Math 323 Solutions

Mar. 28 Assignment

Problem from webpage:

> Suppose that $f(t) = 5$ for $0 < t \leq 10$ and $f(t) = t$ for $t > 10$. Compute the Laplace transform of $f(t)$ in two ways—using the definition, and by expressing f(t) in terms of a Heaviside step function.

(a) Assume that $s > 0$. Then

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \int_0^{10} 5e^{-st}\,dt + \int_{10}^{\infty} te^{-st}\,dt\\ &= \int_0^{10} 5e^{-st}\,dt + \lim_{A\to\infty}\int_{10}^{A} te^{-st}\,dt\\ &= \left.\frac{-5}{s}e^{-st}\right|_0^{10} + \lim_{A\to\infty}\left(-\frac{t}{s}e^{-st} - \left.\frac{1}{s^2}e^{-st}\right|_{10}^{A}\right)\\ &= \frac{-5e^{-10s}}{s} + \frac{5}{s} + \left(\frac{10}{s} + \frac{1}{s^2}\right)e^{-10s} = \frac{5}{s} + \left(\frac{5}{s} + \frac{1}{s^2}\right)e^{-10s}\end{aligned}$$

(b) We can write $f(t) = 5 + (t-5)H_{10}(t)$. Then using linearity and the rule

$$\mathcal{L}\{g(t)H_c(t)\} = e^{-cs}\mathcal{L}\{g(t+c)\},$$

we have

$$\mathcal{L}\{f(t)\} = \frac{5}{s} + e^{-10s}\mathcal{L}\{t + 10 - 5\} = \frac{5}{s} + e^{-10s}\left(\frac{1}{s^2} + \frac{5}{s}\right).$$

p.243 #3: Using a step function, we can express the differential equation as

$$y'' + 4y = 1 - H_4(t).$$

Taking Laplace transforms on both sides gives

$$s^2Y - 3s + 2 + 4Y = \left(1 - e^{-4s}\right)\frac{1}{s}.$$

Solving for Y gives

$$Y = \frac{3s-2}{s^2+4} + \left(1 - e^{-4s}\right)\frac{1}{s(s^2+4)}.$$

Applying partial fractions to the last part of the second term gives

$$\begin{aligned}Y &= \frac{3s-2}{s^2+4} + \frac{1}{4}\left(1 - e^{-4s}\right)\left(\frac{1}{s} - \frac{s}{s^2+4}\right)\\ &= \frac{\frac{11}{4}s - 2}{s^2+4} + \frac{1}{4}\frac{1}{s} + \frac{1}{4}e^{-4s}\left(\frac{s}{s^2+4} - \frac{1}{s}\right)\end{aligned}$$

Taking inverse Laplace transforms gives

$$y = \frac{11}{4}\cos(2t) - \sin(2t) + \frac{1}{4} + \frac{1}{4}\left(\cos(2(t-4)) - 1\right)H_4(t).$$

#9: Using step functions, this differential equation is

$$y'' - 2y' + y = t(H_1(t) - H_2(t)).$$

Taking Laplace transforms on both sides gives

$$s^2Y - 1 - 2sY + Y = e^{-s}\mathcal{L}\{t+1\} - e^{-2s}\mathcal{L}\{t+2\} = e^{-s}\left(\frac{1+s}{s^2}\right) - e^{-2s}\left(\frac{1+2s}{s^2}\right).$$

Solving for Y gives

$$Y = \frac{1}{(s-1)^2} + e^{-s}\frac{1+s}{s^2(s-1)^2} - e^{-2s}\frac{1+2s}{s^2(s-1)^2}.$$

Applying the partial fraction decompositions, we can write

$$Y = \frac{1}{(s-1)^2} + e^{-s}\left(\frac{3}{s} + \frac{1}{s^2} - \frac{3}{s-1} + \frac{2}{(s-1)^2}\right) - e^{-2s}\left(\frac{4}{s} + \frac{1}{s^2} - \frac{4}{s-1} + \frac{3}{(s-1)^2}\right).$$

Taking inverse Laplace transforms gives

$$\begin{aligned} y &= te^t + \left(3 + (t-1) - 3e^{t-1} + 2(t-1)e^{t-1}\right)H_1(t) - \left(4 + (t-2) - 4e^{t-2} + 3(t-2)e^{t-2}\right)H_2(t) \\ &= te^t + \left(2 + t + (2t-5)e^{t-1}\right)H_1(t) - \left(t + 3 + (3t-10)e^{t-2}\right)H_2(t). \end{aligned}$$

(Note that, in the answer in the back of the textbook, the last term is wrong.)

