COINCIDENCE FOR SUBSTITUTIONS OF PISOT TYPE

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Abstract. Let φ be a substitution of Pisot type on d letters. We prove that the Coincidence Conjecture holds for $d = 2$ and provide a partial result for $d \geq 2$.

A substitution φ on an alphabet $\mathcal{A} = \{1, 2, \ldots, d\}$ satisfies the *coincidence condition* if for every $i, j \in \mathcal{A}$, there are integers k, n such that $\varphi^n(i)$ and $\varphi^n(j)$ have the same k^{th} letter, and the prefixes of length $k-1$ of $\varphi^n(i)$ and $\varphi^n(j)$ have the same image under the abelianization map. This condition, first formulated for substitutions of constant length by F. M. Dekking and generalized to the present form by Arnoux and Ito, is satisfied for such a substitution φ if and only if the substitutive dynamical system associated with φ (with the shift map) has discrete spectrum ([Dek]). Hollander has shown that if a primitive substitution on a 2-letter alphabet satisfies the coincidence condition, then the associated dynamical system has discrete spectrum ([Hol]). According to [CS], Host proved (in unpublished work) that if φ is a unimodular Pisot type substitution over two letters satisfying the coincidence condition, then the substitutive system associated with φ is measure-theoretically isomorphic with both an interval exchange and a one-dimensional toral rotation. Arnoux and Ito ([AI]) have generalized this to $d \geq 2$. (Also see [CS]).

No examples of Pisot type substitutions which do not satisfy the coincidence condition are known, and the statement that all such substitutions satisfy the coincidence condition is known as the *Coincidence Conjecture*. In this paper, we prove that the Coincidence Conjecture holds for Pisot substitutions on 2 letters and provide a partial result for substitutions on d letters where $d > 2$.

Theorem 1. *Let φ be a Pisot substitution on an alphabet $\mathcal{A} = \{1, 2, \ldots, d\}$. There are distinct letters $i, j \in \mathcal{A}$ for which there are integers k, n such that $\varphi^n(i)$ and $\varphi^n(j)$ have the same k^{th} letter, and the prefixes of length $k-1$ of $\varphi^n(i)$ and $\varphi^n(j)$ have the same image under the abelianization map.*

We introduce terminology necessary for the proof of the theorem. For $j = 1$ to d, I_j will denote the interval of length 1 in R^d given by $I_j = \{0\} \times \{0\} \times \ldots \times [0, 1] \times \ldots \times \{0\}$, where $[0, 1]$ appears in the j^{th} position. Similarly, for $1 \leq j \leq d$, $\mathbf{e}_j$ will denote the unit vector $(0, 0, \ldots, 1, \ldots, 0)$, where 1 appears in the j^{th} position. If $x \in \mathbb{Z}^d$, the integer lattice in $\mathbb{R}^d$, the vector from the origin to x is denoted by $\mathbf{x}$. For $\mathbf{v} = (v_1, v_2, \dots, v_d)$,

$$I_j + \mathbf{v} = \{v_1\} \times \{v_2\} \times \dots [v_j, v_j + 1] \times \dots \{v_d\}.$$

The collection $\mathcal{M}$ of line segments joining 'adjacent' elements of the integer lattice is then

$$\mathcal{M} = \{I_j + \mathbf{x} : 1 \leq j \leq d, x \in \mathbb{Z}^d\}.$$

The term *segment* will refer to an element of $\mathcal{M}$.

