INTERMEDIATE WAKIMOTO MODULES FOR AFFINE $\mathfrak{sl}(n+1,\mathbb{C})$

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Abstract. We construct certain boson type realizations of affine $\mathfrak{sl}(n+1,\mathbb{C})$ that depend on a parameter $0 \leq r \leq n$ such that when $r=0$ we get a Fock space realization appearing in [Cox04] and when $r=n$ they are the Wakimoto modules described in the work of Feigin and Frenkel [FF88].

1. Introduction

Wakimoto modules for affine Lie algebras were introduced by B. Feigin and E. Frenkel in [FF88] by a homological characterization. These modules admit a remarkable boson realization on a Fock space [Wak86] (for $\hat{\mathfrak{sl}}(2)$), [FF90b] (for $\hat{\mathfrak{sl}}(n)$) which plays an important role in the conformal field theory providing a new bosonization rule for the Wess-Zumino-Witten models. Wakimoto modules have a geometric interpretation as certain sheaves on a semi-infinite flag manifold [FF90a]. They belong to the category $\mathcal{O}$ and generically are isomorphic to corresponding Verma modules.

Affine Lie algebras admit Verma type modules associated with non-standard Borel subalgebras, see [Cox94], [FS93] and [JK85]. Modules associated with the *natural Borel subalgebra* were first introduced by H. Jakobsen and V. Kac in [JK85]. They were studied in [Fut94] under the name of *imaginary Verma modules*.

A Fock space realization of the imaginary Verma modules for $\hat{\mathfrak{sl}}(2)$ were constructed by D. Bernard and G. Felder in [BF90] and then extended in [Cox04] to the case of $\hat{\mathfrak{sl}}(n)$. These realizations are given generically by certain Wakimoto type modules.

The main motivation for our work was a problem of finding suitable boson type realizations for all Verma type modules over $\hat{\mathfrak{sl}}(n+1)$. In Theorem 3.1 we construct such realizations, *intermediate Wakimoto modules*, for a series of generic Verma type modules depending on the parameter $0 \leq r \leq n$. If $r=n$ this construction coincides with the boson realization of Wakimoto modules in [FF88]. On the other hand when $r=0$ the obtained representation gives a Fock space realization described in [Cox04]. One difficulty that arises in the study of Verma type modules that are not induced from a standard Borel subalgebra is that certain of their weight spaces are infinite dimensional. On the other hand the structure of representations that have infinite dimensional weight spaces is an important problem that appears naturally in other contexts. Besides appearing in the repesentation theory of infinite dimensional Heisenberg Lie algebras such representations also arise in the work of [CP87] and [CM01]. Intermediate Wakimoto modules are another family of representations with certain weight spaces being infinite dimensional. We plan to discuss their detailed structure in a subsequent paper using the construction given in this paper. See the concluding remarks for the potential usefulness of our result.

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2. Preliminaries

Fix a positive integer n, $0 \leq r \leq n$, $\gamma \in \mathbb{C}^*$. Set $k = \gamma^2 - (r+1)$. Let $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$ and let E^{ij}, $i, j = 1, \dots, n+1$ be the standard basis for $\mathfrak{gl}(n+1, \mathbb{C})$. Set $H_i := E^{ii} - E^{i+1,i+1}$, $E_i := E^{i,i+1}$, $F_i := E^{i+1,i}$ which is a basis for $\mathfrak{sl}(n+1, \mathbb{C})$. Furthermore we denote the Killing form by $(X|Y) = \operatorname{tr}(XY)$ and $X_m = t^m \otimes X$ for $X, Y \in \mathfrak{g}$ and $m \in \mathbb{Z}$. Let $\{\alpha_1, \dots, \alpha_n\}$ be a base for Δ^+, the positive set of roots for $\mathfrak{g}$, such that $H_i = \check{\alpha}_i$ and let Δ_r be the root system with base $\{\alpha_1, \dots, \alpha_r\}$ ($\Delta_r = \emptyset$, if $r = 0$) of the Lie subalgebra $\mathfrak{g}_r = \mathfrak{sl}(r+1, \mathbb{C})$. A Cartan subalgebra $\mathfrak{H}$ (respectively $\mathfrak{H}_r$) of $\mathfrak{g}$ (respectively $\mathfrak{g}_r$) is spanned by H_i, $i = 1, \dots, n$ (respectively $i = 1, \dots, r$) and set $\mathfrak{H}_0 = 0$.

For any Lie algebra $\mathfrak{a}$, let $L(\mathfrak{a}) = \mathfrak{a} \bigotimes \mathbb{C}[t, t^{-1}]$ be the loop algebra of $\mathfrak{a}$. Then $\hat{\mathfrak{g}} = \hat{\mathfrak{sl}}(n+1, \mathbb{C}) = L(\mathfrak{g}) \oplus \mathbb{C}c \oplus \mathbb{C}d$ and $\hat{\mathfrak{g}}_r = L(\mathfrak{g}_r) \oplus \mathbb{C}c \oplus \mathbb{C}d$ are the associated affine Kac-Moody algebras with $\hat{\mathfrak{H}} = \mathfrak{H} \oplus \mathbb{C}c \oplus \mathbb{C}d$ and $\hat{\mathfrak{H}}_r = \mathfrak{H}_r \oplus \mathbb{C}c \oplus \mathbb{C}d$ respectively.

The algebra $\hat{\mathfrak{g}}$ has generators E_{im}, F_{im}, H_{im}, $i = 1, \dots, n$, $m \in \mathbb{Z}$, and central element c with the product

$$[X_m, Y_n] = t^{m+n}[X, Y] + \delta_{m+n,0}m(X|Y)c.$$

2.1. **Oscillator algebras.** Let $\hat{\mathfrak{a}}$ be the infinite dimensional Heisenberg algebra with generators $a_{ij,m}$, $a^*_{ij,m}$, and $\mathbf{1}$, $1 \leq i \leq j \leq n$ and $m \in \mathbb{Z}$, subject to the relations

$$\begin{aligned} [a_{ij,m}, a_{kl,n}] &= [a^*_{ij,m}, a^*_{kl,n}] = 0, \\ [a_{ij,m}, a^*_{kl,n}] &= \delta_{ik}\delta_{jl}\delta_{m+n,0}\mathbf{1}, \\ [a_{ij,m}, \mathbf{1}] &= [a^*_{ij,m}, \mathbf{1}] = 0. \end{aligned}$$

Such an algebra has a representation $\tilde{\rho} : \hat{\mathfrak{a}} \to \mathfrak{gl}(\mathbb{C}[\mathbf{x}])$ where

$$\mathbb{C}[\mathbf{x}] := \mathbb{C}[x_{ij,m} | i, j, m \in \mathbb{Z}, \, 1 \leq i \leq j \leq n]$$

denotes the algebra over $\mathbb{C}$ generated by the indeterminates $x_{ij,m}$ and $\tilde{\rho}$ is defined by

$$\tilde{\rho}(a_{ij,m}) := \begin{cases} \partial/\partial x_{ij,m} & \text{if} \quad m \geq 0, \text{ and } j \leq r \\ x_{ij,m} & \text{otherwise,} \end{cases}$$

$$\tilde{\rho}(a^*_{ij,m}) := \begin{cases} x_{ij,-m} & \text{if} \quad m \leq 0, \text{ and } j \leq r \\ -\partial/\partial x_{ij,-m} & \text{otherwise.} \end{cases}$$

and $\tilde{\rho}(\mathbf{1}) = 1$. In this case $\mathbb{C}[\mathbf{x}]$ is an $\hat{\mathfrak{a}}$-module generated by $1 =: |0\rangle$, where

$$a_{ij,m}|0\rangle = 0, \quad m \geq 0 \text{ and } j \leq r, \quad a^*_{ij,m}|0\rangle = 0, \quad m > 0 \text{ or } j > r.$$

Let $\hat{\mathfrak{a}}_r$ denote the subalgebra generated by $a_{ij,m}$ and $a^*_{ij,m}$ and $\mathbf{1}$, where $1 \leq i \leq j \leq r$ and $m \in \mathbb{Z}$. If $r = 0$, we set $\hat{\mathfrak{a}}_r = 0$.

Let $A_n = ((\alpha_i|\alpha_j))$ be the Cartan matrix for $\mathfrak{sl}(n+1, \mathbb{C})$ and let $\mathfrak{B}$ be the matrix whose entries are

$$\mathfrak{B}_{ij} := (\alpha_i|\alpha_j)(\gamma^2 - \delta_{i>r}\delta_{j>r}(r+1) + \frac{r}{2}\delta_{i,r+1}\delta_{j,r+1})$$

where

$$\delta_{i>r} = \begin{cases} 1 & \text{if} \quad i > r, \\ 0 & \text{otherwise.} \end{cases}$$

In other words

$$\mathfrak{B} := \gamma^2 A_n - (r+1)\begin{pmatrix} 0 & 0 \\ 0 & A_{n-r} \end{pmatrix} + rE_{r+1,r+1}.$$

We also have the Heisenberg Lie algebra $\hat{\mathfrak{b}}$ with generators b_{im}, $1 \leq i \leq n$, $m \in \mathbb{Z}$, $\mathbf{1}$, and relations $[b_{im}, b_{jp}] = m\,\mathfrak{B}_{ij}\delta_{m+p,0}\mathbf{1}$ and $[b_{im}, \mathbf{1}] = 0$.

For each $1 \leq i \leq n$ fix $\lambda_i \in \mathbb{C}$ and let $\lambda = (\lambda_1, \dots, \lambda_n)$. Then the algebra $\hat{\mathfrak{b}}$ has a representation $\rho_\lambda : \hat{\mathfrak{b}} \to \operatorname{End}(\mathbb{C}[\mathbf{y}]_\lambda)$ where

$$\mathbb{C}[\mathbf{y}] := \mathbb{C}[y_{i,m} | i, m \in \mathbb{N}^*,\ 1 \leq i \leq n]$$

and ρ_λ is defined on $\mathbb{C}[\mathbf{y}]$ defined by

$$\rho_\lambda(b_{i0}) = \lambda_i, \quad \rho_\lambda(b_{i,-m}) = \mathbf{e}_i \cdot \mathbf{y}_m, \quad \rho_\lambda(b_{im}) = m\mathbf{e}_i \cdot \frac{\partial}{\partial \mathbf{y}_m} \quad \text{for} \quad m > 0$$

and $\rho_\lambda(\mathbf{1}) = 1$. Here

$$\mathbf{y}_m = (y_{1m}, \cdots, y_{nm}), \quad \frac{\partial}{\partial \mathbf{y}_m} = \left(\frac{\partial}{\partial y_{1m}}, \cdots, \frac{\partial}{\partial y_{nm}}\right)$$

and $\mathbf{e}_i$ are vectors in $\mathbb{C}^n$ such that $\mathbf{e}_i \cdot \mathbf{e}_j = \mathfrak{B}_{ij}$ where $\cdot$ means the usual dot product.

Note that since $\mathfrak{B}_{ij}$ is symmetric, it is orthogonally diagonalizable, (i.e. there exists an orthogonal matrix P such that $P^t\mathfrak{B}P$ is a diagonal matrix) and hence we can find vectors $\mathbf{e}_i$ in $\mathbb{C}^n$ such that $\mathbf{e}_i \cdot \mathbf{e}_j = \mathfrak{B}_{ij}$. In fact for $m > 0$ and $n < 0$ we get

$$\begin{aligned}
[b_{im}, b_{jn}] &= [m\mathbf{e}_i \cdot \frac{\partial}{\partial \mathbf{y}_m}, \mathbf{e}_j \cdot \mathbf{y}_{-n}] \\
&= m\sum_{k,l}[e_{ik}\frac{\partial}{\partial y_{km}}, e_{jl}y_{l,-n}] \\
&= m\delta_{m+n,0}\sum_k e_{ik}e_{jk} = m\delta_{m+n,0}\mathfrak{B}_{ij}.
\end{aligned}$$

(See also [FF90b].)

2.2. **Formal Distributions.** We need some more notation that will simplify some of the arguments later. This notation follows roughly [Kac98] and [MN99]: A *formal distribution* is an expression of the form

$$a(z, w, \dots) = \sum_{m,n,\dots \in \mathbb{Z}} a_{m,n,\dots} z^m w^n$$

where the $a_{m,n,\dots}$ lie in some fixed vector space V and $z, w, \dots$ are formal variables. We define $\partial a(z) = \partial_z a(z) = \sum_n n a_n z^{n-1}$. We also have expansion about zero: there are two canonical embeddings of vector spaces $\iota_{z,w} : \mathbb{C}(z - w) \to \mathbb{C}[[z, w]]$ and $\iota_{w,z} : \mathbb{C}(z - w) \to \mathbb{C}[[z, w]]$ where $\iota_{z,w}(a(z, w))$ is formal Laurent series expansion in z^{-1} and $-\iota_{w,z}(a(z, w))$ is formal Laurent series expansion in z. The *formal delta function* $\delta(z - w)$ is the formal distribution

$$\delta(z - w) = z^{-1}\sum_{n \in \mathbb{Z}}\left(\frac{z}{w}\right)^n = \iota_{z,w}\left(\frac{1}{z - w}\right) - \iota_{w,z}\left(\frac{1}{z - w}\right).$$

For any sequence of elements $\{a_m\}_{m \in \mathbb{Z}}$ in the ring $\operatorname{End}(V)$, V a vector space, the formal distribution

$$a(z) := \sum_{m \in \mathbb{Z}} a_m z^{-m-1}$$

is called a *field*, if for any $v \in V$, $a_m v = 0$ for $m \gg 0$. If $a(z)$ is a field, then we set

$$a(z)_- := \sum_{m \geq 0} a_m z^{-m-1}, \quad \text{and} \quad a(z)_+ := \sum_{m < 0} a_m z^{-m-1}.$$

In particular

$$\delta(z-w)_-=\iota_{z,w}\left(\frac{1}{z-w}\right),\quad \delta(z-w)_+=-\iota_{w,z}\left(\frac{1}{z-w}\right).$$

Note that

$$-\partial_z\delta(z-w)=\partial_w\delta(z-w)=\iota_{z,w}\left(\frac{1}{(z-w)^2}\right)-\iota_{w,z}\left(\frac{1}{(z-w)^2}\right).$$

The *normal ordered product* of two distributions $a(z)$ and $b(w)$ (and their coefficients) is defined by

$$\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}:a_mb_n:z^{-m-1}w^{-n-1}=:a(z)b(w):=a(z)_+b(w)+b(w)a(z)_-. \tag{2.1}$$

For any $1\le i\le j\le n$, we define

$$a^*_{ij}(z)=\sum_{n\in\mathbb{Z}}a^*_{ij,n}z^{-n},\quad a_{ij}(z)=\sum_{n\in\mathbb{Z}}a_{ij,n}z^{-n-1}$$

and

$$b_i(z)=\sum_{n\in\mathbb{Z}}b_{in}z^{-n-1}.$$

In this case

$$\begin{aligned}[b_i(z),b_j(w)]&=\mathfrak{B}_{ij}\partial_w\delta(z-w),\\ [a_{ij}(z),a^*_{kl}(w)]&=\delta_{ik}\delta_{jl}\mathbf{1}\delta(z-w).\end{aligned}$$

Observe that $a_{ij}(z)$ for $j>r$ is not a field whereas $a^*_{ij}(z)$ is always a field. We will call $a_{ij}(z)$ (resp. $a^*_{ij}(z)$) a *pure creation* (resp. *annihilation*) *operator* if $j>r$. Set

$$\begin{aligned}a_{ij}(z)_+&=a_{ij}(z),\quad a_{ij}(z)_-=0\\ a^*_{ij}(z)_+&=0,\quad a^*_{ij}(z)_-=a^*_{ij}(z),\end{aligned}$$

if $j>r$.

Now we should point out that while $:a^1(z_1)\cdots a^m(z_m):$ is always defined as a formal series, we will only define $:a(z)b(z)::=\lim_{w\to z}:a(z)b(w):$ for certain pairs $(a(z),b(w))$. For example

$$:a_{ij}(z)a^*_{kl}(z):=\sum_{m\in\mathbb{Z}}\left(\sum_{n\in\mathbb{Z}}:a_{ij,n}a^*_{kl,m-n}:\right)z^{-m-1}$$

is well defined as an element in $\mathrm{End}(\mathbb{C}[\mathbf{x}]\otimes\mathbb{C}[\mathbf{y}])[[z,z^{-1}]]$ for all $l>r$ (as $\tilde\rho(a^*_{kl,m}):=-\partial/\partial x_{kl,-m}$ for $l>r$) or if both $l\le r$ and $j\le r$ (see also the remarks after Theorem 3.1).

Then one defines recursively

$$:a^1(z_1)\cdots a^k(z_k):=:a^1(z_1)\left(:a^2(z_2)\left(:\cdots:a^{k-1}(z_{k-1})a^k(z_k):\right)\cdots:\right):,$$

while normal ordered product

$$:a^1(z)\cdots a^k(z):=\lim_{z_1,z_2,\cdots,z_k\to z}:a^1(z_1)\left(:a^2(z_2)\left(:\cdots:a^{k-1}(z_{k-1})a^k(z_k):\right)\cdots\right):$$

will only be defined for certain k-tuples $(a^1,\dots,a^k)$.

