Math 323 Solutions

Jan. 25 Assignment

p.37#6: When the population is small, solutions of the logistic model behave like those of the Malthusian model. Hence, a doubling time T gives $a = \ln(2)/T$; in this case, we get $a = .0173$ when time is measured in minutes. The limiting population is $a/b = 5 \times 10^8$, so $b = 3.464 \times 10^{-11}$. The general solution for the logistic model, with $t_0 = 0$, is

$$p(t) = \frac{ap_0}{bp_0 + (a - bp_0)e^{-at}}.$$

(a) With $p_0 = 10^8$, we get $p(120) = 3.33 \times 10^8$.
(b) With $p_0 = 10^9$, we get $p(120) = 5.33 \times 10^8$.
Note that if time is in hours instead, then $a = \ln(2)/(2/3) = 1.039$ and $b = 2.079 \times 10^{-9}$, and you get the same answers for the populations after 2 hours.

#8: With emigration and murder taken into account, we get

$$\frac{dp}{dt} = \frac{1}{25}p - \frac{1}{25 \times 10^6}p^2 - 10^4 = -\frac{1}{25 \times 10^6}(p - 5 \times 10^5)^2.$$

Note that the right-hand side is a perfect square. Separating gives

$$\frac{1}{(p - 5 \times 10^5)^2}dp = -\frac{1}{25 \times 10^6}dt.$$

Integrating and multiplying through by -1 gives

$$\frac{1}{p - 5 \times 10^5} = \frac{1}{25 \times 10^6}t + C.$$

Taking $t = 0$ to be 1970, we have $p(0) = 8 \times 10^6$, so $C = 1/(7.5 \times 10^6)$. Solving for $p(t)$ gives

$$p(t) = 5 \times 10^5 + \frac{75 \times 10^6}{3t + 10}.$$

This shows that, as $t \to \infty$, the population will approach 5×10^5.

Fish Problem The population of fish in a lake is 1000 in 1980 and 3000 in 1990. Predict the population in 2010 using (a) the Malthusian model, and (b) the logistic model, with the extra information that the population is 4500 in the year 2000.

Measure time in years after 1980. For part (a), we set $p(t) = p_0 e^{at}$ with $p_0 = 1000$, and we find a by solving

$$3000 = p(10) = 1000e^{10a},$$

giving $a = \ln(3)/10 = .10986$. Then the population in 2010 is predicted by $p(30) = 1000e^{30a} = 27000$.

For part (b), we have $p_0 = 1000$, $p_1 = 3000$ and $p_2 = 4500$ for times even separated by $\delta t = 10$ years. Using the formulas given in class, we get

$$a = \frac{1}{\Delta t}\ln\left(\frac{p_2(p_1 - p_0)}{p_0(p_2 - p_1)}\right) = \ln(6)/10$$

and

$$bp_0 = a\left(\frac{p_0/p_1 - e^{-a\Delta t}}{1 - e^{-a\Delta t}}\right) = a/5.$$

Then the population in 2010 is predicted by

$$p(30) = \frac{ap_0}{bp_0 + (a - bp_0)e^{-30a}} = \frac{1000}{\frac{1}{5} + \frac{4}{5}e^{-3\ln 6}} = \frac{5000}{1 + (4/216)} \simeq 4909.$$

p.66 #3: Taking $M = 2t\sin y + y^3 e^t$ and $N = t^2\cos y + 3y^2 e^t$, we see that $M_y - N_t = 0$, so the equation is exact. Using $\phi = \int M\,dt$ gives

$$\phi(t, y) = t^2 \sin y + y^3 e^t + h(y).$$

Substituting this into the equation $\phi_y = N$ gives $h'(y) = 0$, so we can pick $h(y) = 0$. Then $\phi = t^2 \sin y + y^3 e^t$, and the general solution of the equation (in implicit form) is

$$t^2 \sin y + y^3 e^t = C.$$

#4: Taking $M = 1+(1+ty)e^{ty}$ and $N = 1+t^2 e^{ty}$, we compute $M_y - N_t = (2t+t^2 y)e^{ty} - (2t+t^2 y)e^{ty} = 0$, so the equation is exact. Then

$$\phi = \int M\,dt = t + te^{ty} + h(y).$$

Substituting into $\phi_y = N$ gives $h'(y) = 1$, so we may set $h(y) = y$. Then the general solution (in implicit form) is

$$t(1 + e^{ty}) + y = C.$$

#6: Taking $M = \frac{1}{2}y^2 - 2ye^t$ and $N = (y - e^t)$, we compute $M_y - N_t = (y - 2e^t) - e^t = N$. This equation is not exact; but, because $(M_y - N_t)/N$ doesn't depend on y, we can make it exact by multiplying it by an integrating factor μ which depends only on t, given by

$$\mu(t) = \exp\left(\int (M_y - N_t)/N\,dt\right) = e^t.$$

After multiplying by $\mu(t)$, the new coefficients are $M = \dfrac{1}{2}y^2 e^t - 2ye^{2t}$ and $N = ye^t - e^{2t}$. The potential is

$$\phi = \int M\,dt = \frac{1}{2}y^2 e^t - ye^{2t} + h(y).$$

Substituting into $\phi_y = N$ gives $h(y) = 0$, so an implicit general solution is

$$\frac{1}{2}y^2 e^t - ye^{2t} = C.$$

#8: Taking $M = 2t\cos y + 3t^2 y$ and $N = t^3 - t^2 \sin y - y$, we compute $M_y - N_t = (-2t\sin y + 3t^2) - (3t^2 - 2t\sin y) = 0$, so this equation is exact. Then

$$\phi = \int M\,dt = t^2 \cos y + t^3 y + h(y).$$

Substituting into $\phi_y = N$ gives $h'(y) = -y$, so we may set $h(y) = -\frac{1}{2}y^2$. An implicit general solution is

$$t^2 \cos y + t^3 y - \tfrac{1}{2}y^2 = C.$$

Using the initial value $y = 2$ when $t = 0$, we get $C = -2$. So, the implicit solution to the initial value problem is

$$t^2 \cos y + t^3 y - \tfrac{1}{2}y^2 = -2.$$