Let $\mathcal{A} = \{1, 2, \dots, d\}$ be a finite alphabet and $\mathcal{A}^*$ the collection of finite non-empty words formed from the alphabet $\mathcal{A}$. A *substitution* φ is a map $\varphi : \mathcal{A} \to \mathcal{A}^*$; φ has an associated transition matrix $A_\varphi = A = (a_{ij})_{i,j \in \mathcal{A}}$ in which a_{ij} is the number of occurrences of i in the word $\varphi(j)$. (In what follows, we will also use A to denote the linear transformation on vectors of $\mathbb{R}^d$ and the linear function acting on points of $\mathbb{R}^d$.) The substitution φ is *primitive* if there is n so that for each $i, j \in \mathcal{A}$, j appears in $\varphi^n(i)$; φ is *of Pisot type*, or simply *Pisot*, if all eigenvalues of A other than the Perron-Frobenius eigenvalue have modulus strictly between 0 and 1. If φ is Pisot, then φ is primitive and the (hyperbolic) linear map on $\mathbb{R}^d$ defined by the matrix A has stable space E^s of dimension $d-1$ and unstable space E^u of dimension 1 spanned by a positive Perron-Frobenius eigenvector $\mathbf{v}_u$. Also, neither E^s nor E^u contain elements of the integer lattice other than the origin.

Assume for the remainder of this paper that φ is Pisot. We define an order on $\mathbb{Z}^d$ as follows. Given $P_1, P_2 \in \mathbb{Z}^d$, $P_1 \leq P_2$ if there are $t_1 \leq t_2 \in \mathbb{R}$ so that $P_1 \in E^s + t_1 \mathbf{v}_u$, $P_2 \in E^s + t_2 \mathbf{v}_u$ (we use $E^s + t\mathbf{v}_u$ to denote the set of points x such that $\mathbf{x} = \mathbf{w} + t\mathbf{v}_u$ for some $\mathbf{w} \in E^s$). Note that if $P_1, P_2 \in \mathbb{Z}^d$, and $P_1 \leq P_2$, then $AP_1 \leq AP_2$.

The *inflation and substitution map* F_φ is defined on subsets of $\mathcal{M}$ in the following manner. Suppose that $\varphi(j) = a(j,1)a(j,2)\dots a(j, n(j))$, for $1 \leq j \leq d$. Define

$$\mathbf{v}_\varphi(j, 0) = \mathbf{0},$$
$$\mathbf{v}_\varphi(j, i) = \mathbf{v}_\varphi(j, i-1) + \mathbf{e}_{a(j,i)}, \quad 1 \leq i \leq n(j).$$

Then

$$F_\varphi(I_j) = \{I_{a(j,i)} + \mathbf{v}_\varphi(j, i-1) : 1 \leq i \leq n(j)\}$$

and

$$\begin{aligned} F_\varphi(I_j + \mathbf{v}) &= F_\varphi(I_j) + A\mathbf{v} \\ &= \{s + A\mathbf{v} : s \in F_\varphi(I_j)\}. \end{aligned}$$

Finally, if $M \subseteq \mathcal{M}$,

$$F_\varphi(M) = \{F(s) : s \in M\}.$$

(Strictly speaking, in the above, we should write $F_\varphi(\{I_j\})$ rather than $F_\varphi(I_j)$, etc., but we abuse notation and use the latter when convenient.) Note that $F_{\varphi^n} = F_\varphi^n$.

Suppose that $S = \{s_i = I_{s(i)} + \mathbf{v}_i : 1 \leq i \leq k, 1 \leq s(i) \leq d\}$ is a non-empty finite collection of segments, i.e., a subset of $\mathcal{M}$. The set S is a *strand* if for each $1 \leq i \leq k-1$, $\mathbf{v}_{i+1} = \mathbf{v}_i + \mathbf{e}_{s(i)}$. The *word associated* *with* S is $w_S = s(1)s(2)\dots s(k)$. For such a word w, $\mathbf{l}(w) = (w_1, w_2, \dots, w_d)$ where for each $i \in \mathcal{A}$, $w_i = \#$ of occurrences of i in W. The *displacement vector for* S is then $\mathbf{l}(S) = \mathbf{l}(w_S)$ and the *length of* S is $|S| = |\mathbf{l}(S)|$, where $|(v_1, v_2, \dots, v_d)| = \Sigma_{1\le j\le d}|v_j|$. The endpoints of the segments belonging to S will be called *vertices of* S. A strand S has an *initial vertex* P and a *terminal vertex* Q with the property that $P \le R \le Q$ for all vertices R of S; we say that S is a strand from P to Q. The segments containing P and Q are the *initial* and *terminal* segments of S, respectively. Note that if S is a strand from P to Q, then $F_\varphi(S)$ is a strand from AP to AQ. Two strands S_1 and S_2 are *coincident* if $S_1 \cap S_2 \ne \emptyset$ (that is, they share a segment), and *eventually coincident* if $F_\varphi^m(S_1) \cap F_\varphi^m(S_2) \ne \emptyset$ for some $m \in \mathbb{N}$.

