial of the coefficient matrix is

$$\det(A - \lambda I) = (1 - \lambda)((1 - \lambda)^2 + 4) = (1 - \lambda)(\lambda^2 - 2\lambda + 5)$$

with roots  $\lambda = 1$  and  $\lambda = 1 \pm 2i$ .

For  $\lambda = 1$ ,  $A - \lambda I = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & -2 \\ 2 & 2 & 0 \end{pmatrix}$  and an eigenvector is  $\mathbf{v} = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$ , giving the solution

$$\mathbf{x}(t) = e^t \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}.$$

For  $\lambda = 1 + 2i$ ,  $A - \lambda I = \begin{pmatrix} -2i & 0 & 0 \\ 3 & -2i & -2 \\ 2 & 2 & -2i \end{pmatrix}$ , and an eigenvector is  $\mathbf{v} = \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$ , giving the complex

solution

$$\mathbf{x}(t) = e^{(1+2i)t} \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} = e^t (\cos 2t + i \sin 2t) \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} = e^t \begin{bmatrix} 0 \\ -\sin 2t \\ \cos 2t \end{bmatrix} + i e^t \begin{bmatrix} 0 \\ \cos 2t \\ \sin 2t \end{bmatrix}.$$

Since the real and imaginary parts of this are themselves linearly independent real solutions, we get the general solution

$$\mathbf{x}(t) = c_1 e^t \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} + c_2 e^t \begin{bmatrix} 0 \\ -\sin 2t \\ \cos 2t \end{bmatrix} + c_3 e^t \begin{bmatrix} 0 \\ \cos 2t \\ \sin 2t \end{bmatrix}.$$

#6: The characteristic polynomial is

$$\det(A - \lambda I) = (3 - \lambda)(-1 - \lambda) + 8 = \lambda^2 - 2\lambda + 5,$$

with roots  $\lambda = 1 \pm 2i$  again.

Using  $\lambda = 1 + 2i$ ,  $A - \lambda I = \begin{pmatrix} 2 - 2i & -2 \\ 4 & -2 - 2i \end{pmatrix}$ , and an eigenvector is  $\mathbf{v} = \begin{bmatrix} 1 \\ 1 - i \end{bmatrix}$ , giving the complex solution

$$\mathbf{x}(t) = e^{(1+2i)t} \begin{bmatrix} 1 \\ 1 - i \end{bmatrix} = e^t(\cos 2t + i \sin 2t) \begin{bmatrix} 1 \\ 1 - i \end{bmatrix} = e^t \begin{bmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{bmatrix} + ie^t \begin{bmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{bmatrix}.$$

The general solution is

$$\mathbf{x}(t) = c_1 e^t \begin{bmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{bmatrix} + c_2 e^t \begin{bmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{bmatrix} \quad \text{with} \quad \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Imposing  $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$  gives  $c_1 = 1$  and  $c_1 - c_2 = 5$ . So,  $c_1 = 1$ ,  $c_2 = -4$ , and

$$\mathbf{x}(t) = e^t \begin{bmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{bmatrix} - 4e^t \begin{bmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{bmatrix} = e^t \begin{bmatrix} \cos 2t - 4 \sin 2t \\ 5 \cos 2t - 3 \sin 2t \end{bmatrix}.$$

#9: The characteristic polynomial is  $\det(A - \lambda I) = (1 - \lambda)^2(-1 - \lambda) + 2 = (1 - \lambda)(\lambda^2 + 1)$ , with roots  $\lambda = 1$  and  $\lambda = \pm i$ .

For  $\lambda = 1$ ,  $A - \lambda I = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 1 & -1 & -2 \end{pmatrix}$ , with eigenvector  $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ .

For  $\lambda = i$ ,  $A - \lambda I = \begin{pmatrix} 1 - i & 0 & -2 \\ 0 & 1 - i & 0 \\ 1 & -1 & -1 - i \end{pmatrix}$ , with eigenvector  $\mathbf{v} = \begin{bmatrix} 1 + i \\ 0 \\ 1 \end{bmatrix}$ , giving the complex

solution

$$\mathbf{x}(t) = e^{it} \begin{bmatrix} 1 + i \\ 0 \\ 1 \end{bmatrix} = (\cos t + i \sin t) \begin{bmatrix} 1 + i \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos t - \sin t \\ 0 \\ \cos t \end{bmatrix} + i \begin{bmatrix} \sin t + \cos t \\ 0 \\ \sin t \end{bmatrix}.$$

The general solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} \cos t - \sin t \\ 0 \\ \cos t \end{bmatrix} + c_2 \begin{bmatrix} \sin t + \cos t \\ 0 \\ \sin t \end{bmatrix} + c_3 e^t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Notice that this will be periodic only if  $c_3 = 0$ . Thus, the solution will be periodic only for initial conditions of the form

$$\mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

in other words, for initial conditions in which the middle entry of the vector is zero.

p.352 #1: The characteristic polynomial is  $\det(A - \lambda I) = -(\lambda^3 + 4\lambda^2 + 3\lambda^2 + 2\lambda + 2) = -(\lambda + 1)^2(\lambda + 2)$ . For  $\lambda = -2$ ,

$$A - \lambda I = \begin{pmatrix} 2 & -1 & 1 \\ 2 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

with eigenvector  $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ , giving the solution  $\mathbf{x}^1(t) = e^{-2t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ .

