Math 323 Solutions

Feb. 22 Assignment

p.149 #12: Let $y = v(t)y_1 = tv(t)$ be a second solution, linearly independent of $y_1 = t$. Then $y' = tv' + v$, $y'' = tv'' + 2v'$. Substituting these into the differential equation gives

$$(1 - t^2)(tv'' + 2v') - 2t^2 v' = 0.$$

Let $w = dv/dt$; then

$$\frac{1}{w}\frac{dw}{dt} = \frac{-2}{t} + \frac{2t}{1 - t^2},$$

giving $\ln w = -2\ln|t| - \ln|1 - t^2| + C$ and

$$w = C' \frac{1}{t^2(1 - t^2)} = C' \left(\frac{1}{t^2} + \frac{1}{2}\frac{1}{1+t} - \frac{1}{2}\frac{1}{1-t} \right),$$

where $C' \neq 0$. Integrating again gives

$$v = \int w\, dt = C' \left(\frac{-1}{t} + \frac{1}{2}\ln\left|\frac{1+t}{1-t}\right| \right) + c_2.$$

Using $y = tv(t)$ gives the general solution

$$y = c_1 \left(-1 + \frac{t}{2}\ln\left|\frac{1+t}{1-t}\right| \right) + c_2 t.$$

#20: Using the result from problem 18, t^r is a solution to $t^2y'' - ty' + y = 0$ if $r(r-1) - r + 1 = 0$, so $(r-1)^2 = 0$ and $r = 1$. So, $y_1(t) = t$ is one solution. Using reduction of order to find a second solution, let $y = tv(t)$, giving

$$t^2(tv'' + 2v') - t^2v' = t^2(tv'' + v') = 0$$

in the differential equation. Letting $w = dv/dt$, we have $dw/w = -t^{-1}dt$ and $w = t^{-1}$ up to multiple. Then $v = \ln t$ and $y = t\ln t$ gives a second solution. Thus, the general solution is

$$y = c_1 t + c_2 t\ln t.$$

p.156 #4: A fundamental solution set for the homogeneous equation $y'' - 3y' + 2y = 0$ is given by $y_1 = e^t$ and $y_2 = e^{2t}$, with $W[y_1, y_2] = e^{3t}$. For $y_p = u_1y_1 + u_2y_2$, then the right-hand side $g(t) = te^{3t} + 1$ gives

$$u_1' = -y_2\frac{g(t)}{W[y_1, y_2]} = -e^{2t}(t + e^{-3t}) = -te^{2t} - e^{-t},$$

$$u_2' = y_1\frac{g(t)}{W[y_1, y_2]} = e^t(t + e^{-3t}) = te^t + e^{-2t}.$$

Integrating gives $u_1 = -\frac{1}{2}te^{2t} + \frac{1}{4}e^{2t} + e^{-t}$, $u_2 = te^t - e^t - \frac{1}{2}e^{-2t}$, and a particular solution

$$y_p = e^t(-\tfrac{1}{2}te^{2t} + \tfrac{1}{4}e^{2t} + e^{-t}) + e^{2t}(te^t - e^t - \tfrac{1}{2}e^{-2t}) = \tfrac{1}{2} + \tfrac{1}{2}te^{3t} + \tfrac{1}{4}e^{3t}.$$

The general solution is then

$$y = \tfrac{1}{2} + \tfrac{1}{2}te^{3t} + \tfrac{1}{4}e^{3t} + c_1e^t + c_2e^{2t}.$$

#6: Solutions of the homogeneous equation $y'' + 4y' + 4y = 0$ are $y_1 = e^{-2t}$ and $y_2 = te^{-2t}$, with $W[y_1, y_2] = y_1 y_2' - y_2 y_1' = e^{-4t}$. If we let $y_p = u_1 y_1 + u_2 y_2$ be a solution of the nonhomogeneous equation, then

$$u_1' = \frac{-y_2 g(t)}{W[y_1, y_2]} = \frac{-te^{-2t}(t^{5/2}e^{-2t})}{e^{-4t}} = -t^{7/2}$$

and

$$u_2' = \frac{y_1 g(t)}{W[y_1, y_2]} = \frac{e^{-2t}(t^{5/2}e^{-2t})}{e^{-4t}} = t^{5/2}.$$

Thus, we can use $u_1 = -\frac{2}{9}t^{9/2}$ and $u_2 = \frac{2}{7}t^{7/2}$. Then the general solution is

$$y(t) = (c_1 - \tfrac{2}{9}t^{9/2})e^{-2t} + (\tfrac{2}{7}t^{7/2} + c_2)te^{-2t} = (c_1 + c_2 t)e^{-2t} + \tfrac{4}{63}t^{9/2}e^{-2t}$$

with

$$y'(t) = (c_2 - 2c_1 - 2tc_2)e^{-2t} + \tfrac{2}{7}t^{7/2}e^{-2t} - \tfrac{8}{63}t^{9/2}e^{-2t}$$

The initial conditions give $c_1 = 0$ and $c_2 - 2c_1 = 0$, so the solution is simply $y = \frac{4}{63}t^{9/2}e^{-2t}$.