The coincidence conjecture can then be stated as follows:

Coincidence Conjecture: For $1 \le j, r \le d$, I_j and I_r are eventually coincident.

We prove the following:

Theorem 1. *There are* $1 \le j \ne r \le d$ *so that* I_j *and* I_r *are eventually coincident.*

In the following, $N_B(E^u)$ will denote $\{x \in \mathbb{R}^d : dist(x, E^u) < B\}$ where $dist(x, y) = |\mathbf{x} - \mathbf{y}|$.

Lemma 2. *There are* $B > 0$ *and* $n \in \mathbb{N}$ *so that (i) if a segment* s *lies in* $N_B(E^u)$*, then the strand* $F_\varphi^n(s)$ *lies in* $N_B(E^u)$ *and (ii) for any segment* s*, there is* $k \in \mathbb{N}$ *so that* $F_\varphi^{nk}(s)$ *lies in* $N_B(E^u)$*.*

Proof: The fact that φ is Pisot implies that there are λ, $0 < \lambda < 1$, and a constant C so that $|A^n\mathbf{v}| < C\lambda^n|\mathbf{v}|$ for all $\mathbf{v} \in E^s$. It follows that there are n and λ' with $\lambda < \lambda' < 1$ so that $A^n(N_B(E^u)) \subseteq N_{\lambda' B}(E^u)$ for all $B > 0$.

Now suppose that s is any segment lying in $N_B(E^u)$. The initial point of $F_\varphi^n(s)$ lies in $N_{\lambda' B}(E^u)$ and the entire strand $F_\varphi^n(s)$ lies in $N_{\lambda' B+m}(E^u)$, where m is the maximum of the lengths of $\varphi^n(i)$ for $i \in \mathcal{A}$. Choose B large enough so that $\lambda' B + m < \lambda'' B$ for some λ'', $0 < \lambda'' < 1$. For such B and any segment s in $N_B(E^u)$, $F_\varphi^n(s)$ lies in $N_B(E^u)$.

If s is a segment not in $N_B(E^u)$, let B' be large enough so that s lies in $N_{B'}(E^u)$. If $k \ge 1$ is such that $(\lambda'')^k B' \le B$, then $F_\varphi^{kn}(s)$ lies in $N_B(E^u)$. $\square$

Since $F_{\varphi^n} = F_\varphi^n$, I_j and I_r are eventually coincident under F_φ if and only if they are eventually coincident under F_{φ^n}. Thus we assume for the remainder of this paper that the n of Lemma 2 is $n = 1$.

A *configuration of segments* is a collection $\mathcal{C}$ of segments with the property that for some $\mathbf{v}$, $E^s + \mathbf{v}$ intersects each element of $\mathcal{C}$ in an interior point. A *configuration of strands* is a collection $\mathcal{C}$ of strands with the property that both the collections of initial segments and final segments form configurations of segments. The *size* of a configuration $\mathcal{C}$ of strands is the number of strands in $\mathcal{C}$. If P' is the largest of the initial vertices of strands of $\mathcal{C}$ and Q' is the smallest of the terminal vertices of strands of $\mathcal{C}$, we say that $\mathcal{C}$ is a configuration of strands *from* P' *to* Q', or that $\mathcal{C}$ *extends from* P *to* Q, with *length* equal to $|\mathbf{Q}' - \mathbf{P}'|$.

