Math 323 Solutions

Feb. 29 Assignment

p.172 #4: The position y of the object, measured downward from equilibrium, satisfies

$$y'' + 3y' + 2y = 0.$$

The general solution of this equation is $y = c_1 e^{-t} + c_2 e^{-2t}$. The initial conditions are $y(0) = 1/2$ and $y'(0) = 0$, which imply that $c_1 + c_2 = 1/2$ while $-c_1 - 2c_2 = 0$. Then $c_2 = -1/2$ and $c_1 = 1$. Thus,

$$y(t) = e^{-t} - \frac{1}{2}e^{-2t}.$$

This satisfies $\lim_{t\to\infty} y(t) = 0$. To show that it doesn't reach equilibrium at any finite time $t > 0$, solve

$$y(t) = e^{-t}(1 - \frac{1}{2}e^{-t}) = 0,$$

which gives $t = -\ln 2$, a negative number. Thus, the object reaches equilibrium only as $t \to \infty$.

#5: The position satisfies

$$y'' + 2y'' + y = 0.$$

The general solution of this equation is $y = (c_1 + c_2 t)e^{-t}$. The initial conditions are $y(0) = 1/4$ and $y'(0) = -1$ (upward), which imply that $c_1 = 1/4$ and $-c_1 + c_2 = -1$, so that $c_2 = -3/4$. Thus,

$$y(t) = \frac{1}{4}(1 - 3t)e^{-t}.$$

Then $\lim_{t\to\infty} y(t) = 0$. The object also passes through equilibrium ($y = 0$) when $t = 1/3$.

#8: In general, the position satisfies $my'' + cy' + ky = F(t)$. Because $c/m = 6$ and $k/m = 9$, then

$$y'' + 6y' + 9y = \frac{3}{m}\sin 3t.$$

Using the judicious guess $y_p(t) = A\cos 3t + B\sin 3t$ (which satisfies $y'' + 9y = 0$) we get $A = -1/(6m)$ and $B = 0$. The general solution is

$$y(t) = c_1 e^{-3t} + c_2 t e^{-3t} - \tfrac{1}{6m}\cos 3t.$$

Using the initial conditions $y(0) = 0, y'(0) = 0$, we get $c_1 = 1/(6m)$ and $c_2 - 3c_1 = 0$, giving $c_2 = 1/(2m)$. Then

$$y(t) = \frac{1}{6m}(e^{-3t}(1 + 3t) - \cos 3t).$$

The maximum value of $e^{-3t}(1 + 3t)$ is 1, which occurs when $t = 0$; thus,

$$|y(t)| \leq \frac{1}{6m}(|e^{-3t}(1 + 3t)| + |\cos 3t|) \leq \frac{1}{3m}.$$

In order for $|y(t)| < 5$, it suffices that $m > 1/15$ kg.

11: The position satisfies

$$y'' + 4y = 1 + t + \sin 2t.$$

Notice that $\sin(2t)$ is a solution of the homogeneous equation $y'' + 4y = 0$. Thus, we use the judicious guess $y_p(t) = A + Bt + Ct\cos 2t + Dt\sin 2t$. Then $y_p'' + 4y_p = 4(A + Bt) - 4C\sin 2t + 4D\cos 2t$, so that $A = 1/4$, $B = 1/4$, $C = -1/4$ and $D = 0$. Then the general solution is

$$y(t) = c_1\cos(2t) + c_2\sin 2t + \tfrac{1}{4}(1 + t - t\cos 2t).$$

Using initial conditions $y(0) = 0$ and $y'(0) = 0$, we get $c_2 = 0$ and $c_1 = -1/4$. Thus,

$$y(t) = \tfrac{1}{4}(1+t)(1-\cos 2t) = \tfrac{1}{2}(1+t)\sin^2 t.$$

Note that $y(\pi/2) = 1/2+\pi/4$, and at earlier positive times $y(t) < (1+t)/2 < 1/2+\pi/4$; so, $t = \pi/2$ seconds is when the spring breaks.