Let

$$\lfloor ab\rfloor=a(z)b(w)-:a(z)b(w):=[a(z)_-,b(w)], \tag{2.2}$$

(half of $[a(z), b(w)]$) denote the *contraction* of any two formal distributions $a(z)$ and $b(w)$ where $a(z)$, $b(z)$ are free fields or pure creation or annihilation operators. For example if $j, l \leq r$, then

$$\lfloor a_{ij} a^*_{kl} \rfloor = \sum_{m\geq 0} \delta_{ik}\delta_{jl} z^{-m-1} w^m = \delta_{i,k}\delta_{j,l}\delta_-(z-w) = \delta_{ik}\delta_{jl}\, \iota_{z,w}\left(\frac{1}{z-w}\right) \tag{2.3}$$

$$\lfloor a^*_{kl} a_{ij} \rfloor = -\sum_{n<0} \delta_{ik}\delta_{jl} z^{n} w^{-n-1} = -\delta_{ik}\delta_{jl}\delta_+(w-z) = \delta_{ik}\delta_{jl}\, \iota_{z,w}\left(\frac{1}{w-z}\right). \tag{2.4}$$

If $l > r$, then

$$\lfloor a_{ij} a^*_{kl} \rfloor = [a_{ij}(z)_-, a^*_{kl}(w)] = 0 \tag{2.5}$$

$$\lfloor a^*_{kl} a_{ij} \rfloor = [a^*_{kl}(z)_-, a_{ij}(w)] = -\delta_{ik}\delta_{jl}\delta(w-z). \tag{2.6}$$

Theorem 2.1 (Wick's Theorem, [BS83], [Hua98] or [Kac98])**.** *Let $a^i(z)$ and $b^j(z)$ be formal distributions with coefficients in the associative algebra* $\mathrm{End}(\mathbb{C}[\mathbf{x}]\otimes\mathbb{C}[\mathbf{y}])$*, satisfying*

1. *$[\lfloor a^i(z)b^j(w)\rfloor, c^k(x)_\pm] = [\lfloor a^i b^j\rfloor, c^k(x)_\pm] = 0$, for all i, j, k and $c^k(x) = a^k(z)$ or $c^k(x) = b^k(w)$.*
2. *$[a^i(z)_\pm, b^j(w)_\pm] = 0$ for all i and j.*
3. *The products*

$$\lfloor a^{i_1} b^{j_1}\rfloor \cdots \lfloor a^{i_s} b^{i_s}\rfloor : a^1(z)\cdots a^M(z) b^1(w)\cdots b^N(w) :_{(i_1,\dots,i_s;j_1,\dots,j_s)}$$

have coefficients in $\mathrm{End}(\mathbb{C}[\mathbf{x}]\otimes\mathbb{C}[\mathbf{y}])$ *for all subsets $\{i_1,\dots,i_s\}\subset\{1,\dots,M\}$, $\{j_1,\dots,j_s\}\subset\{1,\cdots N\}$. Here the subscript $(i_1,\dots,i_s;j_1,\dots,j_s)$ means that those factors $a^i(z)$, $b^j(w)$ with indices $i\in\{i_1,\dots,i_s\}$, $j\in\{j_1,\dots,j_s\}$ are to be omitted from the product : $a^1\cdots a^M b^1\cdots b^N$: and when $s = 0$ we do not omit any factors.*

Then

$$:a^1(z)\cdots a^M(z) :: b^1(w)\cdots b^N(w) := \sum_{s=0}^{\min(M,N)} \sum_{i_1<\cdots<i_s,\, j_1\neq\cdots\neq j_s} \lfloor a^{i_1} b^{j_1}\rfloor \cdots \lfloor a^{i_s} b^{i_s}\rfloor : a^1(z)\cdots a^M(z) b^1(w)\cdots b^N(w) :_{(i_1,\dots,i_s;j_1,\dots,j_s)}.$$

Proof. Although it is essentially the same proof as in [Hua98] or [Kac98], we will repeat the argument here for the convenience of the reader.

To simplify notation we will write $a^i = a^i(z_i)$ and $b^j = b^j(w_j)$ below, hoping that it will not cause confusion. Moreover we define $\lfloor a^k b^k\rfloor = 0$ when $k > \min\{M, N\}$.

The conclusion is true for $M = N = 1$ as follows from the definition of the contraction, (2.2). Suppose now $N > 1$ and $M = 1$. Then by hypotheses 1 and 2, and induction

$$\begin{aligned}
&a : b^1 \cdots b^N := ab^1_+ : b^2 \cdots b^N : + a : b^2 \cdots b^N : b^1_- \\
&= a_+ b^1_+ : b^2 \cdots b^N : + \lfloor ab^1 \rfloor : b^2 \cdots b^N : + b^1_+ a_- : b^2 \cdots b^N : + a : b^2 \cdots b^N : b^1_- \\
&= a_+ b^1_+ : b^2 \cdots b^N : + \lfloor ab^1 \rfloor : b^2 \cdots b^N : + \sum_{j=2}^{N} \lfloor ab^j \rfloor b^1_+ : b^2 \cdots \widehat{b^j} \cdots b^N : + b^1_+ : b^2 \cdots b^N : a_- \\
&\quad + a_+ : b^2 \cdots b^N : b^1_- + \sum_{j=1}^{N} \lfloor ab^j \rfloor : b^2 \cdots \widehat{b^j} \cdots b^N : b^1_- + : b^2 \cdots \widehat{b^j} \cdots b^N : b^1_- a^1_- \\
&= a_+ : b^1 \cdots b^N : + \lfloor ab^1 \rfloor : b^2 \cdots b^N : + \sum_{j=2}^{N} \lfloor ab^j \rfloor : b^1 \cdots \widehat{b^j} \cdots b^N : + : b^1 \cdots b^N : a_- \\
&\quad + : b^1 \cdots \widehat{b^j} \cdots b^N : a^1_- \\
&=: ab^1 \cdots b^N : + \sum_{j=1}^{N} \lfloor ab^j \rfloor : b^1 \cdots \widehat{b^j} \cdots b^N : .
\end{aligned}$$

The hat above a factor means omit that factor. Note that at each step we are combining summations where all of the coefficents in each summand are in $\mathrm{End}(\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\mathbf{y}])$ when $z_1, \dots, z_M$ get replaced by z.

A similar argument proves the result for $M > 1$ and $N = 1$.

Now let us assume that M and N are greater than 1. Then using the hypothesis 1, 2 and induction we get

$$\begin{aligned}
&\sum_{s \geq 0} \sum_{\substack{1 \leq i_1 < \cdots < i_s, \\ j_1 \neq \cdots \neq j_s}} \lfloor a^{i_1} b^{j_1} \rfloor \cdots \lfloor a^{i_s} b^{i_s} \rfloor : a^1 \cdots a^M b^1 \cdots b^N :_{(i_1, \dots, i_s; j_1, \dots, j_s)} \\
&= a^1_+ \sum_{s \geq 0} \sum_{\substack{2 \leq i_1 < \cdots < i_s \\ j_1 \neq \cdots \neq j_s}} \lfloor a^{i_1} b^{j_1} \rfloor \cdots \lfloor a^{i_s} b^{i_s} \rfloor : a^2 \cdots a^M b^1 \cdots b^N :_{(i_1, \dots, i_s; j_1, \dots, j_s)} \\
&\quad + \sum_{s \geq 0} \sum_{\substack{2 \leq i_1 < \cdots < i_s \\ j_1 \neq \cdots \neq j_s}} \lfloor a^{i_1} b^{j_1} \rfloor \cdots \lfloor a^{i_s} b^{i_s} \rfloor : a^2 \cdots a^M b^1 \cdots b^N :_{(i_1, \dots, i_s; j_1, \dots, j_s)} \, a^1_- \\
&\quad + \sum_{j=1}^{N} \lfloor a^1 b^j \rfloor : a^2 \cdots a^M :: b^1 \cdots \widehat{b^j} \cdots b^N : \\
&= a^1_+ \sum_{s \geq 0} \sum_{\substack{2 \leq i_1 < \cdots < i_s \\ j_1 \neq \cdots \neq j_s}} \lfloor a^{i_1} b^{j_1} \rfloor \cdots \lfloor a^{i_s} b^{i_s} \rfloor : a^2 \cdots a^M b^1 \cdots b^N :_{(i_1, \dots, i_s; j_1, \dots, j_s)} \\
&\quad + : a^2 \cdots a^M :: a^1_- b^1 \cdots b^N : + \sum_{j=1}^{N} \lfloor a^1 b^j \rfloor : a^2 \cdots a^M :: b^1 \cdots \widehat{b^j} \cdots b^N : \\
&= a^1_+ \sum_{s \geq 0} \sum_{\substack{2 \leq i_1 < \cdots < i_s \\ j_1 \neq \cdots \neq j_s}} \lfloor a^{i_1} b^{j_1} \rfloor \cdots \lfloor a^{i_s} b^{i_s} \rfloor : a^2 \cdots a^M b^1 \cdots b^N :_{(i_1, \dots, i_s; j_1, \dots, j_s)} \\
&\quad + : a^2 \cdots a^M : a^1_- : b^1 \cdots b^N : \\
&=: a^1 \cdots a^M :: b^1 \cdots b^N : .
\end{aligned}$$

Since every step involves combining sums that are (by induction assumed to be) well defined when $z_1,\dots,z_M$ and $w_1,\dots w_N$ are replaced by z and w respectively, so is the resulting product. This completes the proof of the theorem. □

We will also need the following two results.

Theorem 2.2 (Taylor's Theorem, [Kac98], 2.4.3)**.** *Let $a(z)$ be a formal distribution. Then in the region $|z-w|<|w|$,*

$$a(z)=\sum_{j=0}^{\infty}\partial_w^{(j)}a(w)(z-w)^j. \tag{2.7}$$

Theorem 2.3 ([Kac98], Theorem 2.3.2)**.** *Let $a(z)$ and $b(z)$ be formal distributions with coefficients in the associative algebra* $\operatorname{End}(\mathbb{C}[\mathbf{x}]\otimes\mathbb{C}[\mathbf{y}])$. *The following are equivalent*

(i) $[a(z),b(w)]=\sum_{j=0}^{N-1}\partial_w^{(j)}\delta(z-w)c^j(w)$, *where* $c^j(w)\in\operatorname{End}(\mathbb{C}[\mathbf{x}]\otimes\mathbb{C}[\mathbf{y}])[[w,w^{-1}]]$.

(ii) $\lfloor ab\rfloor=\sum_{j=0}^{N-1}\iota_{z,w}\left(\frac{1}{(z-w)^{j+1}}\right)c^j(w)$.

In other words the singular part of the *operator product expansion*

$$\lfloor ab\rfloor=\sum_{j=0}^{N-1}\iota_{z,w}\left(\frac{1}{(z-w)^{j+1}}\right)c^j(w)$$

completely determines the bracket of mutually local formal distributions $a(z)$ and $b(w)$. One writes

$$a(z)b(w)\sim\sum_{j=0}^{N-1}\frac{c^j(w)}{(z-w)^{j+1}}.$$

For example

$$b_i(z)b_j(w)\sim\frac{\delta_{ij}}{(z-w)^2}.$$

2.3. **Verma type modules.** For a Lie algebra $\mathfrak{a}$ we denote by $U(\mathfrak{a})$ the universal enveloping algebra of $\mathfrak{a}$.

Let $\mathfrak{g}_\alpha$ be a root subspace of $\mathfrak{g}$ corresponding to a root α, $\mathfrak{n}^\pm=\oplus_{\alpha\in\Delta^+}\mathfrak{g}_{\pm\alpha}$ and $\mathfrak{g}=\mathfrak{n}^-\oplus\mathfrak{H}\oplus\mathfrak{n}^+$ a Cartan decomposition of $\mathfrak{g}$. Denote also $\mathfrak{n}_r^\pm=\mathfrak{n}^\pm\cap\mathfrak{g}_r$, $\mathfrak{n}^+(r)=\mathfrak{n}^+\setminus\mathfrak{n}_r^+$,

$$\bar{B}_r=L(\mathfrak{n}^+(r))\oplus(\mathfrak{n}_r^+\otimes\mathbb{C}[t])\oplus((\mathfrak{n}_r^-)\oplus\mathfrak{H})\otimes\mathbb{C}[t]t).$$

Then $B_r=\bar{B}_r\oplus\hat{\mathfrak{H}}$ is a Borel subalgebra of $\hat{\mathfrak{g}}$ for any $0\le r\le n$.

Fix $\tilde{\lambda}\in\hat{\mathfrak{H}}^*$ and consider a $\hat{\mathfrak{g}}$-module

$$M_r(\tilde{\lambda})=U(\hat{\mathfrak{g}})\otimes_{U(B_r)}\mathbb{C}v_{\tilde{\lambda}}$$

where $\bar{B}_rv_{\tilde{\lambda}}=0$ and $hv_{\tilde{\lambda}}=\tilde{\lambda}(h)v_{\tilde{\lambda}}$ for all $h\in\hat{\mathfrak{H}}$.

Module $M_r(\tilde{\lambda})$ is a particular case of Verma type modules studied in [Cox94] and [FS93]. When $r=n$ it gives a usual Verma module construction. If $r=0$ we get an imaginary Verma modules.

Let $\tilde{\lambda}_r=\tilde{\lambda}|_{\hat{\mathfrak{H}}_r}$. Verma type module $M_r(\tilde{\lambda})$ contains a $\hat{\mathfrak{g}}_r$-submodule $M(\tilde{\lambda}_r)=U(\hat{\mathfrak{g}}_r)(1\otimes v_{\tilde{\lambda}})$ which is isomorphic to a usual Verma module for $\hat{\mathfrak{g}}_r$.

Theorem 2.4 ([Cox94], [FS93])**.** *Let $\tilde\lambda(c)\neq 0$. Then the submodule structure of $M_r(\tilde\lambda)$ is completely determined by the submodule structure of $M(\tilde\lambda_r)$. In particular, $M_r(\tilde\lambda)$ is irreducible if $M(\tilde\lambda_r)$ is irreducible.*

3. Intermediate Wakimoto modules

Define

$$E_i(z)=\sum_{n\in\mathbb{Z}}E_{in}z^{-n-1},\quad F_i(z)=\sum_{n\in\mathbb{Z}}F_{in}z^{-n-1},\quad H_i(z)=\sum_{n\in\mathbb{Z}}H_{in}z^{-n-1},\quad 1\leq i\leq n.$$

The defining relations between the generators of $\hat{\mathfrak{g}}$ can be written as follows

$$[H_i(z),H_j(w)]=(\alpha_i|\alpha_j)c\partial_w\delta(w-z)\tag{R1}$$

$$[H_i(z),E_j(w)]=(\alpha_i|\alpha_j)E_j(z)\delta(w-z)\tag{R2}$$

$$[H_i(z),F_j(w)]=-(\alpha_i|\alpha_j)F_j(z)\delta(w-z)\tag{R3}$$

$$[E_i(z),F_j(w)]=\delta_{i,j}(H_i(z)\delta(w-z)+c\partial_w\delta(w-z))\tag{R4}$$

$$[F_i(z),F_j(w)]=[E_i(z),E_j(w)]=0\quad\text{if}\quad(\alpha_i|\alpha_j)\neq-1\tag{R5}$$

$$[F_i(z_1),F_i(z_2),F_j(w)]=[E_i(z_1),E_i(z_2),E_j(w)]=0\quad\text{if}\quad(\alpha_i|\alpha_j)=-1\tag{R6}$$

where $[X,Y,Z]:=[X,[Y,Z]]$ is the Engel bracket for any three operators X,Y,Z.

Recall that $\mathbb{C}[\mathbf{x}]$ is an $\hat{\mathfrak{a}}$-module with respect to the representation $\tilde\rho$ and $\mathbb{C}[\mathbf{y}]$ is a $\hat{\mathfrak{b}}$-module with respect to ρ_λ. The main result of the paper is the following theorem where we define a representation

$$\rho:\hat{\mathfrak{g}}\to\mathfrak{gl}(\mathbb{C}[\mathbf{x}]\otimes\mathbb{C}[\mathbf{y}]).$$

We use the notation $\rho(X_m):=\rho(X)_m$, for $X\in\mathfrak{g}$.

Theorem 3.1. *Let $\lambda\in\mathfrak{H}^*$ and set $\lambda_i=\lambda(H_i)$. The generating functions*

$$\rho(F_i)(z)=a_{ii}+\sum_{j=i+1}^{n}a_{ij}a^*_{i+1,j},$$

$$\begin{aligned}\rho(H_i)(z)=2:a_{ii}a^*_{ii}:+&\sum_{j=1}^{i-1}\big(:a_{ji}a^*_{ji}:-:a_{j,i-1}a^*_{j,i-1}:\big)\\&+\sum_{j=i+1}^{n}\big(:a_{ij}a^*_{ij}:-:a_{i+1,j}a^*_{i+1,j}:\big)+b_i,\end{aligned}$$

$$\begin{aligned}\rho(E_i)(z)=:a^*_{ii}\left(\sum_{k=1}^{i-1}a_{k,i-1}a^*_{k,i-1}-\sum_{k=1}^{i}a_{ki}a^*_{ki}\right):+&\sum_{k=i+1}^{n}a_{i+1,k}a^*_{ik}-\sum_{k=1}^{i-1}a_{k,i-1}a^*_{ki}\\&-a^*_{ii}b_i-\big(\delta_{i>r}(r+1)+\delta_{i\leq r}(i+1)-\gamma^2\big)\partial a^*_{ii},\end{aligned}$$

$$\rho(c)=\gamma^2-(r+1)$$

define an action of the generators E_{im}, F_{im}, H_{im}, $i=1,\dots,n$, $m\in\mathbb{Z}$ and c, on the Fock space $\mathbb{C}[\mathbf{x}]\otimes\mathbb{C}[\mathbf{y}]$. In the above a_{ij}, a^*_{ij} and b_i denotes $a_{ij}(z)$, $a^*_{ij}(z)$ and $b_i(z)$ respectively.

Remark 3.2. One can see that the normal ordered products $: a_{ij}(z)a^*_{ij}(z):$ are all well defined and thus so is $\rho(H_i)(z)$. Moreover a careful analysis of the other formal distributions show that they too have coffecients in $\mathrm{End}(\mathbb{C}[\mathbf{x}]\otimes\mathbb{C}[\mathbf{y}])$. For example

$$\sum_{k=i+1}^{n} a_{i+1,k}a^*_{ik}$$

has summand $a_{i+1,k}a^*_{ik}$ that is certainly well defined if $k \le r$ and for $k > r$ we have $\tilde{\rho}(a^*_{ik,m}) := -\partial/\partial x_{ik,-m}$ for all $m \in \mathbb{Z}$. Moreover the summation

$$\sum_{k=1}^{i-1} a_{k,i-1}a^*_{ki}$$

is well defined also for $i \le r$ and for $i > r$ we have $\tilde{\rho}(a^*_{ki,m}) := -\partial/\partial x_{ki,-m}$.