Given a configuration $\mathcal{C}$ of strands from P' to Q', there is a unique configuration of strands $\mathcal{C}^{(1)} \subseteq F_\varphi(\mathcal{C})$ with the following properties:

(i) $\mathcal{C}^{(1)}$ has the same size as $\mathcal{C}$;

(ii) $\mathcal{C}^{(1)}$ extends from AP' to AQ'; and

(iii) for each strand $S' \in \mathcal{C}^{(1)}$, there is a strand $S \in \mathcal{C}$ with $S' \subseteq F_\varphi(S)$.

We call $\mathcal{C}^{(1)}$ the first iterate of $\mathcal{C}$; higher iterates are defined by $\mathcal{C}^{(2)} = (\mathcal{C}^{(1)})^{(1)}$, etc. A configuration $\mathcal{C}$ of strands is *not eventually coincident* if each distinct pair of strands of $\mathcal{C}$ is not eventually coincident.

The following is a consequence of the definitions and Lemma 2.

Lemma 3. *Suppose that* $\mathcal{C}$ *is a not eventually coincident configuration of strands of size* m. *The iterates,* $\mathcal{C}^{(k)}$, *of* $\mathcal{C}$ *satisfy the following:*

(i) $\mathcal{C}^{(k)}$ *is a not eventually coincident configuration of strands;*

(ii) length $(\mathcal{C}^{(k)}) \to \infty$ *as* $k \to \infty$; *and*

(iii) there is a positive integer K *so that* $\mathcal{C}^{(k)}$ *lies in* $N_B(E^u)$ *for all* $k \geq K$.

Lemma 4. *There is an integer* M *with the following properties:*

(i) there exists a configuration of segments of size M *that is not eventually coincident; and*

(ii) every configuration of segments of size larger than M *is eventually coincident.*

Proof: For each $t \in \mathbb{R}$, let $n(t)$ be the number of distinct segments lying in $N_B(E^u)$ that meet $E^s + t\mathbf{v}_u$ and let $N = sup_{t\in\mathbb{R}} n(t)$. Then $N < \infty$. Let $\mathcal{C}$ be a configuration of not eventually coincident segments of size m and choose k large enough so that $\mathcal{C}^{(k)}$ lies in $N_B(E^u)$. For some t, $E^s + t\mathbf{v}_u$ contains no vertices of $\mathcal{C}^{(k)}$ and meets a segment from each strand of $\mathcal{C}^{(k)}$. Since the strands of $\mathcal{C}^{(k)}$ are not coincident, $m \leq N$. Then N is an upper bound on the size of not eventually coincident configurations. □

Proof of Theorem 1: Let $\mathcal{C}$ be a not eventually coincident collection of segments of maximum size M as in Lemma 4. Without loss of generality, $\mathcal{C}^{(k)}$ lies in $N_B(E^u)$ for all $k \geq 0$. For each k, let $P_0 < P_1 < \ldots < P_m$ ($P_i = P_i(k)$) be the vertices of $\mathcal{C}^{(k)}$ that lie between the initial vertex P_0 and the terminal vertex P_m. Let $t_0 < t_1 < \ldots < t_m$ ($t_i = t_i(k)$) be such that P_i lies on $E^s + t_i\mathbf{v}_u$. For $1 \leq i \leq m$, let $\mathcal{C}_i^{(k)} = \{s : s$ is a segment of some strand of $\mathcal{C}^{(k)}$ and s meets $E^s + t\mathbf{v}_u$ for $t_{i-1} < t < t_i\}$. Then $\mathcal{C}_i^{(k)}$ is a configuration of segments of size M (one from each strand of $\mathcal{C}^{(k)}$) that is not eventually coincident and that extends from P_{i-1} to P_i. Let $l_i \geq 1$ be the number of segments of $\mathcal{C}_i^{(k)}$ that terminate at P_i. Then exactly l_i of the segments of $\mathcal{C}_{i+1}^{(k)}$ begin at P_i. Let $\mathcal{S}_{i+1}$ denote the collection of segments of $\mathcal{C}_{i+1}^{(k)}$ issuing from P_i, let $\mathcal{S}'_{i+1}$ be any other collection of l_i (distinct) segments beginning at P_i, and consider the collection $\mathcal{C}'_{i+1} = (\mathcal{C}^{(k)}_{i+1} \setminus \mathcal{S}_{i+1}) \cup \mathcal{S}'_{i+1}$. Suppose that for some such $\mathcal{S}'_{i+1}$, $\mathcal{C}'_{i+1}$ is not eventually coincident. The maximality of $\mathcal{C}^{(k)}_{i+1}$ implies that $(\mathcal{C}^{(k)}_{i+1} \cup \mathcal{S}_{i+1})$ is eventually coincident, hence there are $s \in \mathcal{S}_{i+1}$, $s' \in \mathcal{S}'_{i+1}$, $s \neq s'$, such that $\{s, s'\}$ is eventually coincident. Write $s = I_j + \mathbf{v}$, $s' = I_r + \mathbf{v}$. It follows that I_j and I_r are eventually coincident, and the theorem is proved.

