

Math 323 Solutions

FEB. 8 ASSIGNMENT

p.136 #2: Let $L[y] = y'' - 6y' + 5y$. Then

(b) $L[e^{2t}] = 4e^{2t} - 12e^{2t} + 5e^{2t} = -3e^{2t}$

(d) $L[e^{rt}] = r^2e^{rt} - 6re^{rt} + 5e^{rt} = (r^2 - 6r + 5)e^{rt}$

(f) $L[t^2] = 2 - 12t + 5t^2$

#5: (a) Let $L[y] = 2t^2y'' + 3ty' - y$. Then

$$L[\sqrt{t}] = 2t^2(1/2)(-1/2)t^{-3/2} + 3t(1/2)t^{-1/2} - t^{1/2} = (-1/2)t^{1/2} + (3/2)t^{1/2} - t^{1/2} = 0,$$

and $L[1/t] = 2t^2(-1)(-2)t^{-3} + 3t(-1)t^{-2} - t^{-1} = (4 - 3 - 1)t^{-1} = 0$.

(b) $W[y_1, y_2] = W[t^{1/2}, t^{-1}] = t^{1/2}(-1)t^{-2} - t^{-1}(1/2)t^{-1/2} = -3/2t^{-3/2}$. As t approaches zero, this Wronskian approaches $-\infty$.

(c) Because $W[y_1, y_2]$ is nonzero on the interval $(0, \infty)$, then $\{y_1, y_2\}$ is a fundamental solution set on this interval. (This follows from the Corollary to Theorem 4, on p.136.)

(d) The general solution on $(0, \infty)$ is

$$y = c_1\sqrt{t} + c_2(1/t).$$

The initial values $y(1) = 2$, $y'(1) = 1$ imply $c_1 + c_2 = 2$, $(1/2)c_1 - c_2 = 1$. These have solution $c_1 = 2$, $c_2 = 0$, and so $y(t) = 2\sqrt{t}$.

#8: The value of $W[y_1, y_2]$ at $t = 0$ is $3 \times (1/3) - 1 \times 1 = 0$. Using Theorem 4, we conclude that y_1 and y_2 are linearly dependent on the whole interval $(-\infty, \infty)$.

#13: Putting Bessel's equation in standard form gives

$$y'' + \frac{1}{t}y' + \frac{t^2 - n^2}{t^2}y = 0,$$

so that $p(t) = 1/t$. The differential equation satisfied by the Wronskian of two solutions is thus

$$W' + p(t)W = W' + \frac{1}{t}W = 0.$$

Then $\int \frac{dW}{W} = -\int \frac{dt}{t}$ and $W(t) = Ct^{-1}$ for some constant t . Using the values given, we know that $W(1) = y_1(1)y_2'(1) - y_2(1)y_1'(1) = 1$, so that $W(t) = t^{-1}$.

p.140 #2: The characteristic equation is $6r^2 - 7r + 1 = 0$, which has roots $r = 1$ and $r = 1/6$. Thus, the general solution is

$$y = c_1e^t + c_2e^{t/6}.$$

#5: The characteristic equation is $r^2 - 3r - 4 = 0$ which has roots $r = 4$ and $r = -1$. Thus, the general solution is $y = c_1e^{4t} + c_2e^{-t}$ with $y' = 4c_1e^{4t} - c_2e^{-t}$. The initial values imply that $c_1 + c_2 = 1$ and $4c_1 - c_2 = 0$. These have solution $c_1 = 1/5$, $c_2 = 4/5$, so

$$y(t) = (e^{4t} + 4e^{-t})/5$$

solves the initial value problem.