Proof. The rather tedious proof is obtained by a routine application of Wick's Theorem, Taylor's Theorem and Theorem 2.3 above. We have left the details in the appendix. ☐

Theorem 3.1 defines a boson type realization of $\hat{\mathfrak{sl}}(n+1,\mathbb{C})$ and a module structure on the Fock space $\mathbb{C}[\mathbf{x}]\otimes\mathbb{C}[\mathbf{y}]$ that depends on the parameter $0 \le r \le n$. We will call such a module, an *intermediate Wakimoto module* and denote it by $W_{n,r}(\lambda,\gamma)$. The intermediate Wakimoto modules $W_{n,r}(\lambda,\gamma)$ have the property that the subalgebra $\bar{B}_r$ annihilates the vector $1\otimes 1 \in \mathbb{C}[\mathbf{x}]\otimes\mathbb{C}[\mathbf{y}]$, $h(1\otimes 1) = \lambda(h)(1\otimes 1)$ for all $h \in \mathfrak{H}$ and $c(1\otimes 1) = (\gamma^2 - (r+1))(1\otimes 1)$. Consider the $\hat{\mathfrak{g}}_r$-submodule $W = U(\hat{\mathfrak{g}}_r)(1\otimes 1) \simeq W_{r,r}(\lambda,\gamma)$ of $W_{n,r}(\lambda,\gamma)$. Then W is isomorphic to the Wakimoto module $W_{\lambda(r),\tilde{\gamma}}$ ([FF90b]) where $\lambda(r) = \lambda|_{\mathfrak{H}_r}$, $\tilde{\gamma} = \gamma^2 - (r+1)$.

Consider $\tilde{\lambda} \in \hat{\mathfrak{H}}^*$ such that $\tilde{\lambda}|_{\mathfrak{H}} = \lambda$, $\tilde{\lambda}(c) = \gamma^2 - (r+1)$, a Verma type module $M_r(\tilde{\lambda})$ and its $\hat{\mathfrak{g}}_r$-submodule $M(\tilde{\lambda}_r)$. Suppose that $M(\tilde{\lambda}_r)$ is irreducible. In this case the Wakimoto module $W_{\lambda(r),\tilde{\gamma}}$ is isomorphic to $M(\tilde{\lambda}_r)$. Let $\tilde{W} = U(\hat{\mathfrak{g}})W_{\lambda(r),\tilde{\gamma}}$ and assume that $\lambda(c) \neq 0$. Then by Theorem 2.4 the module $M_r(\tilde{\lambda})$ is irreducible and therefore isomorphic to $\tilde{W}$. Hence Theorem 3.1 provides a boson type realization for generic Verma type modules.

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5. Field computations

Set

$$\mathcal{H}_i(z) := 2:a_{ii}a^*_{ii}: + \sum_{j=1}^{i-1}\left(:a_{ji}a^*_{ji}: - :a_{j,i-1}a^*_{j,i-1}:\right)$$
$$+ \sum_{j=i+1}^{n}\left(:a_{ij}a^*_{ij}: - :a_{i+1,j}a^*_{i+1,j}:\right).$$

In the above a_{ij}, and a^*_{ij} denotes $a_{ij}(z)$, $a^*_{ij}(z)$ respectively.

For any $\alpha \in \Delta^+$ we can find unique $1 \le k \le l \le n$ such that

$$\alpha = \alpha_k + \cdots + \alpha_l.$$

Set $a_\alpha := a_{kl}$ and $a^*_\alpha := a^*_{kl}$. Observe that

$$(\alpha_i|\alpha) = \sum_{j=k}^{l}(\alpha_i|\alpha_j) = \left(2\delta_{ik}\delta_{il} + \delta_{k<i}\left(\delta_{il} - \delta_{i-1,l}\right) + \delta_{l>i}\left(\delta_{ik} - \delta_{i+1,k}\right)\right).$$

Moreover we have

$$\lfloor a_\alpha a^*_\beta \rfloor = \begin{cases} \delta_{\alpha,\beta}\, \iota_{z,w}\left(\frac{1}{z-w}\right) & \text{if} \quad \alpha,\beta \in \Delta^+_r, \\ 0 & \text{otherwise} \end{cases} \tag{5.1}$$

$$\lfloor a^*_\alpha a_\beta \rfloor = \begin{cases} -\delta_{\alpha,\beta}\, \iota_{z,w}\left(\frac{1}{z-w}\right) & \text{if} \quad \alpha,\beta \in \Delta^+_r, \\ -\delta_{\alpha,\beta}\, \delta(w-z) & \text{otherwise} \end{cases} \tag{5.2}$$

Since this is the case we can rewrite

$$\mathcal{H}_i(z) := \sum_{\alpha\in\Delta^+}(\alpha_i|\alpha) : a_\alpha a^*_\alpha : .$$

As an example of a computation using fields we have the following

Lemma 5.1. *For $1 \le i \le j \le n$, $\alpha, \beta \in \Delta^+$,*

$$\begin{aligned}
[\mathcal{H}_i(z), a_\alpha(w)] &= -(\alpha_i|\alpha)a_\alpha(z)\delta\left(z-w\right), \\
[\mathcal{H}_i(z), a^*_\alpha(w)] &= (\alpha_i|\alpha)a^*_\alpha(z)\delta(z-w), \\
[\mathcal{H}_i(z), \partial_w a^*_\alpha(w)] &= (\alpha_i|\alpha)a^*_\alpha(z)\partial_w(z-w), \\
[\mathcal{H}_i(z), \mathcal{H}_j(w)] &= -(\alpha_i|\alpha_j)\Big((1-\delta_{i>r}\delta_{j>r})(r+1) + \frac{r}{2}\delta_{i,r+1}\delta_{j,r+1}\Big)\partial_w\delta(z-w), \\
[\mathcal{H}_i(z), : a_\alpha(w)a^*_\beta(w)a^*_\gamma(w) :] &= (\alpha_i|\beta+\gamma-\alpha) : a_\alpha(w)a^*_\beta(w)a^*_\gamma(w) : \delta(z-w) \\
&\quad - \delta_{\alpha\in\Delta^+_r}(\alpha_i|\alpha)\left(\delta_{\alpha,\beta}a^*_\gamma(w) + \delta_{\alpha,\gamma}a^*_\beta(w)\right)\partial_w\delta(z-w), \\
[\mathcal{H}_{\alpha_i}(z), : a_\alpha(w)a_\beta(w)a^*_\gamma(w) :] &= (\alpha_i|\gamma-\alpha-\beta) : a_\alpha(w)a_\beta(w)a^*_\gamma(w) : \delta(z-w) \\
&\quad - \delta_{\gamma\in\Delta^+_r}(\alpha_i|\gamma)\left(\delta_{\gamma,\beta}a_\gamma(w) + \delta_{\alpha,\gamma}a_\beta(w)\right)\partial_w\delta(z-w).
\end{aligned}$$

Proof. Now by (5.1) and (5.2) and by Wick's Theorem

$$\sum_j : a_{ij}(z)a^*_{ij}(z) : a_{kl}(w) \sim \delta_{ik}a_{kl}(z)\lfloor a^*_{ij}a_{kl}\rfloor$$

and if $\alpha = \alpha_k + \cdots + \alpha_l$, then

$$\begin{aligned}
\mathcal{H}_i(z)a_{kl}(w) &= \Big(2 : a_{ii}a^*_{ii} : + \sum_{j=1}^{i-1}\left(: a_{ji}a^*_{ji} : - : a_{j,i-1}a^*_{j,i-1} :\right) \\
&\qquad + \sum_{j=i+1}^{n}\left(: a_{ij}a^*_{ij} : - : a_{i+1,j}a^*_{i+1,j} :\right)\Big)a_{kl}(w) \\
&\sim -\delta_{1\le l\le r}\left(2\delta_{ik}\delta_{il} + \delta_{k<i}\left(\delta_{il}-\delta_{i-1,l}\right) + \delta_{l>i}\left(\delta_{ik}-\delta_{i+1,k}\right)\right)a_{kl}(z)\iota_{z,w}\left(\frac{1}{z-w}\right) \\
&\qquad - \delta_{r<l}\left(2\delta_{ik}\delta_{il} + \delta_{k<i}\left(\delta_{il}-\delta_{i-1,l}\right) + \delta_{l>i}\left(\delta_{ik}-\delta_{i+1,k}\right)\right)a_{kl}(z)\delta(z-w) \\
&\sim \delta_{1\le l\le r}(\alpha_i|\alpha)a_{kl}(z)\iota_{z,w}\left(\frac{1}{w-z}\right) - \delta_{r<l}(\alpha_i|\alpha)a_{kl}(z)\delta(z-w).
\end{aligned}$$

On the other hand

$$a_{kl}(w)\mathcal{H}_i(z) \sim \delta_{1\leq l\leq r}(\alpha_i|\alpha)a_{kl}(w)\iota_{w,z}\left(\frac{1}{z-w}\right).$$

Combining the above operator product expansions we get the first identity. A similar computation yeilds the second identity. The third identity comes from differentiating the second with respect to w.

On the other hand by Wick's Theorem

$$\begin{aligned}
&: a_\nu(z)a^*_\mu(z) :: a_\alpha(w)a^*_\beta(w) : \\
&\qquad =: a_\alpha(w)a^*_\beta(w)a_\nu(z)a^*_\mu(z) : + \lfloor a_\alpha a^*_\mu \rfloor : a_\nu(z)a^*_\beta(w) : \\
&\qquad\quad + \lfloor a^*_\beta a_\nu \rfloor : a_\alpha(w)a^*_\mu(z) : + \lfloor a_\alpha a^*_\mu \rfloor \lfloor a^*_\beta a_\nu \rfloor .
\end{aligned}$$

Thus

$$\begin{aligned}
\mathcal{H}_{\alpha_i}(z)\mathcal{H}_{\alpha_j}(w) &= \sum_{\alpha,\beta\in\Delta^+}(\alpha_i|\alpha)(\alpha_j|\beta) : a_\alpha(z)a^*_\alpha(z) :: a_\beta(w)a^*_\beta(w) : \\
&= \sum_{\alpha,\beta\in\Delta^+}(\alpha_i|\alpha)(\alpha_j|\beta) : a_\alpha(z)a_\beta(w)a^*_\alpha(z)a^*_\beta(w) : \\
&\quad + \sum_{\beta\in\Delta^+}(\alpha_i|\beta)(\alpha_j|\beta) : a_\beta(w)a^*_\beta(z) \lfloor a^*_\beta a_\beta \rfloor \\
&\quad + \sum_{\alpha\in\Delta^+}(\alpha_i|\alpha)(\alpha_j|\alpha) : a_\alpha(z)a^*_\alpha(w) : \lfloor a_\alpha a^*_\alpha \rfloor \\
&\quad + \sum_{\alpha\in\Delta_r^+}(\alpha_i|\alpha)(\alpha_j|\alpha) \lfloor a_\alpha a^*_\alpha \rfloor \lfloor a^*_\alpha a_\alpha \rfloor ,
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
[\mathcal{H}_{\alpha_i}(z), \mathcal{H}_{\alpha_j}(w)] &= \sum_{\alpha\in\Delta_r^+}(\alpha_i|\alpha)(\alpha_j|\alpha)\left(\iota_{w,z}\frac{1}{(w-z)^2} - \iota_{z,w}\frac{1}{(z-w)^2}\right) \\
&= -(\alpha_i|\alpha_j)\left((1-\delta_{i>r}\delta_{j>r})(r+1) + \frac{r}{2}\delta_{i,r+1}\delta_{j,r+1}\right)\partial_w\delta(z-w).
\end{aligned}$$

This follows from the calculation below for root system of $\mathfrak{sl}(r+1)$: If $j \leq r$, then

$$\sum_{\alpha\in\Delta_r^+}(\alpha_j|\alpha)\alpha = (r+1)\alpha_j$$

and

$$\sum_{\alpha\in\Delta_r^+}(\alpha_{r+1}|\alpha)^2 = r.$$

Again by (5.1), (5.2) and Wick's Theorem

$$\begin{aligned}
&: a_\nu(z)a^*_\nu(z) :: a_\alpha(w)a^*_\beta(w)a^*_\gamma(w) : \\
&\qquad =: a_\nu(z)a^*_\nu(z)a_\alpha(w)a^*_\beta(w)a^*_\gamma(w) : \\
&\qquad\quad + \lfloor a^*_\nu a_\alpha \rfloor : a_\nu(z)a^*_\beta(w)a^*_\gamma(w) : \\
&\qquad\quad + \lfloor a_\nu a^*_\beta \rfloor : a_\alpha(w)a^*_\nu(z)a^*_\gamma(w) : \\
&\qquad\quad + \lfloor a_\nu a^*_\gamma \rfloor : a_\alpha(w)a^*_\beta(w)a^*_\nu(z) : \\
&\qquad\quad + \left(\lfloor a^*_\nu a_\alpha \rfloor \lfloor a_\nu a^*_\beta \rfloor a^*_\gamma(w) + \lfloor a^*_\nu a_\alpha \rfloor \lfloor a_\nu a^*_\gamma \rfloor a^*_\beta(w)\right).
\end{aligned}$$

Hence the last identity follows from

$$\begin{aligned}
\mathcal{H}_{\alpha_i}(z) : a_\alpha(w)a^*_\beta(w)a^*_\gamma(w) &:= \sum_{\nu\in\Delta^+} (\alpha_i|\nu) : a_\nu(z)a^*_\nu(z) :: a_\alpha(w)a^*_\beta(w)a^*_\gamma(w) : \\
&\sim \sum_{\nu\in\Delta^+} (\alpha_i|\nu)\Bigg(\Big(\lfloor a^*_\nu a_\alpha\rfloor : a_\nu(z)a^*_\beta(w)a^*_\gamma(w) : \\
&\quad + \lfloor a_\nu a^*_\beta\rfloor : a_\alpha(w)a^*_\nu(z)a^*_\gamma(w) : + \lfloor a_\nu a^*_\gamma\rfloor : a_\alpha(w)a^*_\beta(z)a^*_\nu(z)\Big) : \\
&\quad + \big(\lfloor a^*_\nu a_\alpha\rfloor\lfloor a_\nu a^*_\beta\rfloor a^*_\gamma(w) + \lfloor a^*_\nu a_\alpha\rfloor\lfloor a_\nu a^*_\gamma\rfloor a^*_\beta(w)\big) \\
&\sim \Big((\alpha_i|\alpha)\lfloor a^*_\alpha a_\alpha\rfloor : a_\alpha(z)a^*_\beta(w)a^*_\gamma(w) : + (\alpha_i|\beta)\lfloor a_\beta a^*_\beta\rfloor : a_\alpha(w)a^*_\beta(z)a^*_\gamma(w) : \\
&\quad + (\alpha_i|\gamma)\lfloor a_\gamma a^*_\gamma\rfloor : a_\alpha(z)a^*_\beta(z)a^*_\mu(w)\Big) : \\
&\quad + (\alpha_i|\alpha)\left(\delta_{\alpha,\beta}a^*_\gamma(w) + \delta_{\alpha,\gamma}a^*_\beta(w)\right)\lfloor a^*_\alpha a_\alpha\rfloor\lfloor a_\alpha a^*_\alpha\rfloor.
\end{aligned}$$

□

Lemma 5.2.

$$\begin{aligned}
[a_{ij}(z), a^*_{kl}(w)] &= \delta_{ik}\delta_{jl}\delta(z-w) \\
[a_{ij}(z)a^*_{ij}(z), a_{ij}(w)a^*_{ij}(w)] &= \delta_{1\le i,j\le r}\partial_w\delta(z-w) \\
[a_{ij}(z), \partial_w a^*_{kl}(w)] &= \delta_{ik}\delta_{jl}\partial_w\delta(z-w) \\
\partial_w a^*_{ij}(w)\delta(z-w) &= a^*_{ij}(z)\partial_w\delta(z-w) - a^*_{ij}(w)\partial_w\delta(z-w)
\end{aligned}$$

The following result collects some other computations involving the free fields that will make future calculations less tedious.

Lemma 5.3.

