Math 323 Solutions

Mar. 21 Assignment

p. 216 #2: Putting this ODE in standard form,

$$y'' + \frac{1}{(t-2)^2}y' + \frac{1}{t(t-2)^2}y = 0,$$

shows that $p(t) = 1/(t-2)^2$ and $q(t) = 1/(t(t-2)^2)$. Because $(t-2)p(t)$ is not analytic at $t = 2$, this means that $t = 2$ is not a regular singular point.

#10: Putting the ODE $2t^2y'' - ty' + (1+t)y = 0$ in standard form,

$$y'' - \frac{1}{2t}y' + \frac{1+t}{2t^2}y = 0,$$

shows that $p(t) = -1/(2t)$ and $q(t) = (1+t)/(2t^2)$. Then $t = 0$ is an ordinary singular point, with

$$p_0 = \lim_{t\to 0} tp(t) = -\tfrac{1}{2}, \qquad q_0 = \lim_{t\to 0} t^2q(t) = \tfrac{1}{2}.$$

The indicial equation $r(r-1) + p_0r + q_0 = 0$ (see Theorem 8, p.215) becomes

$$r^2 - \tfrac{3}{2}r + \tfrac{1}{2} = 0,$$

with roots $r = 1$ and $r = 1/2$.

For $r = 1$, substitute $y = \sum_{n=0}^{\infty} a_nt^{n+1}$ (with $a_0 \neq 0$) in the ODE, giving

$$\begin{aligned} 0 &= 2\sum_{n=0}^{\infty} n(n+1)a_nt^{n+1} - \sum_{n=0}^{\infty}(n+1)a_nt^{n+1} + \sum_{n=0}^{\infty} a_nt^{n+1} + \sum_{n=0}^{\infty} a_nt^{n+2} \\ &= \sum_{n=0}^{\infty}(2n^2+n)a_nt^{n+1} - \sum_{n=1}^{\infty} a_{n-1}t^{n+1} = \sum_{n=1}^{\infty}[(2n^2+n)a_n + a_{n-1}]t^{n+1}. \end{aligned}$$

So, we get $a_n = -(a_{n-1})/(n(2n+1))$ for all $n \geq 1$. If $a_0 = 1$, then

$$a_1 = -\tfrac{1}{1\times 3}, \quad a_2 = \tfrac{1}{1\times 2\times 3\times 5}, \quad a_3 = -\tfrac{1}{1\times 2\times 3\times 3\times 5\times 7}, \ldots$$

so that, in general,

$$a_n = \frac{(-1)^n}{n!\frac{(2n+1)!}{2\times 4\times \ldots \times 2n}} = \frac{(-1)^n 2^n}{(2n+1)!}.$$

For $r = 1/2$, substitute $y = \sum_{n=0}^{\infty} a_nt^{n+\frac{1}{2}}$ (with $a_0 \neq 0$) in the ODE, giving

$$\begin{aligned} 0 &= 2\sum_{n=0}^{\infty}(n-\tfrac{1}{2})(n+\tfrac{1}{2})a_nt^{n+\frac{1}{2}} - \sum_{n=0}^{\infty}(n+\tfrac{1}{2})a_nt^{n+\frac{1}{2}} + \sum_{n=0}^{\infty} a_nt^{n+\frac{1}{2}} + \sum_{n=0}^{\infty} a_nt^{n+\frac{3}{2}} \\ &= \sum_{n=0}^{\infty}(2n^2-n)a_nt^{n+\frac{1}{2}} - \sum_{n=1}^{\infty} a_{n-1}t^{n+\frac{1}{2}} = \sum_{n=1}^{\infty}[(2n^2-n)a_n + a_{n-1}]t^{n+\frac{1}{2}}. \end{aligned}$$

So, we get $a_n = -(a_{n-1})/(n(2n-1))$ for all $n \geq 1$. If $a_0 = 1$, then

$$a_1 = \tfrac{-1}{1\times 1}, \quad a_2 = \tfrac{1}{1\times 2\times 1\times 3}, \quad a_3 = -\tfrac{1}{1\times 2\times 3\times 1\times 3\times 5}, \ldots$$

so that, in general,

$$a_n = \frac{(-1)^n}{n!\frac{(2n)!}{2\times 4\times \ldots \times 2n}} = \frac{(-1)^n 2^n}{(2n)!}.$$

The general solution is

$$y = c_1 \sum_{n=0}^{\infty} \frac{(-2)^n}{(2n+1)!} t^n + c_2 \sum_{n=0}^{\infty} \frac{(-2)^n}{(2n)!} t^{n+\frac{1}{2}}.$$

p.232 #4: Using the definition of Laplace transform, we have

$$\mathcal{L}\{e^{at} \sin bt\} = \int_0^{\infty} e^{(a-s)t} \sin bt\, dt.$$

From integral tables,

$$\int e^{(a-s)t} \sin bt\, dt = e^{(a-s)t} \left(\frac{(a-s)\sin bt - b\cos bt}{(s-a)^2 + b^2} \right).$$