Suppose then, for the remainder of this proof, that for each such $\mathcal{S}'_{i+1}$, $\mathcal{C}'_{i+1}$ is eventually coincident. Then $\mathcal{C}^{(k)}_i$ completely determines $\mathcal{C}^{(k)}_{i+1}$.

In the above argument, the location of configurations did not play a role; only the relative position of segments within configurations was considered. In particular, if there are p with $0 \leq i \leq i+p < m$ and $\mathbf{v}$ so that $\mathcal{C}^{(k)}_{i+p} = \mathcal{C}^{(k)}_i + \mathbf{v}$, then $\mathcal{C}^{(k)}_{i+p+1} = \mathcal{C}^{(k)}_{i+1} + \mathbf{v}$.

Up to translation, there are only finitely many configurations of segments that lie in $N_B(E^u)$; let b be an upper bound on the number of equivalence classes (up to translation) of such configurations of segments. Recall that $\mathcal{C}^{(k)}$ extends from P_0 to P_m ($m = m(k)$) and that $P_0 < P_1 < \ldots < P_m$ ($P_i = P_i(k)$) are the vertices of $\mathcal{C}^{(k)}$ that lie between P_0 and P_m. The vectors $\mathbf{P}_{i+1} - \mathbf{P}_i$, $0 \leq i \leq m-1$, are bounded independently of m (by $2B+1$), and $length(\mathcal{C}^{(k)}) = |\mathbf{P}_m - \mathbf{P}_0| \to \infty$ as $k \to \infty$. Thus $m(k) \to \infty$ as $k \to \infty$ and, for $m(k) > b$, there are $i = i(k) \leq b$ and l with $i < l \leq m(k)$ such that $\mathcal{C}^{(k)}_l = \mathcal{C}^{(k)}_i + \mathbf{v}$ for the integer vector $\mathbf{v} = \mathbf{P}_l - \mathbf{P}_i$. It follows from the above that $\mathcal{C}^{(k)}_{i+p(l-i)} = \mathcal{C}^{(k)}_i + p\mathbf{v}$ for all positive integers p with $i + p(l-i) \leq m$. Since $i < l \leq b$ and $\mathbf{v}$ is a nonzero integer vector bounded independently of k, some such vector $\mathbf{v}$ occurs for infinitely many k. For this $\mathbf{v}$, it must be the case that $p\mathbf{v}$ lies in $N_{2B}(E^u)$ for arbitrarily large p. But then $\mathbf{v}$ is a scalar multiple of $\mathbf{v}_u$, so that E^u contains a nonzero point of $\mathbb{Z}^d$, an impossibility. This contradiction proves the theorem. $\square$

References

[AI] P. Arnoux and S. Ito, Pisot substitutions and Rauzy fractals, preprint.

[CS] V. Canterini and A. Siegel, Geometric representation of primitive substitutions of Pisot type, to appear in *Trans. Amer. Math. Soc.*

[Dek] F.M. Dekking, The spectrum of dynamical systems arising from substitutions of constant length, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **41**(1977/78), 221-239.

[Hol] M. Hollander, Linear numeration systems, finite β-expansions, and discrete spectrum of substitution dynamical systems, PhD thesis, University of Washington, 1996.

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