(a)
$$\sum_{k=i+1,l=j+1}^{n}\big[a_{ik}(z)a^*_{i+1,k}(z),a_{jl}(w)a^*_{j+1,l}(w)\big]$$
$$=\Big(\delta_{i,j+1}\sum_{k=j+2}^{n}a_{jk}(z)a^*_{j+2,k}(z)-\delta_{j,i+1}\sum_{k=i+2}^{n}a_{ik}(z)a^*_{i+2,k}(z)\Big)\delta(z-w)$$

(b)
$$\sum_{k=1}^{i-1}\sum_{l=j+1}^{n}\big[a_{k,i-1}(z)a^*_{ki}(z),a_{jl}(w)a^*_{j+1,l}(w)\big]=-\delta_{j,i-1}a_{i-1,i-1}(z)a^*_{ii}(z)\delta(z-w)$$

(c)
$$\sum_{l=j+1}^{n}\Big[:a^*_{ii}\left(\sum_{k=1}^{i-1}a_{k,i-1}a^*_{k,i-1}-\sum_{k=1}^{i}a_{ki}a^*_{ki}\right):,a_{jl}(w)a^*_{j+1,l}(w)\Big]=0$$

(d)
$$\Big[:a^*_{ii}\left(\sum_{k=1}^{i-1}a_{k,i-1}a^*_{k,i-1}-\sum_{k=1}^{i}a_{ki}a^*_{ki}\right):,a_{jj}(w)\Big]$$
$$=-\big(\delta_{j,i-1}a_{i-1,i-1}(z)a^*_{i,i}(z)+2\delta_{i,j}a_{ii}(z)a^*_{ii}(z)\big)\,\delta(z-w)$$

(e)
$$\Big[:a^*_{ii}\Big(\sum_{k=1}^{i-1}a_{k,i-1}a^*_{k,i-1}\Big):,:a^*_{jj}\left(\sum_{l=1}^{j-1}a_{l,j-1}a^*_{l,j-1}\right):\Big]$$
$$\begin{aligned}=\quad&\delta_{j,i-1}:a^*_{ii}(z)a^*_{i-1,i-1}(z)\left(\sum_{k=1}^{i-2}a_{k,i-2}(w)a^*_{k,i-2}(w)\right):\delta(z-w)\\&-\delta_{i,j-1}:a^*_{jj}(w)a^*_{j-1,j-1}(w)\left(\sum_{l=1}^{j-2}a_{l,j-2}(z)a^*_{l,j-2}(z)\right)\delta(z-w)\\&-(i-1)\delta_{1\le i-1\le r}\delta_{ij}:a^*_{ii}(z)a^*_{ii}(w):\partial_w\delta(z-w)\end{aligned}$$

(f)
$$\Big[:a^*_{ii}\Big(\sum_{k=1}^{i-1}a_{k,i-1}a^*_{k,i-1}\Big):,:a^*_{jj}\left(\sum_{l=1}^{j}a_{l,j}a^*_{l,j}\right):\Big]$$
$$\begin{aligned}=&\,\delta_{j,i-1}:a^*_{ii}(z)a^*_{i-1,i-1}(z)\left(\sum_{k=1}^{i-1}a_{k,i-1}(w)a^*_{k,i-1}(w)\right):\delta(z-w)\\&-\delta_{ij}:a^*_{ii}(w)a^*_{ii}(w)\left(\sum_{l=1}^{i-1}a_{l,i-1}(z)a^*_{l,i-1}(z)\right)\delta(z-w)\\&-i\delta_{1\le i-1\le r}\delta_{j,i-1}:a^*_{ii}(z)a^*_{i-1,i-1}(w):\partial_w\delta(z-w)\end{aligned}$$

(g)
$$\Big[:a^*_{ii}\Big(\sum_{k=1}^{i}a_{ki}a^*_{ki}\Big):,:a^*_{jj}\left(\sum_{l=1}^{j}a_{lj}a^*_{lj}\right):\Big]=-(3+i)\delta_{1\le i\le r}\delta_{ij}:a^*_{ii}(z)a^*_{ii}(w):\partial_w\delta(z-w)$$

(h)
$$\sum_{l=j+1}^{n}\Big[: a^*_{ii}\left(\sum_{k=1}^{i-1} a_{k,i-1}a^*_{k,i-1}-\sum_{k=1}^{i} a_{ki}a^*_{ki}\right) :, a_{j+1,l}(w)a^*_{jl}(w)\Big]$$

$$= -\delta_{j,i-1} : a^*_{i-1,i}(w)\left(\sum_{k=1}^{i-1} a_{k,i-1}(z)a^*_{k,i-1}(z)-\sum_{k=1}^{i} a_{ki}(z)a^*_{ki}(z) :\right)\delta(z-w)$$
$$+ \delta_{i\leq r}\delta_{j,i-1}a^*_{i-1,i}(z)\partial_w\delta(z-w)$$

(i)
$$\Big[: a^*_{ii}\left(\sum_{k=1}^{i-1} a_{k,i-1}a^*_{k,i-1}-\sum_{k=1}^{i} a_{ki}a^*_{ki}\right) :, \sum_{l=1}^{j-1} a_{l,j-1}a^*_{lj}\Big]$$
$$= \Bigg(: a^*_{ii}\left(\delta_{i-1,j}\sum_{l=1}^{i-2} a_{l,i-2}a^*_{l,i-1}-2\delta_{ij}\sum_{l=1}^{i-1} a_{l,i-1}a^*_{li}+\delta_{i,j-1}\sum_{l=1}^{i} a_{l,i}a^*_{l,i+1}\right) :$$
$$- \delta_{i,j-1} : a^*_{i,i+1}\left(\sum_{l=1}^{i-1} a_{l,i-1}a^*_{l,i-1}-\sum_{l=1}^{i} a_{li}a^*_{li}\right) : \Bigg)\delta(z-w)$$

(j)
$$\Big[\sum_{k=1}^{i-1} a_{k,i-1}a^*_{ki}, \sum_{l=1}^{j-1} a_{l,j-1}a^*_{lj}\Big]$$
$$= \left(\delta_{j,i-1}\sum_{l=1}^{i-2} : a_{l,i-2}a^*_{li} : -\delta_{i,j-1}\sum_{l=1}^{j-2} : a_{l,j-2}a^*_{lj} :\right)\delta(z-w)$$

(k)
$$\Big[\sum_{k=i+1}^{n} a_{i+1,k}(z)a^*_{ik}(z), \sum_{l=j+1}^{n} a_{j+1,l}(w)a^*_{jl}(w)\Big]$$
$$= \left(\delta_{i,j-1}\sum_{l=i+2}^{n} : a_{i+2,l}a^*_{il} : -\delta_{j,i-1}\sum_{l=j+2}^{n} : a_{j+2,k}a^*_{jk} :\right)\delta(z-w)$$

(l)
$$\Big[\sum_{k=1}^{i-1} a_{k,i-1}a^*_{ki}, \sum_{l=j+1}^{n} a_{j+1,l}a^*_{jl}\Big] = 0$$

Proof. For (e) we have by Wick's theorem

$$: a^*_{ii}\Big(\sum_{k=1}^{i-1} a_{k,i-1}a^*_{k,i-1}\Big) :: a^*_{jj}\left(\sum_{l=1}^{j-1} a_{l,j-1}a^*_{l,j-1}\right) :$$

$$\begin{aligned}
\sim\quad & \delta_{j,i-1} : a^*_{ii}(z)a^*_{i-1,i-1}(z)\left(\sum_{k=1}^{i-2} a_{k,i-2}(w)a^*_{k,i-2}(w)\right) : \lfloor a_{i-1,i-1}a^*_{jj}\rfloor \\
& + \delta_{ij} : a^*_{ii}(z)a^*_{ii}(w)\left(\sum_{l=1}^{i-1} a_{l,i-1}(w)a^*_{l,i-1}(z)\right)\lfloor a_{i-1,i-1}a^*_{j-1,j-1}\rfloor \\
& + \delta_{ij} : a^*_{ii}(z)a^*_{ii}(w)\left(\sum_{l=1}^{i-1} a_{l,i-1}(z)a^*_{l,i-1}(w)\right)\lfloor a^*_{i-1,i-1}a_{j-1,j-1}\rfloor \\
& + \delta_{i,j-1} : a^*_{jj}(w)a^*_{j-1,j-1}(w)\left(\sum_{l=1}^{j-2} a_{l,j-2}(z)a^*_{l,j-2}(z)\right)\lfloor a^*_{i,i}a_{j-1,j-1}\rfloor \\
& + \delta_{ij} : a^*_{ii}(z)a^*_{ii}(w) : \left(\sum_{l=1}^{i-1}\lfloor a_{l,i-1}a^*_{l,j-1}\rfloor\lfloor a^*_{l,i-1}a_{l,j-1}\rfloor\right)
\end{aligned}$$

Hence

$$\Big[: a^*_{ii}\Big(\sum_{k=1}^{i-1} a_{k,i-1}a^*_{k,i-1}\Big) :, : a^*_{jj}\left(\sum_{l=1}^{j-1} a_{l,j-1}a^*_{l,j-1}\right) : \Big]$$

$$\begin{aligned}
=\quad & \delta_{j,i-1} : a^*_{ii}(z)a^*_{i-1,i-1}(z)\left(\sum_{k=1}^{i-2} a_{k,i-2}(w)a^*_{k,i-2}(w)\right) : \delta(z-w) \\
& + \delta_{i,j-1} : a^*_{jj}(w)a^*_{j-1,j-1}(w)\left(\sum_{l=1}^{j-2} a_{l,j-2}(z)a^*_{l,j-2}(z)\right)\delta(z-w) \\
& + (i-1)\delta_{1\le i-1\le r}\delta_{ij} : a^*_{ii}(z)a^*_{ii}(w) : \partial_w\delta(z-w)
\end{aligned}$$

For (f) we have by Wick's theorem

$$
\begin{aligned}
&: a_{ii}^*\Big(\sum_{k=1}^{i-1} a_{k,i-1}a_{k,i-1}^*\Big) :: a_{jj}^*\left(\sum_{l=1}^{j} a_{lj}a_{lj}^*\right) : \\
&\qquad \sim \quad \delta_{j,i-1} : a_{ii}^*(z)a_{i-1,i-1}^*(z)\left(\sum_{k=1}^{i-1} a_{k,i-1}(w)a_{k,i-1}^*(w)\right) : \lfloor a_{i-1,i-1}a_{jj}^*\rfloor \\
&\qquad + \delta_{j,i-1} : a_{ii}^*(z)a_{i-1,i-1}^*(w)\left(\sum_{l=1}^{i-1} a_{l,i-1}(w)a_{l,i-1}^*(z)\right)\lfloor a_{i-1,i-1}a_{j,i}^*\rfloor \\
&\qquad + \delta_{ij} : a_{ii}^*(w)a_{ii}^*(w)\left(\sum_{l=1}^{i-1} a_{l,i-1}(z)a_{l,i-1}^*(z)\right)\lfloor a_{i-1,i-1}^*a_{j-1,j-1}\rfloor \\
&\qquad + \delta_{j,i-1} : a_{ii}^*(z)a_{i-1,i-1}^*(w)\left(\sum_{l=1}^{i-1} a_{l,i-1}(z)a_{l,i-1}^*(w)\right)\lfloor a_{jj}^*a_{i-1,i-1}\rfloor \\
&\qquad + \delta_{j,i-1} : a_{ii}^*(z)a_{i-1,i-1}^*(w) : \lfloor a_{i-1,i-1}a_{i-1,i-1}^*\rfloor\lfloor a_{i-1,i-1}^*a_{i-1,j-1}\rfloor \\
&\qquad + \delta_{j,i-1} : a_{ii}^*(z)a_{i-1,i-1}^*(w) : \left(\sum_{l=1}^{i-1}\lfloor a_{l,i-1}a_{l,j-1}^*\rfloor\lfloor a_{l,i-1}^*a_{l,j-1}\rfloor\right).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\Big[: a_{ii}^*\Big(\sum_{k=1}^{i-1} a_{k,i-1}a_{k,i-1}^*\Big) :, : a_{jj}^*\left(\sum_{l=1}^{j} a_{l,j}a_{l,j}^*\right) : \Big] \\
&\qquad = \delta_{j,i-1} : a_{ii}^*(z)a_{i-1,i-1}^*(z)\left(\sum_{k=1}^{i-1} a_{k,i-1}(w)a_{k,i-1}^*(w)\right) : \delta(z-w) \\
&\qquad + \delta_{ij} : a_{ii}^*(w)a_{ii}^*(w)\left(\sum_{l=1}^{i-1} a_{l,i-1}(z)a_{l,i-1}^*(z)\right)\delta(z-w) \\
&\qquad + i\delta_{1\le i-1\le r}\delta_{j,i-1} : a_{ii}^*(z)a_{i-1,i-1}^*(w) : \partial_w\delta(z-w).
\end{aligned}
$$

For (g) we have by Wick's theorem

$$: a_{ii}^*\Big(\sum_{k=1}^{i} a_{ki}a_{ki}^*\Big) :: a_{jj}^* \left(\sum_{l=1}^{j} a_{lj}a_{lj}^*\right) :$$

$$\begin{aligned}
\sim \quad & \delta_{ij} : a_{ii}^*(z)^2 \left(\sum_{k=1}^{i} a_{ki}(w)a_{ki}^*(w)\right) : \lfloor a_{ii}a_{jj}^* \rfloor \\
& + \delta_{ij} : a_{ii}^*(z)a_{ii}^*(w) \left(\sum_{l=1}^{i} a_{li}(w)a_{li}^*(z)\right) \lfloor a_{ii}a_{jj}^* \rfloor \\
& + \delta_{ij} : a_{ii}^*(w)a_{ii}^*(w) \left(\sum_{l=1}^{i} a_{li}(z)a_{li}^*(z)\right) \lfloor a_{ii}^*a_{jj} \rfloor \\
& + \delta_{ij} : a_{ii}^*(z)a_{ii}^*(w) \left(\sum_{l=1}^{i} a_{li}(z)a_{li}^*(w)\right) \lfloor a_{ii}^*a_{jj} \rfloor \\
& + 3\delta_{ij} : a_{ii}^*(z)a_{ii}^*(w) : \lfloor a_{ii}a_{jj}^* \rfloor \lfloor a_{ii}^*a_{jj} \rfloor \\
& + \delta_{ij} : a_{ii}^*(z)a_{ii}^*(w) : \left(\sum_{l=1}^{i} \lfloor a_{li}a_{lj}^* \rfloor \lfloor a_{li}^*a_{lj} \rfloor\right).
\end{aligned}$$

Hence

$$\Big[: a_{ii}^*\Big(\sum_{k=1}^{i} a_{ki}a_{ki}^*\Big) :, : a_{jj}^* \left(\sum_{l=1}^{j} a_{lj}a_{lj}^*\right) : \Big]$$

$$= (3+i)\delta_{1\le i\le r}\delta_{ij} : a_{ii}^*(z)a_{ii}^*(w) : \partial_w\delta(z-w).$$

Next we have

$$\text{(h)} \qquad \sum_{l=j+1}^{n} \Big[: a_{ii}^* \left(\sum_{k=1}^{i-1} a_{k,i-1}a_{k,i-1}^* - \sum_{k=1}^{i} a_{ki}a_{ki}^*\right) :, a_{j+1,l}(w)a_{jl}^*(w)\Big]$$

$$\begin{aligned}
&= -\delta_{j,i-1} : a_{i-1,i}^*(w) \left(\sum_{k=1}^{i-1} a_{k,i-1}(z)a_{k,i-1}^*(z) - \sum_{k=1}^{i} a_{ki}(z)a_{ki}^*(z) :\right) \delta(z-w) \\
&- \delta_{j+1\le i-1}\, a_{ii}a_{j+1,i-1}(z)a_{j,i-1}^*(z)\delta(z-w) \\
&+ \delta_{j+1\le i}\, a_{ii}a_{j+1,i}(z)a_{j,i}^*(z)\delta(z-w) \\
&+ \delta_{j+1\le i-1}\, a_{ii}a_{j+1,i-1}(z)a_{j,i-1}^*(z)\delta(z-w) \\
&- \delta_{j+1\le i}\, a_{ii}a_{j+1,i}(z)a_{j,i}^*(z)\delta(z-w) \\
&= -\delta_{j,i-1} : a_{i-1,i}^*(w) \left(\sum_{k=1}^{i-1} a_{k,i-1}(z)a_{k,i-1}^*(z) - \sum_{k=1}^{i} a_{ki}(z)a_{ki}^*(z) :\right) \delta(z-w).
\end{aligned}$$

□

6. PROOF OF THEOREM 3.1

We can now check the defining relations.

Lemma 6.1 (R1)**.**

$$[\rho(H_i)(z),\rho(H_j)(w)]=(\alpha_i|\alpha_j)\rho(c)\partial_w\delta(z-w).$$

Proof. We use Lemma 5.1 in the following calculations:

$$\begin{aligned}
[\rho(H_i)(z),\rho(H_j)(w)]&=[\mathcal{H}_i(z)+b_i(z),\ \ \mathcal{H}_j(z)+b_j(z)]\\
&=\Bigg(-(\alpha_i|\alpha_j)\left((1-\delta_{i>r}\delta_{j>r})(r+1)+\frac{r}{2}\delta_{i,r+1}\delta_{j,r+1}\right)\\
&\qquad+(\alpha_i|\alpha_j)\left(\gamma^2-\delta_{i>r}\delta_{j>r}(r+1)+\frac{r}{2}\delta_{i,r+1}\delta_{j,r+1})\right)\Bigg)\partial_w\delta(z-w)\\
&=(\alpha_i|\alpha_j)\rho(c)\partial_w\delta(z-w).
\end{aligned}$$

□

Lemma 6.2 (R2)**.**

$$[\rho(H_i)(z),\rho(E_j)(w)]=(\alpha_i|,\alpha_j)\rho(E_j)(z)\delta(z/w).$$

Proof. We will use Lemma 5.1 repeatedly:

$$\begin{aligned}
&[\mathcal{H}_i(z), \rho(E_j)(w)] = [\mathcal{H}_i(z), \ : a^*_{jj}(\sum_{k=1}^{j-1} a_{k,j-1}a^*_{k,j-1} - \sum_{k=1}^{j} a_{kj}a^*_{kj}) : \\
&\quad + \sum_{k=j+1}^{n} a_{j+1,k}a^*_{jk} - \sum_{k=1}^{j-1} a_{k,j-1}a^*_{kj} - a^*_{jj}b_j - \left(\delta_{j>r}(r+1) + \delta_{j\le r}(j+1) - \gamma^2\right)\partial a^*_{jj}] \\
&= \sum_{k=1}^{j-1}[\mathcal{H}_i(z), \ : a^*_{jj}a_{k,j-1}a^*_{k,j-1} :] - \sum_{k=1}^{j}[\mathcal{H}_i(z), \ : a^*_{jj}a_{kj}a^*_{kj} :] \\
&\quad + \sum_{k=j+1}^{n}[\mathcal{H}_i(z), \ a_{j+1,k}a^*_{jk}] - \sum_{k=1}^{j-1}[\mathcal{H}_i(z), \ a_{k,j-1}a^*_{kj}] \\
&\quad - [\mathcal{H}_i(z), \ a^*_{jj}]b_j - \left(\delta_{j>r}(r+1) + \delta_{j\le r}(j+1) - \gamma^2\right)[\mathcal{H}_i(z), \ \partial_w a^*_{jj}(w)] \\
&= \sum_{k=1}^{j-1}\left((\alpha_i|\alpha_j) : a^*_{jj}a_{k,j-1}a^*_{k,j-1} : \delta(z-w) - \delta_{1\le j-1\le r}(\alpha_i|\alpha_k + \cdots + \alpha_{j-1})a^*_{jj}(w)\partial_w\delta(z-w)\right) \\
&\quad - \sum_{k=1}^{j}\left((\alpha_i|\alpha_j) : a^*_{jj}a_{kj}a^*_{kj} : \delta(z-w) - \delta_{1\le j\le r}(\alpha_i|\alpha_k + \cdots + \alpha_j)(\delta_{jk}a^*_{kk}(w) + a^*_{jj}(w))\partial_w\delta(z-w)\right) \\
&\quad + (\alpha_i|\alpha_j)\sum_{k=j+1}^{n} a_{j+1,k}(z)a^*_{jk}(w)\delta(z-w) - (\alpha_i|\alpha_j)\sum_{k=1}^{j-1} a_{k,j-1}(z)a^*_{kj}(w)\delta(z-w) \\
&\quad - (\alpha_i|\alpha_j)a^*_{jj}(z)b_j(w)\delta(z-w) \\
&\quad - (\alpha_i|\alpha_j)\left(\delta_{j>r}(r+1) + \delta_{j\le r}(j+1) - \gamma^2\right)a^*_{jj}(z)\partial_w\delta(z-w) \\
&= (\alpha_i|\alpha_j)\Bigg(: a^*_{jj}(\sum_{k=1}^{j-1} a_{k,j-1}a^*_{k,j-1} - \sum_{k=1}^{j} a_{kj}a^*_{kj}) : \\
&\quad + \sum_{k=j+1}^{n} a_{j+1,k}a^*_{jk} - \sum_{k=1}^{j-1} a_{k,j-1}a^*_{kj} - a^*_{jj}b_j\Bigg)\delta(z-w) \\
&\quad + \sum_{k=1}^{j}\delta_{1\le j\le r}(\alpha_i|\alpha_k + \cdots + \alpha_j)(\delta_{jk}a^*_{kk}(w) + a^*_{jj}(w))\partial_w\delta(z-w) \\
&\quad - \sum_{k=1}^{j-1}\left(\delta_{1\le j-1\le r}(\alpha_i|\alpha_k + \cdots + \alpha_{j-1})a^*_{jj}(w)\partial_w\delta(z-w)\right) \\
&\quad - (\alpha_i|\alpha_j)(\delta_{j>r}(r+1) + \delta_{j\le r}(j+1) - \gamma^2)a^*_{jj}(z)\partial_w\delta(z-w).
\end{aligned}$$