Extra Problem on Webpage:

> When attached to a spring, a 300g mass stretches the spring by 20cm. If the mass is pulled down by another 1/4 metre, and released with a downward velocity of 1 m/s, find (i) the period of the resulting motion; (ii) the maximum height above equilibrium that the mass reaches; and (iii) the first time that the mass passes through the equilibrium position.

The spring constant $k = mg/\ell = .3\times 9.8/.2 = 14.7$ N/m. The position $y(t)$ satisfies $.3y''+14.7y = 0$, or

$$y'' + 49y = 0.$$

The general solution of this equation is

$$y = c_1\cos(7t) + c_2\sin(7t).$$

The initial conditions $y(0) = 1/4$ and $y'(0) = 1$ give $c_1 = 1/4$ and $7c_2 = 1$, so that

$$y = \frac{1}{4}\cos(7t) + \frac{1}{7}\sin(7t).$$

(i) The period of the motion is $2\pi/7$.

(ii) The amplitude is $\sqrt{(1/4)^2 + (1/7)^2} = \sqrt{65}/28 \simeq .288$ metres, and this is the maximum height reached above equilibrium.

(iii) Any time that the mass passes through equilibrium, $\frac{1}{4}\cos(7t) + \frac{1}{7}\sin(7t) = 0$, implying that $\tan(7t) = -7/4$. The first positive where this occurs is $t = (\pi - \arctan(7/4))/7 \simeq .2986$ seconds.

p.197 #1: Substituting $y = \sum_{n=0}^{\infty} a_n t^n$ in the ODE gives

$$0 = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + t\sum_{n=1}^{\infty} n a_n t^{n-1} + \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} + (n+1)a_n]t^n.$$

This gives

$$a_{n+2} = -\frac{a_n}{n+2}.$$

First, suppose $y(0) = a_0 = 1$ and $y'(0) = a_1 = 0$. Then all odd degree terms in the series are zero, and

$$a_2 = -\frac{1}{2},\quad a_4 = \frac{1}{2\times 4},\quad a_6 = -\frac{1}{2\times 4\times 6},$$

and in general $a_{2k} = (-1)^k/(k!2^k)$. Then this series solution is

$$y_1(t) = \sum_{k=0}^{\infty}\frac{(-\frac{1}{2}t^2)^k}{k!} = e^{-\frac{1}{2}t^2}.$$

For a second linearly independent solution, suppose $a_0 = 0$ and $a_1 = 1$. Then all even degree terms in the series are zero, and

$$a_3 = -\frac{1}{3},\quad a_5 = \frac{1}{3\times 5},\quad a_7 = -\frac{1}{3\times 5\times 7},$$

and in general $a_{2k+1} = k!(-2)^k/(2k+1)!$. So this series solution is

$$y_2(t) = \sum_{k=0}^{\infty} \frac{k!(-2)^k}{(2k+1)!} t^{2k+1}.$$

The general solution is $y(t) = c_1 y_1(t) + c_2 y_2(t)$.

#2: Substituting $y = \sum_{n=0}^{\infty} a_n t^n$ in the ODE gives

$$\begin{aligned} 0 &= \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} - t \sum_{n=0}^{\infty} a_n t^n \\ &= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n - \sum_{n=1}^{\infty} a_{n-1} t^n \\ &= a_2 + \sum_{n=1}^{\infty} [(n+2)(n+1) a_{n+2} - a_{n-1}] t^n. \end{aligned}$$

This gives $a_2 = 0$ and

$$a_{n+2} = \frac{a_{n-1}}{(n+1)(n+2)}$$

for all $n \geq 1$.

