Math 323 Solutions

Feb. 1 Assignment

p.80 #1: $y_0(t) = 0$, and

$$y_1(t) = \int_0^t 2s\,ds = t^2,$$
$$y_2(t) = \int_0^t 2s(1+s^2)\,ds = t^2 + \tfrac{1}{2}t^4,$$
$$y_3(t) = \int_0^t 2s(1+s^2+\tfrac{1}{2}s^4)\,ds = t^2 + \tfrac{1}{2}t^4 + \tfrac{1}{6}t^6.$$

In fact, for any k, $y_k(t) - y_{k-1}(t) = \dfrac{1}{k!}t^{2k}$. To prove this by induction, suppose it holds for $k = n$, and note that

$$y_{n+1}(t) - y_n(t) = \int_0^t f(s, y_n(s)) - f(s, y_{n-1}(s))\,ds = \int_0^t 2s(y_n(s) - y_{n-1}(s))\,ds.$$

Substituting the induction hypothesis in the right-hand side gives

$$y_{n+1}(t) - y_n(t) = \int_0^t \frac{2s}{n!}\cdot s^{2n}\,ds = \frac{2}{n!}\int_0^t s^{2n+1}\,ds = \frac{2}{(2n+2)n!}t^{2n+2} = \frac{1}{(n+1)!}t^{2(n+1)}.$$

Then

$$y(t) = \sum_{k=1}^{\infty} \frac{1}{k!}t^{2k} = e^{t^2} - 1.$$

#2: $y_0(t) = 1$, and

$$y_1(t) = 1 + \int_0^t s^2 + y_0(s)^2\,ds = 1 + \int_0^t s^2 + 1\,ds = 1 + t + \tfrac{1}{3}t^3,$$

$$y_2(t) = 1 + \int_0^t s^2 + y_1(s)^2\,ds = 1 + \int_0^t s^2 + (1 + s + \tfrac{1}{3}s^3)^2\,ds$$
$$= 1 + \int_0^t \tfrac{1}{9}s^6 + \tfrac{2}{3}s^4 + \tfrac{2}{3}s^3 + 2s^2 + 2s + 1\,ds = \tfrac{1}{63}t^7 + \tfrac{2}{15}t^5 + \tfrac{1}{6}t^4 + \tfrac{2}{3}t^3 + t^2 + t + 1$$

#4: Let R be the rectangle $0 \le t \le 1$, $|y| \le b$. Then

$$\max_R |f(t,y)| = \max_R |y^2 + \cos(t^2)| \le 1 + b^2 = M.$$

The width of the interval of existence guaranteed by the Picard theorem (Theorem 2 on p.76) is

$$\alpha = \min(1, \frac{b}{M}) = \min(1, \frac{b}{1+b^2}).$$

We want to choose b so that $b/(1+b^2)$ is no smaller than $1/2$. In fact, the maximum value of $b/(1+b^2)$ occurs at $b = 1$, where it equals $1/2$. So, using $b = 1$ will give us $\alpha = 1/2$, and then Theorem 2 gives existence of a solution on the interval $0 \le t \le 1/2$.

#14: Let R be the rectangle $0 \le t \le a$, $-b \le y \le b$ for some positive a, b. Then

$$\max_R |f(t,y)| = \max_R |e^{-t}\ln(1+y^2)| \le 1 + \ln(1+b^2) = M,$$

and the width of the interval of existence is

$$\alpha = \min(a, \frac{b}{1+\ln(1+b^2)}).$$

Note that

$$\lim_{b\to\infty} \frac{b}{1+\ln(1+b^2)} = \infty.$$

This means that, for any number a, we can find a large enough b so that $\frac{b}{1+\ln(1+b^2)} > a$. Therefore, we can always pick b so that $\alpha = a$, for any a we want. So, the solution will exist on $0 \le t < \infty$.