The last term of $\rho(H_i)(z)$ gives us

$$[b_i(z), \ \rho(E_j)(w)] = -\mathfrak{B}_{ij}a_{jj}(w)\partial_w\delta(z-w).$$

There are three cases to consider:
Case I. $j \le r$: Then

$$
\begin{aligned}
&\sum_{k=1}^{j}(\alpha_i|\alpha_k+\cdots+\alpha_j)(\delta_{jk}a^*_{kk}(w)+a^*_{jj}(w))\partial_w\delta(z-w)\\
&-\sum_{k=1}^{j-1}(\alpha_i|\alpha_k+\cdots+\alpha_{j-1})a^*_{jj}(w)\partial_w\delta(z-w)\\
&-(\alpha_i|\alpha_j)\left(\delta_{j>r}(r+1)+\delta_{j\le r}(j+1)-\gamma^2\right)a^*_{jj}(z)\partial_w\delta(z-w)\\
&-(\alpha_i|\alpha_j)(\gamma^2-\delta_{i>r}\delta_{j>r}(r+1)+\frac{r}{2}\delta_{i,r+1}\delta_{j,r+1})a_{jj}(w)\partial_w\delta(z-w)\\
&=(j+1)(\alpha_i|\alpha_j)a^*_{jj}(w)\partial_w\delta(z-w)\\
&-(\alpha_i|\alpha_j)(j+1-\gamma^2)a^*_{jj}(z)\partial_w\delta(z-w)\\
&-(\alpha_i|\alpha_j)\gamma^2a_{jj}(w)\partial_w\delta(z-w)\\
&=-(\alpha_i|\alpha_j)(j+1-\gamma^2)\partial_wa_{jj}(w)\delta(z-w)
\end{aligned}
$$

by Lemma 5.2.
Case II: $j=r+1$:

$$
\begin{aligned}
&-\sum_{k=1}^{r}\left((\alpha_i|\alpha_k+\cdots+\alpha_r)a^*_{r+1,r+1}(w)\partial_w\delta(z-w)\right)\\
&-(\alpha_i|\alpha_{r+1})\left((r+1)-\gamma^2\right)a^*_{r+1,r+1}(z)\partial_w\delta(z-w)\\
&-(\alpha_i|\alpha_{r+1})(\gamma^2-\delta_{i>r}(r+1)+\frac{r}{2}\delta_{i,r+1})a_{r+1,r+1}(w)\partial_w\delta(z-w)\\
&=-(\alpha_i|\alpha_{r+1})\left((r+1)-\gamma^2\right)(\partial_wa^*_{r+1,r+1}(w))\delta(z-w)
\end{aligned}
$$

which follows from

$$
\begin{aligned}
&-\sum_{k=1}^{r}(\alpha_i|\alpha_k+\cdots+\alpha_r)+(\alpha_i|\alpha_{r+1})(\delta_{i>r}(r+1)-\frac{r}{2}\delta_{i,r+1})\\
&=\begin{cases}-2 & \text{if}\quad 1=i=r\\ 0 & \text{if}\quad 1\le i<r\\ -(r+1) & \text{if}\quad 1<i=r\\ 2(r+1) & \text{if}\quad 1\le i=r+1\\ (\alpha_i|\alpha_{r+1})(r+1) & \text{if}\quad i>r+1\end{cases}\\
&=(\alpha_i|\alpha_{r+1})(r+1)
\end{aligned}
$$

Cases III: $j>r+1$:

$$
\begin{aligned}
&-(\alpha_i|\alpha_j)\left(r+1-\gamma^2\right)a^*_{jj}(z)\partial_w\delta(z-w)\\
&-\mathfrak{B}_{ij}a_{jj}(w)\partial_w\delta(z-w)\\
&=-(\alpha_i|\alpha_j)\left(r+1-\gamma^2\right)a^*_{jj}(z)\partial_w\delta(z-w)+(\alpha_i|\alpha_j)(\gamma^2-\delta_{i>r}(r+1))a_{jj}(w)\partial_w\delta(z-w)\\
&=-(\alpha_i|\alpha_j)\left(r+1-\gamma^2\right)(\partial_wa^*_{jj}(w))\delta(z-w)
\end{aligned}
$$

by Lemma 5.2 and the fact that $(\alpha_i|\alpha_j)=0$ for $i\le r<r+1<j$.

Putting these computations together we get

$$[\rho(H_i)(z),\rho(E_j)(w)] = (\alpha_i|\alpha_j)\rho(E_j)(w)\delta(z-w).$$

□

Lemma 6.3 (R3)**.**

$$[\rho(H_i)(z),\rho(F_j)(w)] = -(\alpha_i|\alpha_j)\rho(F_j)(z)\delta(z-w).$$

Proof. The proof follows from Lemma 5.3 :

$$\begin{aligned}
[\rho(H_i)(z),\rho(F_j)(w)] &= [\mathcal{H}_i(z),\rho(F_j)(w)] \\
&= [\mathcal{H}_i(z), a_{jj}(w) + \sum_{k=j+1}^{n} a_{jk}(w)a^*_{j+1,k}(w)] \\
&= -(\alpha_i|\alpha_j)a_{jj}(w)\delta(z-w) + \sum_{k=j+1}^{n}[\mathcal{H}_i(z),a_{jk}(w)]a^*_{j+1,k}(w) + \sum_{k=j+1}^{n} a_{jk}(w)[\mathcal{H}_i(z),a^*_{j+1,k}(w)] \\
&= \Big(-(\alpha_i|\alpha_j)a_{jj}(w) - \sum_{k=j+1}^{n}(\alpha_i|\alpha_j+\cdots+\alpha_k)a_{jk}(z)a^*_{j+1,k}(w) \\
&\qquad + \sum_{k=j+1}^{n}(\alpha_i|\alpha_{j+1}+\cdots+\alpha_l)a_{jk}(w)a^*_{j+1,k}(z)\Big)\delta(z-w) \\
&= -(\alpha_i|\alpha_j)\rho(F_j)(z)\delta(z-w)
\end{aligned}$$

□

Lemma 6.4 (R4)**.**

$$[\rho(E_i)(z),\rho(F_j)(w)] = \delta_{i,j}(\rho(H_i)(z))\delta(z-w) + \rho(c)\partial_w\delta(z-w))$$

Proof. First we take $i=j$. Now for the convenience of the reader we recall that $\rho(E_i)(z)$ is equal to

$$\begin{aligned}
&: a^*_{ii}\left(\sum_{k=1}^{i-1} a_{k,i-1}a^*_{k,i-1} - \sum_{k=1}^{i} a_{ki}a^*_{ki}\right) : + \sum_{k=i+1}^{n} a_{i+1,k}a^*_{ik} - \sum_{k=1}^{i-1} a_{k,i-1}a^*_{ki} \\
&- a^*_{ii}b_i - \left(\delta_{i>r}(r+1) + \delta_{i\le r}(i+1) - \gamma^2\right)\partial a^*_{ii}
\end{aligned}$$

and thus the first summand of $\rho(F_i)(w) = a_{ii} + \sum_{l=i+1}^{n} a_{il}a^*_{i+1,l}$ brackets with $\rho(E_i)(z)$ to give us

$$\begin{aligned}
&\Big(2 : a_{ii}(z)a^*_{ii}(z) : - : \sum_{k=1}^{i-1}\left(a_{k,i-1}a^*_{k,i-1} - a_{ki}a^*_{ki}\right) : + b_i(z)\Big)\delta(z-w) \\
&+ \left(\delta_{i>r}(r+1) + \delta_{i\le r}(i+1) - \gamma^2\right)\partial_z\delta(z-w).
\end{aligned}$$

The second summation in $\rho(F_i)(w)$ contributes

$$\begin{aligned}\sum_{l=i+1}^{n}\Big[\rho(E_i)(z),\ a_{il}(w)a^*_{i+1,l}(w)\Big] &= \sum_{l=i+1}^{n}\Big[\sum_{k=i+1}^{n} a_{i+1,k}(z)a^*_{ik}(z), a_{il}(w)a^*_{i+1,l}(w)\Big]\\ &= \sum_{l=i+1}^{n}\Big(a_{il}(z)a^*_{il}(z) - a_{i+1,l}(z)a^*_{i+1,l}(z)\Big)\delta(z-w)\\ &\quad - \delta_{i+1\leq r}(r-i)\partial_w\delta(z-w).\end{aligned}$$

Adding these two summations up, we arrive at the desired result.

Now consider the case $|i-j|\geq 1$. Then $\rho(F_j)(w)$ acts by $a_{jj}+\sum_{l=j+1}^{n} a_{jl}a^*_{j+1,l}$. First we have

$$\begin{aligned}[E_i(z), a_{jj}(w)] &= \Big[:a^*_{ii}\left(\sum_{k=1}^{i-1} a_{k,i-1}a^*_{k,i-1} - \sum_{k=1}^{i} a_{ki}a^*_{ki}\right): + \sum_{k=i+1}^{n} a_{i+1,k}a^*_{ik} - \sum_{k=1}^{i-1} a_{k,i-1}a^*_{ki}\\ &= \Big[:a^*_{ii}\left(\sum_{k=1}^{i-1} a_{k,i-1}a^*_{k,i-1} - \sum_{k=1}^{i} a_{ki}a^*_{ki}\right):, a_{jj}(w)\Big]\\ &= -\delta_{j,i-1}a_{i-1,i-1}(z)a^*_{i,i}(z)\delta(z-w)\end{aligned}$$

by Lemma 5.1 (d). Next we have

$$\begin{aligned}[E_i(z), \sum_{l=j+1}^{n} a_{jl}(w)a^*_{j+1,l}(w)] &= \Big[:a^*_{ii}\left(\sum_{k=1}^{i-1} a_{k,i-1}a^*_{k,i-1} - \sum_{k=1}^{i} a_{ki}a^*_{ki}\right): + \sum_{k=i+1}^{n} a_{i+1,k}a^*_{ik} - \sum_{k=1}^{i-1} a_{k,i-1}a^*_{ki}\\ &\qquad - a^*_{ii}b_i + \left(\gamma^2 - \delta_{i+1\leq r}(i+1)\right)\partial a^*_{ii}, \sum_{l=j+1}^{n} a_{jl}(w)a^*_{j+1,l}(w)\Big]\\ &= \Big[:a^*_{ii}\left(\sum_{k=1}^{i-1} a_{k,i-1}a^*_{k,i-1} - \sum_{k=1}^{i} a_{ki}a^*_{ki}\right): - \sum_{k=1}^{i-1} a_{k,i-1}a^*_{ki}, \sum_{l=j+1}^{n} a_{jl}(w)a^*_{j+1,l}(w)\Big]\\ &\qquad\qquad \text{by Lemma 5.1 (a)}\\ &= \delta_{j,i-1}a_{i-1,i-1}(z)a^*_{ii}(z)\delta(z-w)\\ &\qquad\qquad \text{by Lemma 5.1 (b) and (c).}\end{aligned}$$

Adding the last two calculations up finishes the proof of this lemma.

□

We are now left with the Serre relations:

Lemma 6.5 (R5/R6)**.**

$$\begin{gathered}[\rho(F_i)(z), \rho(F_j)(w)] = [\rho(E_i)(z), \rho(E_j)(w)] = 0 \quad \textit{if} \quad (\alpha_i|\alpha_j)\neq -1\\ [\rho(F_i)(z_1), \rho(F_i)(z_2), \rho(F_j)(w)] = [\rho(E_i)(z_1), \rho(E_i)(z_2), \rho(E_j)(w)] = 0 \quad \textit{if} \quad (\alpha_i|\alpha_j) = -1.\end{gathered}$$

Proof. Let us check the relations for $\rho(F_i)$. (The Serre relations were already known to hold true for the F_i, see [FF90b], but we provide a proof for the convenience of the reader.) A straight forward calculation shows

$$\Big[a_{ii}(z),\ a_{jj}+\sum_{l=j+1}^{n}a_{jl}a^*_{j+1,l}\Big]=\delta_{i,j+1}a_{j,j+1}(w)\delta(z-w).$$

Moreover

$$\begin{aligned}\Big[&\sum_{k=i+1}^{n}a_{ik}(z)a^*_{i+1,k}(z),\ a_{jj}(w)+\sum_{l=j+1}^{n}a_{jl}(w)a^*_{j+1,l}(w)\Big]\\&=\Big(\delta_{i,j+1}\sum_{k=j+2}^{n}a_{jk}(z)a^*_{j+2,k}(z)-\delta_{j,i+1}\sum_{k=i+2}^{n}a_{ik}(z)a^*_{i+2,k}(z)-\delta_{j,i+1}a_{i,i+1}(z)\Big)\delta(z-w)\end{aligned}$$

by Lemma 5.3 (a). Thus

$$\begin{aligned}[\rho(F_i)(z),\rho(F_j)(w)]&=\left(\delta_{i,j+1}a_{j,j+1}(w)-\delta_{j,i+1}a_{i,i+1}(z)\right)\delta(z-w)\\&\quad+\left(\delta_{i,j+1}\sum_{k=j+2}^{n}a_{jk}(z)a^*_{j+2,k}(z)-\delta_{j,i+1}\sum_{k=i+2}^{n}a_{ik}(z)a^*_{i+2,k}(z)\right)\delta(z-w).\end{aligned}\tag{6.1}$$

Note the above is zero if $|i-j|\neq 1$ which is precisely when $(\alpha_i|\alpha_j)\neq -1$. As the first index in a_{kl} (resp. a^*_{kl}) in $\rho(F_i)(z)$ is i (resp. $i+1$) we also get

$$[\rho(F_i)(z_1),\rho(F_i)(z_1),\rho(F_j)(w)]=0.$$

This completes the proof of the relations R5 and R6 for $\rho(F_i)(z)$.

