Math 323 Solutions

April 18 Assignment

p.366 #2: First we obtain solutions of the homogeneous system $\mathbf{x}'(t) = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}\mathbf{x}$. The only eigenvalue is $\lambda = 1$, with eigenvector $\mathbf{v}^1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. This gives the solution $\mathbf{x}(t) = e^t\mathbf{v}^1$ for the homogeneous system.

A generalized eigenvector is obtained by solving $(A - I)\mathbf{v}^2 = \mathbf{v}^1$, which has solution $\mathbf{v}^2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. A second linearly independent solution of the homogeneous system is

$$\mathbf{x}(t) = e^t(\mathbf{v}^2 + t(A - I)\mathbf{v}^2) = e^t \begin{bmatrix} 1 + 2t \\ t \end{bmatrix}.$$

We use this as the first column of a fundamental solution matrix

$$X(t) = e^t \begin{pmatrix} 1 + 2t & 2 \\ t & 1 \end{pmatrix}.$$

Since $\det X(t) = e^{2t}$, shortcuts for 2-by-2 matrices give

$$X^{-1}(t) = e^{-t} \begin{pmatrix} 1 & -2 \\ -t & 1 + 2t \end{pmatrix}.$$

Using variation of parameters, suppose the solution of the given initial value problem is of the form $\mathbf{x}(t) = X(t)\mathbf{u}(t)$. Then a formula developed in class gives

$$\mathbf{u}(t) = X^{-1}(0)\mathbf{x}(0) + \int_0^t X^{-1}(s)\mathbf{f}(s)\,ds,$$

where $\mathbf{f}(t)$ is the non-homogeneous term in the equation; in this case, $\mathbf{f}(t) = \begin{bmatrix} e^t \\ e^t \end{bmatrix}$. We calculate

$$\int_0^t e^{-s} \begin{pmatrix} 1 & -2 \\ -s & 1 + 2s \end{pmatrix} \begin{bmatrix} e^s \\ e^s \end{bmatrix} ds = \int_0^t \begin{bmatrix} -1 \\ 1 + s \end{bmatrix} ds = \begin{bmatrix} -t \\ t + \frac{1}{2}t^2 \end{bmatrix}.$$

Using $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ we get

$$\mathbf{u}(t) = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -t \\ t + \frac{1}{2}t^2 \end{bmatrix} = \begin{bmatrix} -1 - t \\ 1 + t + \frac{1}{2}t^2 \end{bmatrix}.$$

Multiplying by $X(t)$ gives

$$\mathbf{x}(t) = e^t \begin{bmatrix} 1 - t - t^2 \\ 1 - \frac{1}{2}t^2 \end{bmatrix}.$$

#6: The characteristic polynomial of the matrix

$$A = \begin{pmatrix} -1 & -1 & -2 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{pmatrix}$$

is $-(\lambda-1)^3$. The eigenvalue $\lambda = 1$ has eigenvector

$$\mathbf{v}^1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \Longrightarrow \quad \mathbf{x}(t) = e^t \mathbf{v}^1.$$

Solving for a generalized eigenvector $\mathbf{v}^2$ such that $(A-I)\mathbf{v}^2 = \mathbf{v}^1$ gives

$$\mathbf{v}^2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad \mathbf{x}(t) = e^t(\mathbf{v}^2 + t\mathbf{v}^1) = e^t \begin{bmatrix} t \\ -1 \\ -t \end{bmatrix}.$$

Solving for a generalized eigenvector $\mathbf{v}^3$ such that $(A-I)\mathbf{v}^3 = \mathbf{v}^2$ gives

$$\mathbf{v}^3 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \quad \Longrightarrow \quad \mathbf{x}(t) = e^t(\mathbf{v}^3 + t\mathbf{v}^2 + \tfrac{1}{2}t^2\mathbf{v}^1) = e^t \begin{bmatrix} \frac{1}{2}t^2 \\ 2-t \\ -1-\frac{1}{2}t^2 \end{bmatrix}.$$

Using these solutions for the homogeneous system gives a fundamental matrix

$$X(t) = e^t \begin{pmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & -1 & 2-t \\ -1 & -t & -1-\frac{1}{2}t^2 \end{pmatrix}, \text{ with } X^{-1}(t) = e^{-t} \begin{pmatrix} -\frac{1}{2}t^2+2t+1 & t & -\frac{1}{2}t^2+2t \\ t-2 & -1 & t-2 \\ -1 & 0 & -1 \end{pmatrix}.$$