p.164 #1: For $y'' + 3y = t^3 - 1$, try a particular solution of the form

$$y_p = At^3 + Bt^2 + Ct + D.$$

Then $y_p' = 3At^2 + 2Bt + C$ and $y_p'' = 6At + 2B$. Substituting into the differential equation gives

$$6At + 2B + 3(At^3 + Bt^2 + Ct + D) = t^3 - 1.$$

This implies $3A = 1$, $3B = 0$, $6A + 3C = 0$ and $2B + 3D = -1$. The solution is $A = 1/3$, $B = 0$, $C = -2/3$ and $D = -1/3$. The particular solution is

$$y_p = \tfrac{1}{3}(t^3 - 2t - 1).$$

#5: For $y'' + 2y' + y = e^{-t}$, the solutions of the homogeneous equation are linear combinations of e^{-t} and te^{-t}. Therefore, we will use the judicious guess

$$y_p = At^2 e^{-t},$$

giving $y_p' = A(2t - t^2)e^{-t}$ and $y_p'' = A(2 - 4t + t^2)e^{-t}$. Substituting into the differential equation gives

$$A(2 - 4t + t^2 + 2(2t - t^2) + t^2)e^{-t} = e^{-t},$$

implying that $A = 1/2$. (Note that the total coefficients of t and t^2 on the left are zero.) The particular solution is

$$y_p = \frac{1}{2}t^2 e^{-t}.$$

#6: For $y'' + 5y' + 4y = t^2 e^{7t}$, the solutions of the homogenous equation are linear combinations of e^{-t} and e^{-4t}. Therefore, we may go ahead and use

$$y_p = (At^2 + Bt + C)e^{7t},$$

giving

$$y_p' = (7At^2 + (7B + 2A)t + 7C + B)e^{7t},$$
$$y_p'' = (49At^2 + (49B + 28A)t + 49C + 14B + 2A)e^{7t}.$$

Substituting these into the differential equation gives

$$(49At^2+(49B+28A)t+49C+14B+2A+5(7At^2+(7B+2A)t+7C+B)+4(At^2+Bt+C))e^{7t} = t^2e^{7t}.$$

This implies the equations $88A = 1$, $88B + 38A = 0$ and $88C + 19B + 2A = 0$. The solution of these equations is $A = 1/88$, $B = -38/(88)^2 = -19/3872$ and $C = 546/(88)^3 = 273/340736$. The particular solution is then

$$y = \left(\frac{1}{88}t^2 - \frac{19}{3872}t + \frac{273}{340736}\right)e^{7t}.$$

#11: For $y''+y'-6y = \sin t+te^{2t}$, the solutions of the homogeneous equation are linear combinations of e^{-3t} and e^{2t}. Therefore, we will use a particular solution of the form

$$y_p = A\cos t + B\sin t + (Ct^2 + Dt)e^{2t}.$$

(If the last term had been $(Ct + D)e^{2t}$, this would overlap with solutions of the homogenous equation.) Then

$$y_p' = -A\sin t + B\cos t + (2Ct^2 + (2D + 2C)t + D)e^{2t},$$
$$y_p'' = -A\cos t - B\sin t + (4Ct^2 + (4D + 8C)t + 4D + 2C)e^{2t}.$$

Substituting these into the differential equation gives

$$(-A + B - 6A)\cos t + (-B - A - 6B)\sin t + (10Ct + 2C + 5D)e^{3t} = \sin t + te^{3t}.$$

This implies $B - 7A = 0$, $-A - 7B = 1$, $10C = 1$ and $2C + 5D = 0$. The solutions are $A = -1/50$, $B = -7/50$, $C = 1/10$ and $D = -1/25$. Then the particular solution is

$$y_p = \frac{-1}{50}(\cos t + 7\sin t) + (\frac{1}{10}t^2 - \frac{1}{25}t)e^{2t}.$$

#13: For $y'' - 3y' + 2y = e^t + e^{2t}$, both the terms on the right-hand side are solutions of the corresponding homogeneous equation. We therefore guess a particular solution of the form

$$y_p = Ate^t + Bte^{2t}.$$

Then $y_p' = A(t + 1)e^t + B(2t + 1)e^{2t}$ and $y_p'' = A(t + 2)e^t + B(4t + 6)e^{2t}$. Substituting these in the differential equation gives

$$(A(t + 2) - 3A(t + 1) + 2At)e^t + (B(4t + 6) - 3B(2t + 1) + 2Bt) = -Ae^t + 3Be^{2t} = e^t + e^{2t}.$$

Thus, $A = -1$ and $B = 1/3$, and a particular solution is

$$y_p = -te^t + \frac{1}{3}te^{2t}.$$

p.172 #1: We compute $k = mg/\ell = 1 \times 9.8 \times 320/49 = 64$, and $\omega = \sqrt{k/m} = 8$. The initial value problem for the position is

$$y'' + 64y = 0, \qquad y(0) = .25, \qquad y'(0) = 0,$$

with solution $y = .25cos(8t)$. The amplitude is .25, the period is $2\pi/8 = \pi/4$ and the frequency $\omega = 8$.