Now we break up $\rho(E_i)(z)$ into three summands

$$\begin{aligned}\rho(E_i^1)(z)&:=:a^*_{ii}\left(\sum_{k=1}^{i-1}a_{k,i-1}a^*_{k,i-1}-\sum_{k=1}^{i}a_{ki}a^*_{ki}\right):\\\rho(E_i^2)(z)&:=\sum_{k=i+1}^{n}a_{i+1,k}a^*_{ik}-\sum_{k=1}^{i-1}a_{k,i-1}a^*_{ki}\\\rho(E_i^3)(z)&:=-a^*_{ii}b_i-\left(\delta_{i>r}(r+1)+\delta_{i\leq r}(i+1)-\gamma^2\right)\partial a^*_{ii}.\end{aligned}$$

By Lemma 5.3 (e), (f) and (g) we have

$$\begin{aligned}
&\Big[\rho(E^1_i)(z),\rho(E^1_j)(w)\Big] \\
&= \delta_{j,i-1} : a^*_{ii}(z)a^*_{i-1,i-1}(z)\left(\sum_{k=1}^{i-2} a_{k,i-2}(w)a^*_{k,i-2}(w)\right) : \delta(z-w) \\
&- \delta_{i,j-1} : a^*_{jj}(w)a^*_{j-1,j-1}(w)\left(\sum_{l=1}^{j-2} a_{l,j-2}(z)a^*_{l,j-2}(z)\right)\delta(z-w) \\
&- \delta_{j,i-1} : a^*_{ii}(z)a^*_{i-1,i-1}(z)\left(\sum_{k=1}^{i-1} a_{k,i-1}(w)a^*_{k,i-1}(w)\right) : \delta(z-w) \\
&+ \delta_{i,j-1} : a^*_{jj}(z)a^*_{j-1,j-1}(z)\left(\sum_{k=1}^{j-1} a_{k,j-1}(w)a^*_{k,j-1}(w)\right) : \delta(z-w) \\
&- (i-1)\delta_{1\le i-1\le r}\delta_{ij} : a^*_{ii}(z)a^*_{ii}(w) : \partial_w\delta(z-w) \\
&+ i\delta_{1\le i-1\le r}\delta_{j,i-1} : a^*_{ii}(z)a^*_{i-1,i-1}(w) : \partial_w\delta(z-w) \\
&- j\delta_{1\le j-1\le r}\delta_{i,j-1} : a^*_{jj}(w)a^*_{j-1,j-1}(z) : \partial_z\delta(z-w) \\
&- (3+i)\delta_{1\le i\le r}\delta_{ij} : a^*_{ii}(z)a^*_{ii}(w) : \partial_w\delta(z-w).
\end{aligned}$$

By Lemma 5.3 (h) and (i),

$$\begin{aligned}
&\Big[\rho(E_i^1)(z),\rho(E_j^2)(w)\Big]+\Big[\rho(E_i^2)(z),\rho(E_j^1)(w)\Big]\\
&=-\delta_{j,i-1}:a_{i-1,i}^*(w)\left(\sum_{k=1}^{i-1}a_{k,i-1}(z)a_{k,i-1}^*(z)-\sum_{k=1}^{i}a_{ki}(z)a_{ki}^*(z):\right)\delta(z-w)\\
&\quad+\delta_{i,j-1}:a_{j-1,j}^*(z)\left(\sum_{k=1}^{j-1}a_{k,j-1}(w)a_{k,j-1}^*(w)-\sum_{k=1}^{j}a_{kj}a_{kj}^*:\right)\delta(z-w)\\
&\quad-\Bigg(:a_{ii}^*\left(\delta_{i-1,j}\sum_{l=1}^{i-2}a_{l,i-2}a_{l,i-1}^*+\delta_{i,j-1}\sum_{l=1}^{i}a_{l,i}a_{l,i+1}^*\right)\\
&\quad-\delta_{i,j-1}a_{i,i+1}^*\left(\sum_{l=1}^{i-1}a_{l,i-1}a_{l,i-1}^*-\sum_{l=1}^{i}a_{li}a_{li}^*\right):\Bigg)\delta(z-w)\\
&\quad+\Bigg(:a_{jj}^*\left(\delta_{j-1,i}\sum_{l=1}^{j-2}a_{l,j-2}a_{l,j-1}^*+\delta_{j,i-1}\sum_{l=1}^{j}a_{l,j}a_{l,j+1}^*\right)\\
&\quad-\delta_{j,i-1}a_{j,j+1}^*\left(\sum_{l=1}^{j-1}a_{l,j-1}a_{l,j-1}^*-\sum_{l=1}^{j}a_{lj}a_{lj}^*\right):\Bigg)\delta(z-w)\\
&\quad+a_{i-1,i}^*(z)\partial_w\delta(z-w)\\
&=-\delta_{j,i-1}:a_{i-1,i}^*\left(\sum_{k=1}^{i-2}a_{k,i-2}a_{k,i-2}^*-\sum_{k=1}^{i}a_{ki}a_{ki}^*:\right)\delta(z-w)\\
&\quad+\delta_{i,j-1}:a_{i,i+1}^*\left(\sum_{k=1}^{i-1}a_{k,i-1}a_{k,i-1}^*-\sum_{k=1}^{i+1}a_{k,i+1}a_{k,i+1}^*:\right)\delta(z-w)\\
&\quad-:a_{ii}^*\left(\delta_{i-1,j}\sum_{l=1}^{i-2}a_{l,i-2}a_{l,i-1}^*+\delta_{i,j-1}\sum_{l=1}^{i}a_{l,i}a_{l,i+1}^*\right)\delta(z-w)\\
&\quad+:a_{jj}^*\left(\delta_{j-1,i}\sum_{l=1}^{j-2}a_{l,j-2}a_{l,j-1}^*+\delta_{j,i-1}\sum_{l=1}^{j}a_{l,j}a_{l,j+1}^*\right)\delta(z-w)\\
&\quad+\delta_{i\le r}\delta_{j,i-1}a_{i-1,i}^*(z)\partial_w\delta(z-w)\\
&\quad-\delta_{j\le r}\delta_{i,j-1}a_{j-1,j}^*(w)\partial_z\delta(z-w).
\end{aligned}$$

Similarly

$$\begin{aligned}
&\Big[\rho(E_i^1)(z)\rho(E_j^3)(w)\Big]\\
&=\Big[:a_{ii}^*\left(\sum_{k=1}^{i-1}a_{k,i-1}a_{k,i-1}^*-\sum_{k=1}^{i}a_{ki}a_{ki}^*\right):,-a_{jj}^*b_j-\left(\delta_{j>r}(r+1)+\delta_{j\le r}(j+1)-\gamma^2\right)\partial a_{jj}^*\Big]\\
&=\left(-\delta_{i-1,j}a_{ii}^*a_{i-1,i-1}^*+\delta_{ij}a_{ii}^*a_{ii}^*\right)b_j\delta(z-w)\\
&+\left(\delta_{j>r}(r+1)+\delta_{j\le r}(j+1)-\gamma^2\right)\left(-\delta_{i-1,j}a_{ii}^*(z)a_{i-1,i-1}^*(z)+\delta_{ij}a_{ii}^*(z)a_{ii}^*(z)\right)\partial_w\delta(z-w)
\end{aligned}$$

so that

$$\begin{aligned}
&\Big[\rho(E_i^1)(z)\rho(E_j^3)(w)\Big] + \Big[\rho(E_i^3)(z), \rho(E_j^1)(w)\Big] \\
&= \left(-\delta_{i-1,j}a_{ii}^*a_{i-1,i-1}^*b_j + \delta_{j-1,i}a_{jj}^*a_{j-1,j-1}^*b_i\right)\delta(z-w) \\
&+ \left(\delta_{j>r}(r+1) + \delta_{j\leq r}(j+1) - \gamma^2\right)\left(-\delta_{i-1,j}a_{ii}^*(z)a_{i-1,i-1}^*(z) + \delta_{ij}a_{ii}^*(z)a_{ii}^*(z)\right)\partial_w\delta(z-w) \\
&- \left(\delta_{i>r}(r+1) + \delta_{i\leq r}(i+1) - \gamma^2\right)\left(-\delta_{j-1,i}a_{jj}^*(w)a_{j-1,j-1}^*(w) + \delta_{ij}a_{jj}^*(w)a_{jj}^*(w)\right)\partial_z\delta(z-w).
\end{aligned}$$

By Lemma 5.3 (j), (k) and (l) we have

$$\begin{aligned}
&\Big[\rho(E_i^2)(z), \rho(E_j^2)(w)\Big] \\
&\qquad = \left(\delta_{j,i-1}\sum_{l=1}^{i-2} : a_{l,i-2}a_{li}^* : -\delta_{i,j-1}\sum_{l=1}^{j-2} : a_{l,j-2}a_{lj}^* :\right)\delta(z-w) \\
&\qquad + \left(\delta_{i,j-1}\sum_{l=i+2}^{n} : a_{i+2,l}a_{il}^* : -\delta_{j,i-1}\sum_{l=j+2}^{n} : a_{j+2,k}a_{jk}^* :\right)\delta(z-w).
\end{aligned}$$

Next we have

$$\begin{aligned}
&\Big[\rho(E_i^2)(z), \rho(E_j^3)(w)\Big] \\
&= \Big[\sum_{k=i+1}^{n} a_{i+1,k}a_{ik}^* - \sum_{k=1}^{i-1} a_{k,i-1}a_{ki}^*, -a_{jj}^*b_j - \left(\delta_{j>r}(r+1) + \delta_{j\leq r}(j+1) - \gamma^2\right)\partial a_{jj}^*\Big] \\
&= \left(-\delta_{j,i+1}a_{i,i+1}^*b_{i+1} + \delta_{j,i-1}a_{i-1,i}^*b_{i-1}\right)\delta(z-w) \\
&\quad - \left(\delta_{j>r}(r+1) + \delta_{j\leq r}(j+1) - \gamma^2\right)\left(\delta_{j,i+1}a_{i,i+1}^*(z) - \delta_{j,i-1}a_{i-1,i}^*(z)\right)\partial_w\delta(z-w)
\end{aligned}$$

so that

$$\begin{aligned}
&\Big[\rho(E_i^2)(z), \rho(E_j^3)(w)\Big] + \Big[\rho(E_i^3)(z), \rho(E_j^2)(w)\Big] \\
&= \left(-\delta_{j,i+1}a_{i,i+1}^*b_{i+1} + \delta_{j,i-1}a_{i-1,i}^*b_{i-1}\right)\delta(z-w) \\
&\quad - \left(\delta_{j>r}(r+1) + \delta_{j\leq r}(j+1) - \gamma^2\right)\left(\delta_{j,i+1}a_{i,i+1}^*(z) - \delta_{j,i-1}a_{i-1,i}^*(z)\right)\partial_w\delta(z-w) \\
&- \left(-\delta_{i,j+1}a_{j,j+1}^*b_{j+1} + \delta_{i,j-1}a_{j-1,j}^*b_{j-1}\right)\delta(z-w) \\
&\quad + \left(\delta_{i>r}(r+1) + \delta_{i\leq r}(i+1) - \gamma^2\right)\left(\delta_{i,j+1}a_{j,j+1}^*(w) - \delta_{i,j-1}a_{j-1,j}^*(w)\right)\partial_z\delta(z-w).
\end{aligned}$$

Lastly we have

$$\begin{aligned}
&\Big[\rho(E_i^3)(z),\ \ \rho(E_j^3)(w)\Big] \\
&= \Big[-a_{ii}^*b_i - \left(\delta_{i>r}(r+1) + \delta_{i\leq r}(i+1) - \gamma^2\right)\partial a_{ii}^*, -a_{jj}^*b_j - \left(\delta_{j>r}(r+1) + \delta_{j\leq r}(j+1) - \gamma^2\right)\partial a_{jj}^*\Big] \\
&= a_{ii}^*a_{jj}^*(\alpha_i|\alpha_j)\left(\gamma^2 - \delta_{i>r}\delta_{j>r}(r+1) + \frac{r}{2}\delta_{i,r+1}\delta_{j,r+1}\right)\partial_w\delta(z-w).
\end{aligned}$$

Now we add to obtain

$$\begin{aligned}
&\Big[\rho(E_i)(z),\ \ \rho(E_j)(w)\Big]\\
&= \delta_{j,i-1} : a^*_{ii}(z)a^*_{i-1,i-1}(z)\left(\sum_{k=1}^{i-2} a_{k,i-2}(w)a^*_{k,i-2}(w)\right) : \delta(z-w)\\
&- \delta_{i,j-1} : a^*_{jj}(w)a^*_{j-1,j-1}(w)\left(\sum_{l=1}^{j-2} a_{l,j-2}(z)a^*_{l,j-2}(z)\right)\delta(z-w)\\
&- \delta_{j,i-1} : a^*_{ii}(z)a^*_{i-1,i-1}(z)\left(\sum_{k=1}^{i-1} a_{k,i-1}(w)a^*_{k,i-1}(w)\right) : \delta(z-w)\\
&+ \delta_{i,j-1} : a^*_{jj}(z)a^*_{j-1,j-1}(z)\left(\sum_{k=1}^{j-1} a_{k,j-1}(w)a^*_{k,j-1}(w)\right) : \delta(z-w)\\
&- (i-1)\delta_{1\le i-1\le r}\delta_{ij} : a^*_{ii}(z)a^*_{ii}(w) : \partial_w\delta(z-w)\\
&+ i\delta_{1\le i-1\le r}\delta_{j,i-1} : a^*_{ii}(z)a^*_{i-1,i-1}(w) : \partial_w\delta(z-w)\\
&- j\delta_{1\le j-1\le r}\delta_{i,j-1} : a^*_{jj}(w)a^*_{j-1,j-1}(z) : \partial_z\delta(z-w)\\
&- (3+i)\delta_{1\le i\le r}\delta_{ij} : a^*_{ii}(z)a^*_{ii}(w) : \partial_w\delta(z-w)
\end{aligned}$$

$$\begin{aligned}
&- \delta_{j,i-1} : a^*_{i-1,i}\left(\sum_{k=1}^{i-2} a_{k,i-2}a^*_{k,i-2} - \sum_{k=1}^{i} a_{ki}a^*_{ki} :\right)\delta(z-w)\\
&+ \delta_{i,j-1} : a^*_{i,i+1}\left(\sum_{k=1}^{i-1} a_{k,i-1}a^*_{k,i-1} - \sum_{k=1}^{i+1} a_{kj}a^*_{k,i+1} :\right)\delta(z-w)\\
&- : a^*_{ii}\left(\delta_{i-1,j}\sum_{l=1}^{i-2} a_{l,i-2}a^*_{l,i-1} + \delta_{i,j-1}\sum_{l=1}^{i} a_{l,i}a^*_{l,i+1}\right)\delta(z-w)\\
&+ : a^*_{jj}\left(\delta_{j-1,i}\sum_{l=1}^{j-2} a_{l,j-2}a^*_{l,j-1} + \delta_{j,i-1}\sum_{l=1}^{j} a_{l,j}a^*_{l,j+1}\right)\delta(z-w)\\
&+ \delta_{j,i-1}a^*_{i-1,i}(z)\partial_w\delta(z-w)\\
&- \delta_{i,j-1}a^*_{j-1,j}(w)\partial_z\delta(z-w)
\end{aligned}$$

$$\begin{aligned}
&+ \left(-\delta_{i-1,j}a^*_{ii}a^*_{i-1,i-1}b_j + \delta_{j-1,i}a^*_{jj}a^*_{j-1,j-1}b_i\right)\delta(z-w)\\
&+ \left(\delta_{j>r}(r+1) + \delta_{j\le r}(j+1) - \gamma^2\right)\left(-\delta_{i-1,j}a^*_{ii}(z)a^*_{i-1,i-1}(z) + \delta_{ij}a^*_{ii}(z)a^*_{ii}(z)\right)\partial_w\delta(z-w)\\
&- \left(\delta_{i>r}(r+1) + \delta_{i\le r}(i+1) - \gamma^2\right)\left(-\delta_{j-1,i}a^*_{jj}(w)a^*_{j-1,j-1}(w) + \delta_{ij}a^*_{jj}(w)a^*_{jj}(w)\right)\partial_z\delta(z-w)
\end{aligned}$$

$$\begin{aligned}
&+ \left(\delta_{j,i-1}\sum_{l=1}^{i-2} : a_{l,i-2}a^*_{li} : -\delta_{i,j-1}\sum_{l=1}^{j-2} : a_{l,j-2}a^*_{lj} :\right)\delta(z-w)\\
&+ \left(\delta_{i,j-1}\sum_{l=i+2}^{n} : a_{i+2,l}a^*_{il} : -\delta_{j,i-1}\sum_{l=j+2}^{n} : a_{j+2,k}a^*_{jk} :\right)\delta(z-w)\\
&+ \left(-\delta_{j,i+1}a^*_{i,i+1}b_{i+1} + \delta_{j,i-1}a^*_{i-1,i}b_{i-1}\right)\delta(z-w)\\
&- \left(\delta_{j>r}(r+1) + \delta_{j\le r}(j+1) - \gamma^2\right)\left(\delta_{j,i+1}a^*_{i,i+1}(z) - \delta_{j,i-1}a^*_{i-1,i}(z)\right)\partial_w\delta(z-w)\\
&- \left(-\delta_{i,j+1}a^*_{j,j+1}b_{j+1} + \delta_{i,j-1}a^*_{j-1,j}b_{j-1}\right)\delta(z-w)\\
&+ \left(\delta_{i>r}(r+1) + \delta_{i\le r}(i+1) - \gamma^2\right)\left(\delta_{i,j+1}a^*_{j,j+1}(w) - \delta_{i,j-1}a^*_{j-1,j}(w)\right)\partial_z\delta(z-w)\\
&+ a^*_{ii}(z)a^*_{jj}(w)(\alpha_i|\alpha_j)\left(\gamma^2 - \delta_{i>r}\delta_{j>r}(r+1) + \frac{r}{2}\delta_{i,r+1}\delta_{j,r+1}\right)\partial_w\delta(z-w).
\end{aligned}$$

When $i=j$ this reduces to

$$\begin{aligned}
&\Big[\rho(E_i)(z),\ \ \rho(E_i)(w)\Big]\\
&=-(i-1)\delta_{1\le i-1\le r} :a_{ii}^*(z)a_{ii}^*(w): \partial_w\delta(z-w)\\
&-(3+i)\delta_{1\le i\le r} :a_{ii}^*(z)a_{ii}^*(w): \partial_w\delta(z-w)\\
&+\left(\delta_{i>r}(r+1)+\delta_{i\le r}(i+1)-\gamma^2\right)a_{ii}^*(z)a_{ii}^*(z)\partial_w\delta(z-w)\\
&-\left(\delta_{i>r}(r+1)+\delta_{i\le r}(i+1)-\gamma^2\right)a_{ii}^*(w)a_{ii}^*(w)\partial_z\delta(z-w)\\
&+2a_{ii}^*(z)a_{ii}^*(w)\left(\gamma^2+(\delta_{1\le i\le r}-1)(r+1)+\delta_{i,r+1}\frac{r}{2}\right)\partial_w\delta(z-w)\\
&=-\left((i-1)\delta_{1\le i-1\le r}+(3+i)\delta_{1\le i\le r}-\delta_{i,r+1}r\right):a_{ii}^*(z)a_{ii}^*(w):\partial_w\delta(z-w)\\
&+\delta_{i\le r}(i+1)\left(a_{ii}^*(z)a_{ii}^*(z)+a_{ii}^*(w)a_{ii}^*(w)\right)\partial_z\delta(z-w)\\
&+(r+1)\delta_{i>r}\left(a_{ii}^*(z)a_{ii}^*(z)+a_{ii}^*(w)a_{ii}^*(w)-2a_{ii}^*(z)a_{jj}^*(w)\right)\partial_w\delta(z-w)\\
&+\gamma^2\left(2a_{ii}^*(z)a_{jj}^*(w)-a_{ii}^*(z)a_{ii}^*(z)-a_{ii}^*(w)a_{ii}^*(w)\right)\partial_w\delta(z-w)=0.
\end{aligned}$$

This proves the result for $i\neq j\pm 1$.