#16: (a) $\max_R |t^2 + y^2| = a^2 + b^2$, so the interval of existence in Picard's theorem has width

$$\alpha = \min(a, \frac{b}{a^2+b^2}).$$

(b) For fixed a, $\frac{b}{a^2+b^2}$ has limit zero as b tends to zero or to ∞. Its maximum will therefore occur at a critical point, obtained by

$$0 = \frac{\partial}{\partial b}\frac{b}{a^2+b^2} = \frac{a^2-b^2}{(a^2+b^2)^2}.$$

The critical point is at $b = a$, where $b/(a^2+b^2) = 1/(2a)$ (note parentheses!).
(c) Thus, the largest α we can get for a fixed a is $\alpha = \min(a, 1/(2a))$. **Note that Braun has a misprint in this formula for α!** We want to maximize this α. The graphs of $f(a) = a$ and $g(a) = 1/(2a)$ cross where $2a^2 = 1$, or $a = 1/\sqrt{2}$. To the left of this point, $f(a) = a$ is lower, and to the right of this point $g(a) = 1/(2a)$ is lower. Therefore, $a = 1/\sqrt{2}$ gives the largest α.
(d) Using $a = 1/\sqrt{2}$ and $b = a$ gives $\alpha = 1/\sqrt{2}$, and we conclude that the solution exists for $0 \le t \le 1/\sqrt{2}$.

#18: (Take $a = 2/3$ in this problem.) The *trivial* solution to $y' = ty^{2/3}$, $y(0) = 0$ is the identically zero function $y(t) \equiv 0$. But this is a separable equation, and

$$\int \frac{1}{y^{2/3}} dy = \int t\, dt$$

yields $3y^{1/3} = \frac{12}{t}^2$ and $y = (1/216)t^6$, which is a non-trivial solution to the same IVP.
The reason that this doesn't contradict Theorem 2' on page 77 is that the theorem cannot be applied. The theorem requires that we find a rectangle around the origin where $f(t, y) = ty^{2/3}$ is continuous and $\partial f/\partial y$ is continuous. However $\partial f/\partial y = \frac{2}{3}ty^{-1/3}$ fails to exist at points on the t axis near the origin, so will not be continuous on any such rectangle.

p.105 #2: For this differential equation, $f(t, y) = t - y^4$. Formula (6) on p.102 gives the estimate

$$E_k \le \frac{D}{2L}(e^{\alpha L} - 1)h,$$

for the size of the error at the kth step. We will obtain L and D as the maximum size of $|\partial f/\partial y|$ and $|\partial f/\partial t + f\partial f/\partial y|$ respectively on the rectangle R defined by $0 \le t \le 1$, $-1 \le y \le 1$. (On this rectangle, $M = \max|f| = 1$, so $\alpha = \min(1, 1/M) = 1$.) We have

$$\frac{\partial f}{\partial y} = -4y^3, \qquad \frac{\partial f}{\partial t} + f\frac{\partial f}{\partial y} = 1 - (t - y^3)(-4y^3),$$

so we'll use $L = 4$ and $D = 1 + 4 = 5$, giving

$$\frac{D}{2L}(e^{\alpha L} - 1) \approx 33.5.$$

Thus, $E_k \leq 33.5h$.

#4: For this differential equation, $f(t, y) = e^t - y^2$. Formula (6) on p.102 gives the estimate

$$E_k \leq \frac{D}{2L}(e^{\alpha L} - 1)h,$$

for the size of the error at the kth step. We will obtain L and D on the same rectangle R as before, $0 \leq t \leq 1$, $-1 \leq y \leq 1$. (On this rectangle, $M = \max |f| = e$, so $\alpha = \min(1, 1/M) = 1/e$.) We have

$$\frac{\partial f}{\partial y} = -2y, \qquad \frac{\partial f}{\partial t} + f\frac{\partial f}{\partial y} = e^t + (e^t - y^2)(-2y),$$

so we'll use $L = 2$ and $D = 3e$. Then we get

$$\frac{D}{2L}(e^{\alpha L} - 1) \approx 2.216.$$

Thus, $E_k \leq 2.216h$. So, if we want $E_k \leq .0001$, then it suffices to make $h \leq .00004512$.