(The inverse can be obtained by row reduction or Cramer's rule.) Using the variation of parameters formula gives

$$\mathbf{u}(t) = \int_0^t X^{-1}(s) \begin{bmatrix} e^s \\ 0 \\ 0 \end{bmatrix} ds = \int_0^t \begin{bmatrix} -\frac{1}{2}s^2+2s+1 \\ s-2 \\ -1 \end{bmatrix} ds = \begin{bmatrix} -\frac{1}{6}t^3+t^2+t \\ \frac{1}{2}t^2-2t \\ -t \end{bmatrix}.$$

Multiplying by $X(t)$ gives

$$\mathbf{x}(t) = e^t \begin{bmatrix} -\frac{1}{6}t^3 - t^2 + t \\ \frac{1}{2}t^2 \\ \frac{1}{6}t^3 + t^2 \end{bmatrix}.$$

#8: Let $x_1 = y$, $x_2 = y'$ and $x_3 = y''$. Then the single ODE $y''' + y' = \sec t \tan t$ translates to the system

$$\mathbf{x}'(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} + \begin{bmatrix} 0 \\ 0 \\ \sec t \tan t \end{bmatrix}.$$

Let A be the 3-by-3 matrix in this system. It has eigenvalues $\lambda = 0$ and $\pm\mathrm{i}$. The $\lambda = 0$ eigenvector gives the constant solution $\mathbf{x}(t) = {}^t[1,0,0]$ for the homogenous system. An eigenvector for $\lambda = \mathrm{i}$ is ${}^t[1,\mathrm{i},-1]$, and this gives the complex solution

$$\mathbf{x}(t) = e^{\mathrm{i}t} \begin{bmatrix} 1 \\ \mathrm{i} \\ -1 \end{bmatrix} = \begin{bmatrix} \cos t \\ -\sin t \\ -\cos t \end{bmatrix} + \mathrm{i} \begin{bmatrix} \sin t \\ \cos t \\ -\sin t \end{bmatrix}.$$

A fundamental solution matrix for the homogeneous system, and its inverse, are given by

$$X(t) = \begin{pmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{pmatrix} \qquad X^{-1}(t) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -\sin t & -\cos t \\ 0 & \cos t & -\sin t \end{pmatrix}.$$

Because the initial condition for our desired solution is the zero vector, the variation of parameters formula gives

$$\mathbf{x}(t) = X(t)\int_0^t X^{-1}(s)\begin{bmatrix}0\\0\\\sec s\tan s\end{bmatrix}ds = X(t)\int_0^t \begin{bmatrix}\sec s\tan s\\-\tan s\\-\tan^2 s\end{bmatrix}ds = X(t)\begin{bmatrix}-1+\sec t\\\ln|\cos(t)|\\t-\tan t\end{bmatrix}.$$

The top entry in this product is

$$y(t) = t\sin t - 1 + (1+\ln|\cos t|)\cos t.$$

p.377 #1: To find equilibria, set

$$\begin{aligned}0 &= x' = x(1-x-2y)\\0 &= y' = y(2-2y-3x).\end{aligned}$$

From the first equation, either

$$x = 0 \Longrightarrow y(2-2y) = 0 \text{ (from 2nd equation) } \Longrightarrow y = 0 \text{ or } y = 1,$$

giving the equilibrium points $(0,0)$, $(0,1)$, OR

$$x = 1-2y \Longrightarrow y(4y-1) = 0 \Longrightarrow y = 0 \text{ or } y = 1/4,$$

giving two more equilibrium points $(1,0)$ and $(1/2, 1/4)$.

#4: To find equilibria, set

$$\begin{aligned}0 &= x' = -x(1+y^2),\\0 &= y' = -y(1+x^2).\end{aligned}$$

The only real solution is $(0,0)$.

#8 To find equilibria, set

$$\begin{aligned}0 &= x - y^2,\\0 &= x^2 - y,\\0 &= e^z - x.\end{aligned}$$

From the first two equations, $x = y^2 = (x^2)^2 = x^4$. Because $x = e^z \neq 0$, then $1 = x^3$, and $x = 1$, giving $y = x^2 = 1$ and $z = \ln(x) = 0$. Thus, $(1,1,0)$ is the only equilibrium point.