p. 271 #2: Letting $x_1 = y$, $x_2 = y'$, and $x_3 = y''$, we get the following system:

$$\begin{cases} x_1' = x_2, \\ x_2' = x_3, \\ x_3' = e^t - \cos(x_1). \end{cases}$$

#4: As directed in the problem, let $x_1 = y$, $x_2 = y'$, $x_3 = z$, $x_4 = z'$. Then we get the system

$$\begin{cases} x_1' = x_2, \\ x_2' = -3x_4 - 2x_1, \\ x_3' = x_4, \\ x_4' = -3x_2 - 2x_3. \end{cases}$$

#5:(a) Suppose $y'' + y' + y = 0$. If $x_1 = y$, $x_2 = y'$, then $dx_1/dt = x_2$ and $dx_2/dt = y'' = -y - y' = -x_1 - x_2$. Thus,

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

(b) Suppose $\mathbf{x}(t)$ satisfies

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}\mathbf{x}.$$

Then $y = x_1(t)$ satisfies $y' = x_2(t)$ (from the top row of the system), and $y'' = dx_2/dt = -x_1 - x_2$ (from the bottom row of the system). Therefore $y'' = -y - y'$, and so $y'' + y' + y = 0$.

p. 296 #1: The given system encodes the second-order ODE $y'' + y' + y = 0$ (where $x_1 = y$ and $x_2 = y'$). The characteristic equation for this ODE has roots $r = (-1 \pm \sqrt{3})/2$, so a fundamental solution set for this ODE is

$$y_1 = e^{-t/2}\cos(\sqrt{3}t/2), \qquad y_2 = e^{-t/2}\sin(\sqrt{3}t/2).$$

Substituting each of these into $x_1 = y$ and $x_2 = y'$, we get two vector solutions to the original system:

$$\mathbf{x}^1(t) = \begin{bmatrix} e^{-t/2}\cos(\sqrt{3}t/2) \\ e^{-t/2}\left(-\frac{1}{2}\cos(\sqrt{3}t/2) - \frac{\sqrt{3}}{2}\sin(\sqrt{3}t/2)\right) \end{bmatrix}, \mathbf{x}^2(t) = \begin{bmatrix} e^{-t/2}\sin(\sqrt{3}t/2) \\ e^{-t/2}\left(-\frac{1}{2}\sin(\sqrt{3}t/2) + \frac{\sqrt{3}}{2}\cos(\sqrt{3}t/2)\right) \end{bmatrix}.$$

The fact that the Wronskian of y_1 and y_2 is nonzero (verified in problem #10 on p.140) implies that $\mathbf{x}^1(t)$ and $\mathbf{x}^2(t)$ are linearly independent solutions.

#2: As in #1, the system encodes a linear constant-coefficient ODE,

$$y''' - 2y'' + y' - 2y = 0.$$

The characteristic equation of this ODE is $0 = r^3 - 2r^2 + r - 2 = (r-2)(r^2+1)$, with roots $r = 2$ and $r = \pm \mathrm{i}$. Thus a basis for the solutions is $y_1 = e^{2t}$, $y_2 = \cos t$, $y_3 = \sin t$. The corresponding vector solutions to the original system are

$$\mathbf{x}^1(t) = \begin{bmatrix} e^{2t} \\ 2e^{2t} \\ 4e^{2t} \end{bmatrix}, \qquad \mathbf{x}^2(t) = \begin{bmatrix} \cos t \\ -\sin t \\ -\cos t \end{bmatrix}, \qquad \mathbf{x}^3(t) = \begin{bmatrix} \sin t \\ \cos t \\ -\sin t \end{bmatrix}.$$

#6: First, we check that the given vector-valued functions are solutions of the system:

$$\frac{d\mathbf{x}^1}{dt} = 2e^{2t}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

while

$$A\mathbf{x}^1(t) = e^{2t}\begin{pmatrix} 4 & -2 & 2 \\ -1 & 3 & 1 \\ 1 & -1 & 5 \end{pmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = e^{2t}\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix},$$

which is the same as $d\mathbf{x}^1/dt$. The other two solutions can be verified in the same way.

Evaluating the given vector-valued functions at $t = 0$ gives

$$\mathbf{x}^1(0) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{x}^2(0) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{x}^3(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

To determine if the given functions are a basis for the solution space, we just have to verify that these vectors are linearly independent. That can be done by making a matrix whose columns are the given vectors and checking that the determinant is nonzero:

$$\det\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = 2.$$

Alternatively, the same matrix can be row-reduced to the identity matrix.