Assume that $s > a$; then the limit of this as $t \to +\infty$ is zero, and we are left with

$$\mathcal{L}\{e^{at} \sin bt\} = \frac{b}{(s-a)^2 + b^2}, \qquad \text{for } s > a.$$

#17: Taking $\mathcal{L}$ on both sides, and using the initial values, we get

$$(s^2 Y - s - 3) + 2(sY - 1) + Y = \frac{1}{s+1}.$$

Solving for Y gives

$$Y = \frac{s+5}{s^2 + 2s + 1} + \frac{1}{(s+1)(s^2 + 2s + 1)} = \frac{s^2 + 6s + 6}{(s+1)^3}.$$

#19: Taking $\mathcal{L}$ on both sides and using the initial values gives

$$s^2 Y - 2 + 3sY + 7Y = \frac{s}{s^2 + 1}.$$

Solving for Y gives

$$Y = \frac{2}{s^2 + 3s + 7} + \frac{s}{(s^2+1)(s^2 + 3s + 7)} = \frac{2s^2 + s + 2}{(s^2+1)(s^2 + 3s + 7)}.$$

p.237 #3: Using the t-property (also known as Property 1, on page 233),

$$\mathcal{L}\{t \sin at\} = \left(-\frac{d}{ds}\right) \mathcal{L}(\sin at) = -\frac{d}{ds} \frac{a}{s^2 + a^2} = \frac{2as}{(s^2 + a^2)^2}.$$

#4: Using the t-property,

$$\mathcal{L}\{t^2 \cos at\} = \left(-\frac{d}{ds}\right)^2 \mathcal{L}(\cos at) = \left(\frac{d}{ds}\right)^2 \frac{s}{s^2 + a^2} = \frac{d}{ds} \frac{a^2 - s^2}{(s^2 + a^2)^2} = \frac{2s^3 - 6a^2 s}{(s^2 + a^2)^3}$$

#10: Because

$$\frac{1}{s(s^2+4)} = \frac{1}{4} \left(\frac{1}{s} - \frac{s}{s^2+4} \right),$$

then

$$\mathcal{L}^{-1}\left(\frac{1}{s(s^2+4)}\right) = \frac{1}{4} - \frac{1}{4}\cos(2t).$$

#13: It is easy to rewrite

$$\frac{3s}{(s+1)^4} = 3\frac{s+1-1}{(s+1)^4} = \frac{3}{(s+1)^3} - \frac{3}{(s+1)^4}.$$

Then,

$$\mathcal{L}^{-1}\left(\frac{3s}{(s+1)^4}\right) 3\left(\frac{1}{2}t^2e^{-t} - \frac{1}{6}t^3e^{-t}\right) = \frac{3}{2}t^2e^{-t} - \frac{1}{2}t^3e^{-t}.$$

#15: The partial fractions decomposition is

$$\frac{s}{(s+1)^2(s^2+1)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{Cs+D}{s^2+1} = \frac{-1}{2}\frac{1}{(s+1)^2} + \frac{1}{2}\frac{1}{s^2+1}.$$

Then

$$\mathcal{L}^{-1}\left(\frac{s}{(s+1)^2(s^2+1)}\right) = \frac{-1}{2}te^{-t} + \frac{1}{2}\sin t.$$

(The answer in the back of the book is incorrect; the cosine should be a sine.)

#23: Taking the Laplace transform of both sides gives

$$(s^2Y - 3s + 5) + (sY - 3) + Y = \frac{1}{s} + \frac{1}{s+1} = \frac{2s+1}{s(s+1)}.$$

Then

$$(s^2+s+1)Y = 3s - 2 + \frac{2s+1}{s(s+1)}$$

and

$$Y = \frac{(3s-2)s(s+1) + 2s + 1}{s(s+1)(s^2+s+1)} = \frac{3s^3+s^2+1}{s(s+1)(s^2+s+1)}.$$

The partial fractions decomposition is

$$Y = \frac{A}{s} + \frac{B}{s+1} + \frac{C(s+\frac{1}{2})+D}{(s+\frac{1}{2})^2+\frac{3}{4}} = \frac{1}{s} + \frac{1}{s+1} + \frac{s+\frac{1}{2}-\frac{7}{2}}{(s+\frac{1}{2})^2+\frac{3}{4}}$$

Then

$$y = \mathcal{L}^{-1}(Y) = 1 + e^{-t} + e^{-t/2}\left(\cos(\frac{\sqrt{3}}{2}t) - \frac{7}{\sqrt{3}}\sin(\frac{\sqrt{3}}{2}t)\right).$$