#6: The characteristic equation is $2r^2 + r - 10 = 0$, which has roots $r = 4$ and $r = -5$. Thus, the general solution is $y = c_1e^{4t} + c_2e^{-5t}$ with $y' = 4c_1e^{4t} - 5c_2e^{-5t}$. The initial values imply that $c_1e^4 + c_2e^{-5} = 5$ and $4c_1e^4 - 5c_2e^{-5} = 2$. Thus, $c_1e^4 = 3$ and $c_2e^{-5} = 2$, and

$$y = 3e^{4(t-1)} + 2e^{-5(t-1)}.$$

#10: Substituting $y = t^r$ into the differential equation $t^2y'' + \alpha ty' + \beta y = 0$ gives

$$(r(r-1) + \alpha r + \beta)t^r = 0.$$

Thus, $y = t^r$ is a solution if and only if $r^2 + (\alpha - 1)r + \beta = 0$.

#12: Comparing with Exercise 10, the differential equation

$$t^2y'' - ty' - 2y = 0$$

has $\alpha = -1$ and $\beta = -2$. Thus, $y = t^r$ is a solution if and only if $r^2 - 2r - 2 = 0$. The roots of this equation are $r = -1 \pm \sqrt{3}$. The general solution is

$$y = c_1t^{1+\sqrt{3}} + c_2t^{1-\sqrt{3}},$$

with

$$y' = (1 + \sqrt{3})c_1t^{\sqrt{3}} + (1 - \sqrt{3})c_2t^{-\sqrt{3}}.$$

The initial value $y(1) = 0$ implies that $c_1 + c_2 = 0$, so $c_2 = -c_1$. Then $1 = y'(1) = 2\sqrt{3}c_1$. Thus,

$$y = \frac{t^{1+\sqrt{3}} - t^{1-\sqrt{3}}}{2\sqrt{3}}.$$

p.144 #3: For $y'' + 2y' + 3y = 0$, the characteristic equation is $r^2 + 2r + 3 = 0$, which has roots $r = -1 \pm \sqrt{2}$. The general solution is thus

$$y = c_1e^{-t} \cos(\sqrt{2}t) + c_2e^{-t} \sin(\sqrt{2}t).$$

#6: The characteristic equation is $r^2 + 2r + 5 = 0$, which has roots $r = -1 \pm 2i$. The general solution is

$$y = e^{-t}(c_1 \cos(2t) + c_2 \sin(2t))$$

with

$$y' = e^{-t}((2c_2 - c_1) \cos(2t) + (-2c_1 - c_2) \sin(2t)).$$

The initial values imply that $c_1 = 0$ and $2c_2 - c_1 = 2$, so $c_2 = 1$ and the solution is

$$y = e^{-t} \sin(2t).$$

#10:

$$\begin{aligned} W[e^{\alpha t} \cos \beta t, e^{\alpha t} \sin \beta t] &= \begin{vmatrix} e^{\alpha t} \cos \beta t & e^{\alpha t} \sin \beta t \\ e^{\alpha t}(\alpha \cos \beta t - \beta \sin \beta t) & e^{\alpha t}(\alpha \sin \beta t + \beta \cos \beta t) \end{vmatrix} \\ &= e^{2\alpha t} \cos \beta t(\alpha \sin \beta t + \beta \cos \beta t) - e^{2\alpha t} \sin \beta t(\alpha \cos \beta t - \beta \sin \beta t) \\ &= \beta e^{2\alpha t}(\cos^2 \beta t + \sin^2 \beta t) = \beta e^{2\alpha t} \end{aligned}$$

p.149 #2: The characteristic equation is $4r^2 - 12r + 9 = 0$, which has repeated root $r = 3/2$. Thus, the general solution of the differential equation is

$$y = e^{3t/2}(c_1 + c_2t).$$

#6: The characteristic equation is $r^2 + 2r + 1 = 0$, which has repeated root $r = -1$. The general solution is

$$y = e^{-t}(c_1 + c_2t)$$

with

$$y' = e^{-t}(c_2 - c_1 - c_2t).$$

The initial values $y(2) = 1$, $y'(2) = -1$ imply that

$$c_1 + 2c_2 = e^2, \quad -c_1 - c_2 = -e^2.$$

This means that $c_2 = 0$ and $c_1 = e^2$. Thus, $y = e^{2-t}$.