If $i=j+1$ then we get

$$\begin{aligned}
&\Big[\rho(E_i)(z),\ \ \rho(E_{i-1})(w)\Big]\\
&=:a_{ii}^*(z)a_{i-1,i-1}^*(z)\left(\sum_{k=1}^{i-2}a_{k,i-2}(w)a_{k,i-2}^*(w)-\sum_{k=1}^{i-1}a_{k,i-1}(w)a_{k,i-1}^*(w)\right):\delta(z-w)\\
&-:a_{i-1,i}^*\left(\sum_{k=1}^{i-2}a_{k,i-2}a_{k,i-2}^*-\sum_{k=1}^{i}a_{ki}a_{ki}^*:\right)\delta(z-w)\\
&-:a_{ii}^*\left(\sum_{l=1}^{i-2}a_{l,i-2}a_{l,i-1}^*\right)\delta(z-w)+:a_{i-1,i-1}^*\left(\sum_{l=1}^{i-1}a_{l,i-1}a_{li}^*\right)\delta(z-w)\\
&+\left(\sum_{l=1}^{i-2}:a_{l,i-2}a_{li}^*:\right)\delta(z-w)-\left(\sum_{l=i+1}^{n}:a_{i+1,k}a_{i-1,k}^*:\right)\delta(z-w)\\
&\\
&-a_{ii}^*a_{i-1,i-1}^*b_{i-1}\delta(z-w)+a_{i-1,i}^*(b_{i-1}+b_i)\delta(z-w)\\
&+\left(\delta_{i>r}(r+1)+\delta_{i\le r}(i+1)-\gamma^2\right)\partial_w a_{i-1,i}^*(w)\delta(z-w)\\
&+a_{ii}^*(z)\partial_w a_{i-1,i-1}^*(w)\left(\gamma^2-\delta_{i-1\le r}i-\delta_{i-1>r}(r+1)\right)\delta(z-w).
\end{aligned}$$

Thus

$$\begin{aligned}
&\Big[\rho(E_i^1)(z_1),\ \ \rho(E_i)(z_2),\ \ \rho(E_{i-1})(w)\Big]\\
&=: a_{ii}^* a_{i-1,i}\left(\sum_{k=1}^{i-2} a_{k,i-2}a_{k,i-2}^* :\right)\delta(z_1-w)\delta(z_2-w)\\
&+ : a_{ii}^* a_{i-1,i}^*\left(\sum_{k=1}^{i} a_{ki}a_{ki}^* : -\sum_{k=1}^{i-1} a_{k,i-1}a_{k,i-1}^*\right)\delta(z_1-w)\delta(z_2-w)\\
&- : a_{ii}^* a_{i-1,i}^*\left(\sum_{k=1}^{i} a_{ki}a_{ki}^* :\right)\delta(z_1-w)\delta(z_2-w)\\
&+ i\delta_{1\le i-1\le r} : a_{ii}^*(z_1)a_{ii}^*(z_2)a_{i-1,i-1}^*(z_2) : \delta(z_2-w)\partial_{z_2}\delta(z_1-z_2)\\
&+ (i+2)\delta_{1\le i\le r}a_{ii}^*(z_1)a_{i-1,i}^*(z_2)\delta(z_2-w)\partial_{z_2}\delta(z_1-z_2)\\
&+ \delta_{1\le i\le r}a_{ii}^*(z_2)a_{i-1,i}^*(z_1)\delta(z_2-w)\partial_{z_2}\delta(z_1-z_2)\\
&- \delta_{1\le i-1\le r}a_{ii}^*(z_1)a_{i-1,i}^*(z_2)\delta(z_2-w)\partial_{z_2}\delta(z_1-z_2)\\
&- : a_{ii}^*\sum_{l=1}^{i-2} a_{l,i-2}a_{li}^* : \delta(z_1-w)\delta(z_2-w)\\
&- : a_{ii}^* a_{i-1,i}^* : (b_{i-1}+b_i)\delta(z_1-z_2)\delta(z_2-w)\\
&- \big(\delta_{i>r}(r+1)+\delta_{i\le r}(i+1)-\gamma^2\big) : a_{ii}^*(z_1)a_{i-1,i}^*(z_1) : \delta(z_2-w)\partial_w\delta(z_1-w)\\
&- \big(\gamma^2-\delta_{i-1\le r}i-\delta_{i-1>r}(r+1)\big) : a_{ii}^*(z_1)a_{ii}^*(z_1)\partial_w a_{i-1,i-1}^*(w) : \delta(z_1-z_2)\delta(z_2-w)\\
&+ \big(\gamma^2-\delta_{i-1\le r}i-\delta_{i-1>r}(r+1)\big) : a_{ii}^*(z_1)a_{ii}^*(z_2)a_{i-1,i-1}^*(z_1) : \partial_w\delta(z_1-w)\delta(z_2-w).
\end{aligned}$$

Next we have

$$\begin{aligned}
&\Big[\rho(E_i^2)(z_1),\ \ \rho(E_i)(z_2),\ \ \rho(E_{i-1})(w)\Big]\\
&= \Bigg(- : a_{ii}^*(z_2)a_{i-1,i}^*(z_2)\left(\sum_{k=1}^{i-2} a_{k,i-2}(w)a_{k,i-2}^*(w)\right) :\\
&\qquad + : a_{ii}^* a_{i-1,i}^*\left(\sum_{k=1}^{i-1} a_{k,i-1}a_{k,i-1}^*\right) : + : a_{i-1,i}^*\sum_{k=1}^{i-1} a_{k,i-1}a_{ki}^*\\
&\qquad + : a_{ii}^*\left(\sum_{l=1}^{i-2} a_{l,i-2}a_{l,i}^*\right) : - : a_{i-1,i}^*\left(\sum_{l=1}^{i-1} a_{l,i-1}a_{li}^*\right) + a_{ii}^* a_{i-1,i}^* b_{i-1}\Bigg)\delta(z_1-z_2)\delta(z_2-w)\\
&- \big(\gamma^2-\delta_{i-1\le r}i-\delta_{i-1>r}(r+1)\big) : a_{ii}^*(z_2)a_{i-1,i}^*(z_1) : \partial_w\delta(z_1-w)\delta(z_2-w).
\end{aligned}$$

The third summation contributes

$$\begin{aligned}
&\Big[\rho(E_i^3)(z_1),\ \ \rho(E_i)(z_2),\ \ \rho(E_{i-1})(w)\Big] \\
&=: a^*_{i-1,i}a^*_{ii}b_i\delta(z_1-z_2)\delta(z_2-w) \\
&+\left(\delta_{i>r}(r+1)+\delta_{i\leq r}(i+1)-\gamma^2\right)a^*_{ii}(z_2)a^*_{i-1,i}(z_2)\partial_{z_1}\delta(z_1-z_2)\delta(z_2-w) \\
&+\Bigg(-a^*_{ii}(z_1)a^*_{i-1,i}(z_2)\left(\gamma^2-\delta_{i>r}(r+1)-\delta_{i,r+1}\right) \\
&\quad -a^*_{ii}(z_1)a^*_{ii}(z_2)a^*_{i-1,i-1}(z_2)\left(\gamma^2-\delta_{i-1>r}(r+1)\right)\Bigg)\delta(z_2-w)\partial_{z_2}\delta(z_1-z_2).
\end{aligned}$$

Consequently

$$\begin{aligned}
&\Big[\rho(E_i)(z_1),\rho(E_i)(z_2),\ \ \rho(E_{i-1})(w)\Big]\\
&=: a_{ii}^*a_{i-1,i}\left(\sum_{k=1}^{i-2}a_{k,i-2}a_{k,i-2}^*:\right)\delta(z_1-w)\delta(z_2-w)\\
&+:a_{ii}^*a_{i-1,i}^*\left(\sum_{k=1}^{i}a_{ki}a_{ki}^*:-\sum_{k=1}^{i-1}a_{k,i-1}a_{k,i-1}^*\right)\delta(z_1-w)\delta(z_2-w)\\
&-:a_{ii}^*a_{i-1,i}^*\left(\sum_{k=1}^{i}a_{ki}a_{ki}^*:\right)\delta(z_1-w)\delta(z_2-w)\\
&+i\delta_{1\le i-1\le r}:a_{ii}^*(z_1)a_{ii}^*(z_2)a_{i-1,i-1}^*(z_2):\delta(z_2-w)\partial_{z_2}\delta(z_1-z_2)\\
&+(i+2)\delta_{1\le i\le r}a_{ii}^*(z_1)a_{i-1,i}^*(z_2)\delta(z_2-w)\partial_{z_2}\delta(z_1-z_2)\\
&+\delta_{1\le i\le r}a_{ii}^*(z_2)a_{i-1,i}^*(z_1)\delta(z_2-w)\partial_{z_2}\delta(z_1-z_2)\\
&-\delta_{1\le i-1\le r}a_{ii}^*(z_1)a_{i-1,i}^*(z_2)\delta(z_2-w)\partial_{z_2}\delta(z_1-z_2)\\
&-:a_{ii}^*\sum_{l=1}^{i-2}a_{l,i-2}a_{li}^*:\delta(z_1-w)\delta(z_2-w)\\
&-:a_{ii}^*a_{i-1,i}^*:(b_{i-1}+b_i)\delta(z_1-z_2)\delta(z_2-w)\\
&-\big(\delta_{i>r}(r+1)+\delta_{i\le r}(i+1)-\gamma^2\big):a_{ii}^*(z_1)a_{i-1,i}^*(z_1):\delta(z_2-w)\partial_w\delta(z_1-w)\\
&-\big(\gamma^2-\delta_{i-1\le r}i-\delta_{i-1>r}(r+1)\big):a_{ii}^*(z_1)a_{ii}^*(z_1)\partial_w a_{i-1,i-1}^*(w):\delta(z_1-z_2)\delta(z_2-w)\\
&+\big(\gamma^2-\delta_{i-1\le r}i-\delta_{i-1>r}(r+1)\big):a_{ii}^*(z_1)a_{ii}^*(z_2)a_{i-1,i-1}^*(z_1):\partial_w\delta(z_1-w)\delta(z_2-w)\\
&+\Biggl(-:a_{ii}^*(z_2)a_{i-1,i}^*(z_2)\left(\sum_{k=1}^{i-2}a_{k,i-2}(w)a_{k,i-2}^*(w)\right):\\
&\qquad+:a_{ii}^*a_{i-1,i}^*\left(\sum_{k=1}^{i-1}a_{k,i-1}a_{k,i-1}^*\right):+:a_{i-1,i}^*\sum_{k=1}^{i-1}a_{k,i-1}a_{ki}^*\\
&\qquad+:a_{ii}^*\left(\sum_{l=1}^{i-2}a_{l,i-2}a_{l,i}^*\right):-:a_{i-1,i}^*\left(\sum_{l=1}^{i-1}a_{l,i-1}a_{li}^*\right)+a_{ii}^*a_{i-1,i}^*b_{i-1}\Biggr)\delta(z_1-z_2)\delta(z_2-w)\\
&-\big(\gamma^2-\delta_{i-1\le r}i-\delta_{i-1>r}(r+1)\big):a_{ii}^*(z_2)a_{i-1,i}^*(z_1):\partial_w\delta(z_1-w)\delta(z_2-w)\\
&+:a_{i-1,i}^*a_{ii}^*b_i\delta(z_1-z_2)\delta(z_2-w)\\
&+\big(\delta_{i>r}(r+1)+\delta_{i\le r}(i+1)-\gamma^2\big)a_{ii}^*(z_2)a_{i-1,i}^*(z_2)\partial_{z_1}\delta(z_1-z_2)\delta(z_2-w)\\
&+\Biggl(-a_{ii}^*(z_1)a_{i-1,i}^*(z_2)\big(\gamma^2-\delta_{i>r}(r+1)-\delta_{i,r+1}\big)\\
&\qquad-a_{ii}^*(z_1)a_{ii}^*(z_2)a_{i-1,i-1}^*(z_2)\big(\gamma^2-\delta_{i-1>r}(r+1)\big)\Biggr)\delta(z_2-w)\partial_{z_2}\delta(z_1-z_2)
\end{aligned}$$

$$
\begin{aligned}
&= \Big(i\delta_{1\leq i-1\leq r} :a^*_{ii}(z_1)a^*_{ii}(z_2)a^*_{i-1,i-1}(z_2): \\
&- \big(\gamma^2 - \delta_{i-1\leq r} i - \delta_{i-1>r}(r+1)\big) :a^*_{ii}(z_1)a^*_{ii}(z_1)\partial_w a^*_{i-1,i-1}(w): \\
&+ \big(\gamma^2 - \delta_{i-1\leq r} i - \delta_{i-1>r}(r+1)\big) :a^*_{ii}(z_1)a^*_{ii}(z_2)a^*_{i-1,i-1}(z_1): \\
&- \big(\gamma^2 - \delta_{i-1>r}(r+1)\big)\, a^*_{ii}(z_1)a^*_{ii}(z_2)a^*_{i-1,i-1}(z_2)\Big)\delta(z_2-w)\partial_w\delta(z_1-w) \\
&+ (i+2)\delta_{1\leq i\leq r} a^*_{ii}(z_1)a^*_{i-1,i}(z_2)\delta(z_2-w)\partial_{z_2}\delta(z_1-z_2) \\
&- \delta_{1\leq i-1\leq r} a^*_{ii}(z_1)a^*_{i-1,i}(z_2)\delta(z_2-w)\partial_{z_2}\delta(z_1-z_2) \\
&+ \delta_{1\leq i\leq r} a^*_{ii}(z_2)a^*_{i-1,i}(z_1)\delta(z_2-w)\partial_{z_2}\delta(z_1-z_2) \\
&- \big(\delta_{i>r}(r+1) + \delta_{i\leq r}(i+1) - \gamma^2\big) :a^*_{ii}(z_1)a^*_{i-1,i}(z_1): \delta(z_2-w)\partial_w\delta(z_1-w) \\
&- \big(\gamma^2 - \delta_{i-1\leq r} i - \delta_{i-1>r}(r+1)\big) :a^*_{ii}(z_2)a^*_{i-1,i}(z_1): \partial_w\delta(z_1-w)\delta(z_2-w) \\
&- \big(\delta_{i>r}(r+1) + \delta_{i\leq r}(i+1) - \gamma^2\big)\, a^*_{ii}(z_2)a^*_{i-1,i}(z_2)\delta(z_2-w)\partial_w\delta(z_1-w) \\
&- a^*_{ii}(z_1)a^*_{i-1,i}(z_2)\big(\gamma^2 - \delta_{i>r}(r+1) - \delta_{i,r+1}\big)\,\delta(z_2-w)\partial_w\delta(z_1-w)
\end{aligned}
$$

$$
\begin{aligned}
&= \Big(- \big(\delta_{i>r}(r+1) + \delta_{i\leq r}(i+1) - \gamma^2\big) :a^*_{ii}(z_1)a^*_{i-1,i}(z_1): \\
&- a^*_{ii}(z_1)a^*_{i-1,i}(z_2)\big(\gamma^2 - (i+1)\delta_{1\leq i\leq r} - \delta_{i>r}(r+1)\big)\Big)\delta(z_2-w)\partial_w\delta(z_1-w) \\
&- \big(\delta_{i>r}(r+1) + \delta_{i\leq r}(i+1) - \gamma^2\big)\, a^*_{ii}(z_2)a^*_{i-1,i}(z_2)\delta(z_2-w)\partial_w\delta(z_1-w) \\
&- \big(\gamma^2 - \delta_{i-1\leq r} i - \delta_{i-1>r}(r+1)\big) :a^*_{ii}(z_2)a^*_{i-1,i}(z_1): \partial_w\delta(z_1-w)\delta(z_2-w) \\
&+ \delta_{1\leq i\leq r} a^*_{ii}(z_2)a^*_{i-1,i}(z_1)\delta(z_2-w)\partial_w\delta(z_1-w)
\end{aligned}
$$

$$
\begin{aligned}
&= \Big(- \big(\delta_{i>r}(r+1) + \delta_{i\leq r}(i+1) - \gamma^2\big) :a^*_{ii}(z_1)a^*_{i-1,i}(z_1): \\
&+ \big(\delta_{i>r}(r+1) + \delta_{i\leq r}(i+1) - \gamma^2\big)\, a^*_{ii}(z_1)a^*_{i-1,i}(z_2) \\
&- \big(\delta_{i>r}(r+1) + \delta_{i\leq r}(i+1) - \gamma^2\big)\, a^*_{ii}(z_2)a^*_{i-1,i}(z_2) \\
&+ \big(\delta_{i>r}(r+1) + \delta_{i\leq r}(i+1) - \gamma^2\big) :a^*_{ii}(z_2)a^*_{i-1,i}(z_1): \Big)\delta(z_2-w)\partial_w\delta(z_1-w) \\
&= 0.
\end{aligned}
$$

Now we turn to the last series of computations:

$$\begin{aligned}
&\Big[\rho(E^1_{i-1})(z_1),\ \ \rho(E_i)(z_2),\ \ \rho(E_{i-1})(w)\Big]\\
&= -(i-2)\delta_{i-2\le r} : a^*_{ii}(z_2)a^*_{i-1,i-1}(z_1)a^*_{i-1,i-1}(z_2)\delta(z_2-w)\partial_{z_2}\delta(z_1-z_2)\\
&-(i+2)\delta_{i-1\le r} : a_{ii}(z_2)a_{i-1,i-1}(z_1)a_{i-1,i-1}(z_2) : \delta(z_2-w)\partial_{z_2}\delta(z_1-z_2)\\
&+(i-2)\delta_{i-2\le r} : a^*_{i-1,i-1}(z_1)a^*_{i-1,i}(z_2) : \delta(z_2-w)\partial_{z_2}\delta(z_1-z_2)\\
&+2\delta_{i-1\le r} : a^*_{i-1,i-1}(z_1)a^*_{i-1,i}(z_2) : \delta(z_2-w)\partial_{z_2}\delta(z_1-z_2)\\
&+2 : a^*_{i-1,i-1}a^*_{ii}\left(\sum_{l=1}^{i-2} a_{l,i-2}a^*_{l,i-1}\right)\delta(z_1-z_2)\delta(z_2-w)\\
&- : a^*_{i-1,i-1}a^*_{i-1,i}\left(\sum_{k=1}^{i-2} a_{k,i-2}a^*_{k,i-2}\right) : \delta(z_1-z_2)\delta(z_2-w)\\
&+a^*_{i-1,i}a^*_{i-1,i-1}\left(\sum_{k=1}^{i-1} a_{k,i-1}a^*_{k,i-1}\right) : \delta(z_1-z_2)\delta(z_2-w)\\
&- : a^*_{i-1,i-1}\Big(\sum_{l=1}^{i-2} a_{l,i-2}a^*_{li} : \Big)\delta(z_1-z_2)\delta(z_2-w)\\
&+ : a^*_{i-1,i-1}a^*_{i-1,i-1}a^*_{ii}b_{i-1}\delta(z_1-z_2)\delta(z_2-w)\\
&-\big(\gamma^2-\delta_{i-1\le r}i-\delta_{i-1>r}(r+1)\big) : a^*_{i-1,i-1}(z_1)a^*_{i-1,i-1}(z_1)a^*_{ii}(z_2) : \delta(z_2-w)\partial_w\delta(z_1-w).
\end{aligned}$$