For  $\lambda = -1$ ,

$$A - \lambda I = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -2 & 1 \\ 1 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

with a one-dimensional eigenspace spanned by  $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ , giving the solution  $\mathbf{x}^2(t) = e^{-t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ .

Therefore, we need to find a generalized eigenvector  $\mathbf{u}$  such that  $(A - (-1)I)\mathbf{u} = \mathbf{v}$ . Since

$$\left( \begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 2 & -2 & 1 & 1 \\ 1 & -1 & 0 & 0 \end{array} \right) \sim \left( \begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right),$$

one generalized eigenvector is  $\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ . Using this we get a third solution

$$\mathbf{x}^3(t) = e^{-t}(\mathbf{u} + t\mathbf{v}) = e^{-t} \begin{bmatrix} t \\ t \\ 1 \end{bmatrix}.$$

The general solution is

$$\mathbf{x}(t) = c_1 e^{-2t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} t \\ t \\ 1 \end{bmatrix}.$$

#6: The characteristic polynomial is  $\det(A - \lambda I) = -\lambda^3 + 6\lambda^2 - 12\lambda + 8 = -(\lambda - 2)^3$ . For  $\lambda = 2$ , the only eigenvector (up to multiple) is

$$\mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \text{ giving the solution } \mathbf{x}^1(t) = e^{2t} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}.$$

Find a generalized eigenvector  $\mathbf{u}$  such that  $(A - 2I)\mathbf{u} = \mathbf{v}$ . Since

$$\left( \begin{array}{ccc|c} -6 & -4 & 0 & 2 \\ 10 & 7 & 1 & -3 \\ -4 & -3 & -1 & 1 \end{array} \right) \sim \left( \begin{array}{ccc|c} 3 & 2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right),$$

one generalized eigenvector is  $\mathbf{u}^1 = {}^t(-1, 1, 0)$ ; others come by adding multiples of  $\mathbf{v}$  to this one. This gives the solution

$$\mathbf{x}^2(t) = e^{\lambda t}(\mathbf{u}^1 + t(A - \lambda I)\mathbf{u}^1) = e^{2t} \begin{bmatrix} -1 + 2t \\ 1 - 3t \\ t \end{bmatrix}.$$

Because we still don't have enough independent vectors, we need to look for generalized eigenvectors of order 2, such that  $(A - 2I)\mathbf{u} = \mathbf{u}^1$ . Since

$$\left( \begin{array}{ccc|c} -6 & -4 & 0 & -1 \\ 10 & 7 & 1 & 1 \\ -4 & -3 & -1 & 0 \end{array} \right) \sim \left( \begin{array}{ccc|c} 3 & 2 & 0 & \frac{1}{2} \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{array} \right),$$

then one generalized eigenvector is  $\mathbf{u}^2 = {}^t(\frac{3}{2}, -2, 0)$ . (Again, others differ by adding  $\mathbf{v}$ .) This gives a third solution

$$\mathbf{x}^3(t) = e^{\lambda t}(\mathbf{u}^2 + t(A - \lambda I)\mathbf{u}^2 + \frac{t^2}{2}(A - \lambda I)^2\mathbf{u}^2) = e^{2t} \begin{bmatrix} \frac{3}{2} - t + t^2 \\ -2 + t - \frac{3}{2}t^2 \\ \frac{1}{2}t^2 \end{bmatrix}$$

The general solution is  $\mathbf{x}(t) = c_1\mathbf{x}^1(t) + c_2\mathbf{x}^2(t) + c_3\mathbf{x}^3(t)$ . In order to get the initial value  $\mathbf{x}(0) = {}^t(2, 1, -1)$ , we use  $c_1 = -1$ ,  $c_2 = -10$ ,  $c_3 = -4$ , giving

$$\mathbf{x}(t) = e^{2t} \begin{bmatrix} 2 - 16t - 4t^2 \\ 1 + 26t + 6t^2 \\ -1 - 10t + 2t^2 \end{bmatrix}.$$

#17:(a)

$$A^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$$

(b) Since  $A^2 = -I$ , then  $A^3 = -A$ ,  $A^4 = I$ ,  $A^5 = A$ , and so on, giving

$$\begin{aligned} e^{At} &= I + tA + \frac{1}{2}t^2A^2 + \frac{1}{3!}t^3A^3 + \frac{1}{4!}t^4A^4 + \frac{1}{5!}t^5A^5 + \frac{1}{6!}t^6A^6 \dots \\ &= I + tA - \frac{1}{2}t^2I - \frac{1}{3!}t^3A + \frac{1}{4!}t^4I + \frac{1}{5!}t^5A - \frac{1}{6!}t^6I \dots \end{aligned}$$

Both diagonal entries of this sum of matrices are equal to

$$1 - \frac{1}{2}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6 \dots = \cos t.$$

The top right entry of the sum is equal to

$$t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 \dots = \sin t,$$

and the bottom left entry is  $-1$  times this. Therefore,

$$e^{At} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$