#11(b): Suppose $\psi(t)$ is any solution with the initial condition $\psi(0) \neq {}^t[1,0,0]$, and let

$$\mathbf{z}(t) = \psi(t) - e^{-t}\begin{bmatrix}1\\0\\0\end{bmatrix}.$$

Then $\mathbf{z}(t)$ is a solution (with nonzero initial conditions) for the homogeneous system $\mathbf{z}'(t) = A\mathbf{z}$, where

$$A = \begin{pmatrix}1 & 1 & 1\\2 & 1 & -1\\-3 & 2 & 4\end{pmatrix}.$$

The characteristic polynomial of A is $-(\lambda-2)^3$, so that a basis of solutions for the homogeneous system consists of vector-valued functions of the form

$$\mathbf{z}(t) = e^{2t}(\mathbf{v} + t(A-2I)\mathbf{v} + \frac{1}{2}t^2(A-2I)^2\mathbf{v} + \ldots),$$

where $\mathbf{v}$ is either an eigenvector or generalized eigenvector for $\lambda = 2$. Because of the exponential factor in front, all such solutions become unbounded as $t \to \infty$.

p.393 #3: We find equilibria by setting $x^2 + y^2 - 1 = 0$ and $2xy = 0$. From the second equation, either $x = 0$ or $y = 0$. If $x = 0$, then $y = \pm 1$ from the first equation, and similarly if $y = 0$ then $x = \pm 1$. Thus, we have 4 equilibrium points $(0, 1)$, $(0, -1)$, $(1, 0)$, $(-1, 0)$.
Let $\mathbf{f}(x, y)$ be the vector-valued function whose components are the right-hand sides of the system. Then the Jacobian matrix of $\mathbf{f}$ is

$$\nabla \mathbf{f} = \begin{pmatrix} 2x & 2y \\ 2y & 2x \end{pmatrix}.$$

At each equilibrium, we evaluate this matrix, giving a constant matrix A whose eigenvalues tell us about the stability of the solution at that point.
For $(0, 1)$, $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, with eigenvalues $1, -1$; thus, this point is **unstable**.
For $(0, -1)$, $A = \left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right)$, with eigenvalues $1, -1$; thus, this point is **unstable** too.
For $(1, 0)$, $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, with eigenvalues $1, 1$; thus, this point is **unstable**.
For $(-1, 0)$, $A = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, with eigenvalues $-1, -1$; thus, this point is **asymptotically stable**.

#4: We find equilibria by simultaneously solving

$$0 = 2x(3 - 3x - y),$$
$$0 = 2y(2 - 2y - x).$$

From the first equation, either $x = 0$ or $y = 3(1 - x)$. In the first case, substituting $x = 0$ in the second equation gives $0 = 4y(1 - y)$, so $y = 0$ or $y = 1$, and we have equilibrium points $(0, 0)$ and $(0, 1)$. In the second case, substituting $y = 3(1-x)$ in the second equation gives $0 = 6(1-x)(-4+5x)$, so that $x = 1$ or $x = 4/5$. Setting $y = 3(1 - x)$ in each of those cases gives the equilibrium points $(1, 0)$ and $(4/5, 3/5)$.
The Jacobian of the right hand sides is

$$\nabla \mathbf{f} = \begin{pmatrix} 6 - 12x - 2y & -2x \\ -2y & 4 - 8y - 2x \end{pmatrix}.$$

For $(0, 0)$, $A = \left(\begin{smallmatrix} 6 & 0 \\ 0 & 4 \end{smallmatrix}\right)$, with eigenvalues $6, 4$, so this point is **unstable**.
For $(0, 1)$, $A = \left(\begin{smallmatrix} 4 & 0 \\ -2 & -4 \end{smallmatrix}\right)$, with eigenvalues $4, -4$, so this point is **unstable**.
For $(1, 0)$, $A = \left(\begin{smallmatrix} -6 & -2 \\ 0 & 2 \end{smallmatrix}\right)$, with eigenvalues $-6, 2$, so this point is **unstable**.
For $(4/5, 3/5)$, $A = \frac{2}{5}\left(\begin{smallmatrix} -12 & -4 \\ -3 & -6 \end{smallmatrix}\right)$, with eigenvalues $2/5(-9 \pm \sqrt{21})$, both of which are negative; so, this point is **asymptotically stable**.

#14: Since the right-hand sides are polynomials in the coordinates, with no constant term, the origin is an equilibrium point. Write the system as

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} z^2 \\ -x^2 \\ y^2 \end{bmatrix},$$

and let A be the matrix on the right-hand side. The characteristic polynomial of A is $(\lambda - 2)(\lambda^2 - \lambda + 1)$. Because $\lambda = 2$ is an eigenvalue of A, the origin is **unstable**.