Next we have

$$\begin{aligned}
&\Big[\rho(E^2_{i-1})(z_1),\ \ \rho(E_i)(z_2),\ \ \rho(E_{i-1})(w)\Big]\\
&=: a^*_{i-1,i}a^*_{i-1,i-1}\left(\sum_{k=1}^{i-2} a_{k,i-2}a^*_{k,i-2}\right) : \delta(z_1-z_2)\delta(z_2-w)\\
&-2 : a^*_{ii}a^*_{i-1,i-1}\Big(\sum_{k=1}^{i-2} a_{k,i-2}a^*_{k,i-1} : \Big)\delta(z_1-z_2)\delta(z_2-w)\\
&-a^*_{i-1,i}a^*_{i-1,i-1}\left(\sum_{k=1}^{i-1} a_{k,i-1}a^*_{k,i-1}\right) : \delta(z_1-z_2)\delta(z_2-w)\\
&+ : a^*_{i-1,i-1}\left(\sum_{l=1}^{i-2} a_{l,i-2}a^*_{li}\right)\delta(z_1-z_2)\delta(z_2-w)\\
&-a^*_{i-1,i}a^*_{i-1,i-1}b_{i-1}\delta(z_1-z_2)\delta(z_2-w)\\
&+\big(\gamma^2-\delta_{i-1\le r}i-\delta_{i-1>r}(r+1)\big)\, a^*_{i-1,i}(z_1)\partial_w a^*_{i-1,i-1}(w)\delta(z_1-z_2)\delta(z_2-w).
\end{aligned}$$

The third summation contributes

$$\begin{aligned}
&\Big[\rho(E_{i-1}^3)(z_1),\ \ \rho(E_i)(z_2),\ \ \rho(E_{i-1})(w)\Big] \\
&= - : a_{ii}^* a_{i-1,i-1}^* a_{i-1,i-1}^* : b_{i-1}\delta(z_1 - z_2)\delta(z_2 - w) \\
&+ \big(\delta_{i-1>r}(r+1) + \delta_{i-1\leq r} i - \gamma^2\big) : a_{ii}^*(z_2) a_{i-1,i-1}^*(z_2) a_{i-1,i-1}^*(z_2) : \delta(z_2 - w)\partial_{z_1}\delta(z_1 - z_2) \\
&+ : a_{i-1,i-1}^* a_{i-1,i}^* : b_{i-1}\delta(z_1 - z_2)\delta(z_2 - w) \\
&+ \big(\delta_{i-1>r}(r+1) + \delta_{i-1\leq r} i - \gamma^2\big) : a_{i-1,i-1}^*(z_2) a_{i-1,i}^*(z_2) : \delta(z_2 - w)\partial_{z_1}\delta(z_1 - z_2) \\
&- (\gamma^2 - (r+1)\delta_{i>r+2} - \delta_{i,r+2}) a_{i-1,i-1}^*(z_1) a_{i-1,i}^*(z_2)\delta(z_2 - w)\partial_{z_2}\delta(z_1 - z_2) \\
&+ 2(\gamma^2 - (r+1)\delta_{i>r+1} + \frac{r}{2}\delta_{i,r+2}) a_{i-1,i-1}^*(z_1) a_{i-1,i-1}^*(z_2) a_{i,i}^*(z_2)\delta(z_2 - w)\partial_{z_2}\delta(z_1 - z_2).
\end{aligned}$$

Adding these all up we get

$$
\begin{aligned}
&\Big[\rho(E_{i-1})(z_1),\ \ \rho(E_i)(z_2),\ \ \rho(E_{i-1})(w)\Big]\\
&= -(i-2)\delta_{i-2\le r} : a^*_{ii}(z_2)a^*_{i-1,i-1}(z_1)a^*_{i-1,i-1}(z_2)\delta(z_2-w)\partial_{z_2}\delta(z_1-z_2)\\
&- (i+2)\delta_{i-1\le r} : a_{ii}(z_2)a_{i-1,i-1}(z_1)a_{i-1,i-1}(z_2) : \delta(z_2-w)\partial_{z_2}\delta(z_1-z_2)\\
&+ (i-2)\delta_{i-2\le r} : a^*_{i-1,i-1}(z_1)a^*_{i-1,i}(z_2) : \delta(z_2-w)\partial_{z_2}\delta(z_1-z_2)\\
&+ 2\delta_{i-1\le r} : a^*_{i-1,i-1}(z_1)a^*_{i-1,i}(z_2) : \delta(z_2-w)\partial_{z_2}\delta(z_1-z_2)\\
&+ 2 : a^*_{i-1,i-1}a^*_{ii}\left(\sum_{l=1}^{i-2} a_{l,i-2}a^*_{l,i-1}\right)\delta(z_1-z_2)\delta(z_2-w)\\
&- : a^*_{i-1,i-1}a^*_{i-1,i}\left(\sum_{k=1}^{i-2} a_{k,i-2}a^*_{k,i-2}\right) : \delta(z_1-z_2)\delta(z_2-w)\\
&+ a^*_{i-1,i}a^*_{i-1,i-1}\left(\sum_{k=1}^{i-1} a_{k,i-1}a^*_{k,i-1}\right) : \delta(z_1-z_2)\delta(z_2-w)\\
&- : a^*_{i-1,i-1}\Big(\sum_{l=1}^{i-2} a_{l,i-2}a^*_{li} : \Big)\delta(z_1-z_2)\delta(z_2-w)\\
&+ : a^*_{i-1,i-1}a^*_{i-1,i-1}a^*_{ii}b_{i-1}\delta(z_1-z_2)\delta(z_2-w)\\
&- \big(\gamma^2-\delta_{i-1\le r}i-\delta_{i-1>r}(r+1)\big) : a^*_{i-1,i-1}(z_1)a^*_{i-1,i-1}(z_1)a^*_{ii}(z_2) : \delta(z_2-w)\partial_w\delta(z_1-w)\\
&+ : a^*_{i-1,i}a^*_{i-1,i-1}\left(\sum_{k=1}^{i-2} a_{k,i-2}a^*_{k,i-2}\right) : \delta(z_1-z_2)\delta(z_2-w)\\
&- 2 : a^*_{ii}a^*_{i-1,i-1}\Big(\sum_{k=1}^{i-2} a_{k,i-2}a^*_{k,i-1} : \Big)\delta(z_1-z_2)\delta(z_2-w)\\
&- a^*_{i-1,i}a^*_{i-1,i-1}\left(\sum_{k=1}^{i-1} a_{k,i-1}a^*_{k,i-1}\right) : \delta(z_1-z_2)\delta(z_2-w)\\
&+ : a^*_{i-1,i-1}\left(\sum_{l=1}^{i-2} a_{l,i-2}a^*_{li}\right)\delta(z_1-z_2)\delta(z_2-w)\\
&- a^*_{i-1,i}a^*_{i-1,i-1}b_{i-1}\delta(z_1-z_2)\delta(z_2-w)\\
&+ \big(\gamma^2-\delta_{i-1\le r}i-\delta_{i-1>r}(r+1)\big)\, a^*_{i-1,i}(z_1)\partial_w a^*_{i-1,i-1}(w)\delta(z_1-z_2)\delta(z_2-w)\\
&- : a^*_{ii}a^*_{i-1,i-1}a^*_{i-1,i-1} : b_{i-1}\delta(z_1-z_2)\delta(z_2-w)\\
&+ \big(\delta_{i-1>r}(r+1)+\delta_{i-1\le r}i-\gamma^2\big) : a^*_{ii}(z_2)a^*_{i-1,i-1}(z_2)a^*_{i-1,i-1}(z_2) : \delta(z_2-w)\partial_{z_1}\delta(z_1-z_2)\\
&+ : a^*_{i-1,i-1}a^*_{i-1,i} : b_{i-1}\delta(z_1-z_2)\delta(z_2-w)\\
&+ \big(\delta_{i-1>r}(r+1)+\delta_{i-1\le r}i-\gamma^2\big) : a^*_{i-1,i-1}(z_2)a^*_{i-1,i}(z_2) : \delta(z_2-w)\partial_{z_1}\delta(z_1-z_2)\\
&- (\gamma^2-(r+1)\delta_{i>r+2}-\delta_{i,r+2})a^*_{i-1,i-1}(z_1)a^*_{i-1,i}(z_2)\delta(z_2-w)\partial_{z_2}\delta(z_1-z_2)\\
&+ 2(\gamma^2-(r+1)\delta_{i>r+1}+\frac{r}{2}\delta_{i,r+2})a^*_{i-1,i-1}(z_1)a^*_{i-1,i-1}(z_2)a^*_{i,i}(z_2)\delta(z_2-w)\partial_{z_2}\delta(z_1-z_2)
\end{aligned}
$$

$$\begin{aligned}
&= -(i-2)\delta_{i-2\leq r} : a_{ii}^*(z_2)a_{i-1,i-1}^*(z_1)a_{i-1,i-1}^*(z_2)\delta(z_2-w)\partial_{z_2}\delta(z_1-z_2) \\
&- (i+2)\delta_{i-1\leq r} : a_{ii}(z_2)a_{i-1,i-1}(z_1)a_{i-1,i-1}(z_2) : \delta(z_2-w)\partial_{z_2}\delta(z_1-z_2) \\
&+ (i-2)\delta_{i-2\leq r} : a_{i-1,i-1}^*(z_1)a_{i-1,i}^*(z_2) : \delta(z_2-w)\partial_{z_2}\delta(z_1-z_2) \\
&+ 2\delta_{i-1\leq r} : a_{i-1,i-1}^*(z_1)a_{i-1,i}^*(z_2) : \delta(z_2-w)\partial_{z_2}\delta(z_1-z_2) \\
&- \big(\gamma^2 - \delta_{i-1\leq r}i - \delta_{i-1>r}(r+1)\big) : a_{i-1,i-1}^*(z_1)a_{i-1,i-1}^*(z_1)a_{ii}^*(z_2) : \delta(z_2-w)\partial_w\delta(z_1-w) \\
&+ \big(\gamma^2 - \delta_{i-1\leq r}i - \delta_{i-1>r}(r+1)\big)\, a_{i-1,i}^*(z_1)\partial_w a_{i-1,i-1}^*(w)\delta(z_1-z_2)\delta(z_2-w) \\
&+ \big(\delta_{i-1>r}(r+1) + \delta_{i-1\leq r}i - \gamma^2\big) : a_{ii}^*(z_2)a_{i-1,i-1}^*(z_2)a_{i-1,i-1}^*(z_2) : \delta(z_2-w)\partial_{z_1}\delta(z_1-z_2) \\
&+ \big(\delta_{i-1>r}(r+1) + \delta_{i-1\leq r}i - \gamma^2\big) : a_{i-1,i-1}^*(z_2)a_{i-1,i}^*(z_2) : \delta(z_2-w)\partial_{z_1}\delta(z_1-z_2) \\
&- (\gamma^2 - (r+1)\delta_{i>r+2} - \delta_{i,r+2})a_{i-1,i-1}^*(z_1)a_{i-1,i}^*(z_2)\delta(z_2-w)\partial_{z_2}\delta(z_1-z_2) \\
&+ 2(\gamma^2 - (r+1)\delta_{i>r+1} + \frac{r}{2}\delta_{i,r+2})a_{i-1,i-1}^*(z_1)a_{i-1,i-1}^*(z_2)a_{i,i}^*(z_2)\delta(z_2-w)\partial_{z_2}\delta(z_1-z_2)
\end{aligned}$$

$$\begin{aligned}
&= -(i-2)\delta_{i-2\leq r} : a_{ii}^*(z_2)a_{i-1,i-1}^*(z_1)a_{i-1,i-1}^*(z_2)\delta(z_2-w)\partial_{z_2}\delta(z_1-z_2) \\
&- (i+2)\delta_{i-1\leq r} : a_{ii}(z_2)a_{i-1,i-1}(z_1)a_{i-1,i-1}(z_2) : \delta(z_2-w)\partial_{z_2}\delta(z_1-z_2) \\
&- \big(\gamma^2 - \delta_{i-1\leq r}i - \delta_{i-1>r}(r+1)\big) : a_{i-1,i-1}^*(z_1)a_{i-1,i-1}^*(z_1)a_{ii}^*(z_2) : \delta(z_2-w)\partial_w\delta(z_1-w) \\
&+ 2(\gamma^2 - (r+1)\delta_{i>r+1} + \frac{r}{2}\delta_{i,r+2})a_{i-1,i-1}^*(z_1)a_{i-1,i-1}^*(z_2)a_{i,i}^*(z_2)\delta(z_2-w)\partial_{z_2}\delta(z_1-z_2) \\
&+ \big(\delta_{i-1>r}(r+1) + \delta_{i-1\leq r}i - \gamma^2\big) : a_{ii}^*(z_2)a_{i-1,i-1}^*(z_2)a_{i-1,i-1}^*(z_2) : \delta(z_2-w)\partial_{z_1}\delta(z_1-z_2) \\
&+ (i-2)\delta_{i-2\leq r} : a_{i-1,i-1}^*(z_1)a_{i-1,i}^*(z_2) : \delta(z_2-w)\partial_{z_2}\delta(z_1-z_2) \\
&+ \big(\delta_{i-1>r}(r+1) + \delta_{i-1\leq r}i - \gamma^2\big) : a_{i-1,i-1}^*(z_2)a_{i-1,i}^*(z_2) : \delta(z_2-w)\partial_{z_1}\delta(z_1-z_2) \\
&+ 2\delta_{i-1\leq r} : a_{i-1,i-1}^*(z_1)a_{i-1,i}^*(z_2) : \delta(z_2-w)\partial_{z_2}\delta(z_1-z_2) \\
&- (\gamma^2 - (r+1)\delta_{i>r+2} - \delta_{i,r+2})a_{i-1,i-1}^*(z_1)a_{i-1,i}^*(z_2)\delta(z_2-w)\partial_{z_2}\delta(z_1-z_2) \\
&+ \big(\gamma^2 - \delta_{i-1\leq r}i - \delta_{i-1>r}(r+1)\big)\, a_{i-1,i}^*(z_1)\partial_w a_{i-1,i-1}^*(w)\delta(z_1-z_2)\delta(z_2-w)
\end{aligned}$$

$$\begin{aligned}
&= -\big(\gamma^2 - \delta_{i-1\leq r}i - \delta_{i-1>r}(r+1)\big) : a_{i-1,i-1}^*(z_1)a_{i-1,i-1}^*(z_1)a_{ii}^*(z_2) : \delta(z_2-w)\partial_w\delta(z_1-w) \\
&+ 2(\gamma^2 - (r+1)\delta_{i>r+1} - i\delta_{i\leq r+1})a_{i-1,i-1}^*(z_1)a_{i-1,i-1}^*(z_2)a_{i,i}^*(z_2)\delta(z_2-w)\partial_{z_2}\delta(z_1-z_2) \\
&+ \big(\delta_{i-1>r}(r+1) + \delta_{i-1\leq r}i - \gamma^2\big) : a_{ii}^*(z_2)a_{i-1,i-1}^*(z_2)a_{i-1,i-1}^*(z_2) : \delta(z_2-w)\partial_{z_1}\delta(z_1-z_2) \\
&+ \big(\delta_{i-1>r}(r+1) + \delta_{i-1\leq r}i - \gamma^2\big) : a_{i-1,i-1}^*(z_2)a_{i-1,i}^*(z_2) : \delta(z_2-w)\partial_{z_1}\delta(z_1-z_2) \\
&- \big(\gamma^2 - \delta_{i-1\leq r}i - \delta_{i-1>r}(r+1)\big)\, a_{i-1,i-1}^*(z_1)a_{i-1,i}^*(z_2)\delta(z_2-w)\partial_{z_2}\delta(z_1-z_2) \\
&+ \big(\gamma^2 - \delta_{i-1\leq r}i - \delta_{i-1>r}(r+1)\big)\, a_{i-1,i}^*(z_1)\partial_w a_{i-1,i-1}^*(w)\delta(z_1-z_2)\delta(z_2-w). \\
&= 0
\end{aligned}$$

□

7. Concluding Remarks

We gave a realization of the intermediate Wakimoto module for $\hat{\mathfrak{sl}}(n+1,\mathbb{C})$. We list below related problems and planned future work:

(1) We believe that generically Verma type modules and intermediate Wakimoto modules are isomorphic. Preliminary calculations using (6.1) above give a proof of this for $\hat{\mathfrak{sl}}(n+1,\mathbb{C})$ when $r = n-1$. This should be explored further.
(2) Verma type modules have a complicated structure when the center c acts by zero (see for example [Fut94]). The realization above given in Theorem 3.1 yields information about the structure of these modules at least in the case of $\hat{\mathfrak{sl}}(2,\mathbb{C})$. These calculations are inspired by work done in [FF90b] when $c = -\check{h}$ is at the singular hyperplane.
(3) A similar realization must exist for all Verma type modules over $\hat{\mathfrak{sl}}(n+1,\mathbb{C})$ and other affine Lie algebras. It would be of interest to us, if one could give a characterization and proof of the existence of intermediate Wakimoto modules using semi-infinite flag manifolds and their cohomology (see [FF90a] and [Vor99]).

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