First, suppose $y(0) = a_0 = 1$ and $y'(0) = a_1 = 0$. Then

$$a_2 = 0, \quad a_3 = \frac{1}{2 \times 3}, \quad a_4 = 0, \quad a_5 = 0, \quad a_6 = \frac{1}{2 \times 3 \times 5 \times 6},$$

and so on. Only the terms whose degree in t is divisible by 3 are nonzero. This solution is

$$y_1 = 1 + \sum_{k=1}^{\infty} \frac{1}{2 \times 3 \times 5 \times 6 \times \ldots \times (3k-1) \times (3k)} t^{3k}.$$

For a second solution, suppose $a_0 = 0$ and $a_1 = 1$. Then only terms whose degree is congruent to 1 modulo 3 are nonzero, and

$$a_4 = \frac{1}{3 \times 4}, \quad a_7 = \frac{1}{3 \times 4 \times 6 \times 7},$$

and so on. This series solution is

$$y_2(t) = \sum_{k=0}^{\infty} \frac{1}{1 \times 3 \times 4 \times 6 \times 7 \times \ldots \times 3k \times (3k+1)} t^{3k+1}.$$

The general solution is $y(t) = c_1 y_1(t) + c_2 y_2(t)$.

#3: Substituting $y = \sum_{n=0}^{\infty} a_n t^n$ in the ODE gives

$$\begin{aligned}
0 &= (2+t^2)\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} - t\sum_{n=1}^{\infty} n a_n t^{n-1} - 3\sum_{n=0}^{\infty} a_n t^n \\
&= 2\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n + \sum_{n=2}^{\infty} n(n-1)a_n t^n - \sum_{n=0}^{\infty}(n+3)a_n t^n \\
&= \sum_{n=0}^{\infty}[2(n+2)(n+1)a_{n+2} + (n^2-2n-3)a_n]t^n.
\end{aligned}$$

This gives

$$a_{n+2} = -\frac{(n-3)}{2(n+2)}a_n.$$

First, suppose $y(0) = a_0 = 1$ and $y'(0) = a_1 = 0$. Then all odd degree terms are zero, and

$$a_2 = -(-3)/4, \quad a_4 = (-3/4)\times(-1/8), \quad a_6 = -(-3/4)\times(-1/8)\times(1/12)$$

and so on. In general,

$$a_{2k} = (-1)^k \frac{(-3)\times(-1)\times 1\times 3\times \ldots \times(2k-5)}{k!2^{2k}},$$

and this solution is

$$y_1(t) = 1 + \sum_{k=1}^{\infty}(-1)^k \frac{(-3)\times(-1)\times 1\times 3\times \ldots \times(2k-5)}{k!2^{2k}} t^{2k}.$$

For a second solution, suppose $a_0 = 0$ and $a_1 = 1$. Then all even degree terms are zero, and

$$a_3 = 1/3, \quad a_5 = 0,$$

and all other odd degree coefficients are zero. Thus, this "series solution" is actually the polynomial

$$y_2(t) = t + \tfrac{1}{3}t^3.$$

The general solution is $y(t) = c_1 y_1(t) + c_2 y_2(t)$.

#7: Substituting $y = \sum_{n=0}^{\infty} a_n t^n$ in the ODE gives

$$\begin{aligned}
0 &= \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} - t^3\sum_{n=0}^{\infty} a_n t^n \\
&= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n - \sum_{n=3}^{\infty} a_{n-3}t^n \\
&= 2a_2 + 6a_3 t + 12a_4 t^2 + \sum_{n=3}^{\infty}[(n+2)(n+1)a_{n+2} - a_{n-3}]t^n.
\end{aligned}$$

This gives $a_2 = a_3 = a_4 = 0$ and

$$a_{n+2} = \frac{a_{n-3}}{(n+1)(n+2)}$$

for all $n \geq 3$. Since $y(0) = a_0 = 0$ and $y'(0) = a_1 = -2$, then

$$a_5 = 0, \quad a_6 = -2/(5\times 6), a_7 = a_8 = a_9 = a_{10} = 0, a_{11} = -2/(5\times 6\times 10\times 11),$$

and so on, with every coefficient zero except for those whose index is congruent to 1 modulo 5. The solution is

$$y = -2\left(t + \frac{t^6}{5 \times 6} + \frac{t^{11}}{5 \times 6 \times 10 \times 11} + \dots\right).$$