

Math 323 Solutions

JAN. 18 ASSIGNMENT

p.10 #3: Multiplying by the integrating factor $\mu(t) = \exp(\int 2t/(1+t^2) dt) = 1+t^2$ gives

$$\frac{d}{dt}(y(1+t^2)) = 1.$$

Integrating gives $y(1+t^2) = t + C$, and

$$y = \frac{t+C}{1+t^2}.$$

#4: Multiplying by the integrating factor $\mu(t) = e^t$ gives

$$\frac{d}{dt}(ye^t) = te^{2t}.$$

Integrating (using integration by parts on the right-hand side) gives

$$ye^t = \frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} + C,$$

and solving for y , we get

$$y = \frac{1}{2}te^t - \frac{1}{4}e^t + Ce^{-t},$$

where C is an arbitrary constant.

#14: Multiplying by the integrating factor $\mu(t) = \exp(\int -2t dt) = \exp(-t^2)$ gives

$$\frac{d}{dt}(ye^{-t^2}) = e^{-t^2}.$$

Unfortunately, an antiderivative for the right-hand side cannot be expressed in terms of elementary functions. We will integrate each side from 0 to t , using the initial value $y(0) = 1$, giving

$$ye^{-t^2} - 1 = \int_0^t e^{-s^2} ds$$

and so

$$y = e^{t^2} \left(1 + \int_0^t e^{-s^2} ds \right).$$

#16: Before calculating the integrating factor, it's necessary to put the equation in standard form, by dividing by $1+t^2$:

$$\frac{dy}{dt} + \frac{4t}{1+t^2}y = \frac{t}{1+t^2}.$$

The integrating factor is $\mu(t) = \exp(\int 4t/(1+t^2) dt) = \exp(2\ln(1+t^2)) = (1+t^2)^2$. Multiplying through by it yields

$$\frac{d}{dt}(y(1+t^2)^2) = t(1+t^2).$$

Integrating gives

$$y(1+t^2)^2 = \frac{1}{4}(1+t^2)^2 + C.$$

When $t = 0$, $y = 1/4$, so that $C = 0$. Then $y(t) = 1/4$, a constant solution.

Extra Initial Value Problem: $y' + (\sin t)y = 0$, $y(0) = 1$.

This is a separable equation, which we rewrite as

$$\frac{1}{y} \frac{dy}{dt} = -\sin t.$$

Integrating both sides gives $\ln y = \cos t + C$, and exponentiating gives

$$y = e^{\cos t + C}.$$

Because $y = 1 = e^0$ when $t = 0$, we must have $C = -\cos 0 = -1$, and

$$y = e^{\cos t - 1}.$$

Radioactive Isotope Problem: The amount $A(t)$ of the first isotope satisfies $dA/dt = -\lambda_1 A$, where $\lambda_1 = \ln(2)/3$. Because $A(0) = 100$,

$$A(t) = 100e^{-\lambda_1 t}.$$

The amount $B(t)$ of the second isotope satisfies

$$dB/dt = -\lambda_2 B - A'(t), \quad \lambda_2 = \ln(2)/10.$$

Rewriting this as a linear equation for B gives

$$dB/dt + \lambda_2 B = -A'(t) = 100\lambda_1 e^{-\lambda_1 t}.$$

Multiplying through by the integrating factor $\exp(\lambda_2 t)$ and integrating gives

$$e^{\lambda_2 t} B(t) = \frac{100\lambda_1}{\lambda_2 - \lambda_1} e^{(\lambda_2 - \lambda_1)t} + C$$

and

$$B(t) = \frac{100\lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} + C e^{-\lambda_2 t}.$$

Using $B(0) = 50$ gives $C = 100\lambda_1/(\lambda_1 - \lambda_2) + 50 \simeq 192.86$ and $B(7) \simeq 90.37$ grams.

p.24 #2: Separating gives $\frac{1}{1+y} dy = (1+t)dt$, and integrating each side gives $\ln|1+y| = t + \frac{1}{2}t^2 + C$.

Exponentiating each side (and absorbing the ambiguity of sign into the constant $C_1 = \pm e^C$) gives

$$y = C_1 \exp\left(t + \frac{1}{2}t^2\right) - 1.$$

#6: Separating gives

$$\frac{2y}{1+y^2} \frac{dy}{dt} = -t^2.$$

Integrating from 0 to t , and using $y(0) = 1$, gives

$$\ln(1+y^2) - \ln(2) = -\frac{1}{3}t^3.$$

Solving for y gives

$$y = \left(2 \exp\left(-\frac{1}{3}t^3\right) - 1\right)^{1/2},$$

which is defined as long as $\exp(-\frac{1}{3}t^3) \geq 1/2$, i.e., as long as $t \leq (3 \ln 2)^{1/3}$.

#8: This separable equation can be rewritten as

$$\frac{1}{y^3} \frac{dy}{dt} = \frac{t}{1+t^2}.$$

Let $F(y) = \int y^{-3} dy = -\frac{1}{2}y^{-2}$. Then the initial condition $y(0) = 1$ implies that

$$F(y) - F(1) = \int_0^t \frac{s}{1+s^2} ds = \frac{1}{2} \ln(1+t^2).$$

Solving for y gives $y^{-2} = 1 - \ln(1+t^2)$, and

$$y = \frac{1}{\sqrt{1 - \ln(1+t^2)}}.$$

The domain is the set of t -values for which $\ln(1+t^2) < 1$, equivalently $|t| < \sqrt{e-1}$.