College of Charleston
Department of Mathematics

Name:

Examination in: Advanced Linear Algebra

Math Course Number	Math 502
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Total number of problems: 5

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Results available by:

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Permitted aids: Proctor, No use of calculators are permitted on this exam.

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2	
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1. Define two linear maps $\phi : \mathbb{R}^2 \to \mathbb{R}^4$ and $\psi : \mathbb{R}^4 \to \mathbb{R}^2$ by

$$\phi(\chi^1, \chi^2) := (3\chi^1 - 2\chi^2, 2\chi^1 - \chi^2, \chi^1, \chi^2)$$
$$\psi(\xi^1, \xi^2, \xi^3, \xi^4) := (-\xi^1 + 2\xi^2 - \xi^3, -\xi^2 + 2\xi^3 - \xi^4).$$

a) What is the kernel of ϕ?

$$\begin{aligned}(\chi^1, \chi^2) \in \ker\phi &\Longrightarrow (0,0,0,0) = \phi(\chi^1, \chi^2) = (3\chi^1 - 2\chi^2, 2\chi^1 - \chi^2, \chi^1, \chi^2)\\ &\Longrightarrow (\chi^1, \chi^2) = (0,0).\end{aligned}$$

Hence $\ker\phi = \{(0,0)\}$.

b) Is ψ surjective?

Sln: Lets show that it is surjective. Observe that

$$\begin{aligned}\psi(-1,0,0,0) &= (1,0) = \mathbf{e}_1\\ \psi(0,0,0,-1) &= (0,1) = \mathbf{e}_2.\end{aligned}$$

Hence $\mathbf{e}_1, \mathbf{e}_2 \in \operatorname{im}\psi$, which implies that $\mathbb{R}^2 = \operatorname{im}\psi$ so that ψ is surjective.

c) What is a basis for the kernel of ψ?

Sln:

$$\begin{aligned}\psi(\xi^1, \xi^2, \xi^3, \xi^4) &= (-\xi^1 + 2\xi^2 - \xi^3, -\xi^2 + 2\xi^3 - \xi^4) = (0,0)\\ &\Longrightarrow \xi^1 = 2\xi^2 - \xi^3, \quad \xi^2 = 2\xi^3 - \xi^4\\ &\Longrightarrow \xi^1 = 2(2\xi^3 - \xi^4) - \xi^3 = 3\xi^3 - 2\xi^4, \quad \xi^2 = 2\xi^3 - \xi^4\\ &\Longrightarrow (\xi^1, \xi^2, \xi^3, \xi^4) = (3\xi^3 - 2\xi^4, 2\xi^3 - \xi^4, \xi^3, \xi^4) \qquad (1)\\ &\Longrightarrow (\xi^1, \xi^2, \xi^3, \xi^4) = \xi^3(3,2,1,0) + \xi^4(-2,-1,0,1)\\ &\Longrightarrow (\xi^1, \xi^2, \xi^3, \xi^4) \in \langle(3,2,1,0), (-2,-1,0,1)\rangle.\end{aligned}$$

Hence $\ker\psi \subset \langle(3,2,1,0), (-2,-1,0,1)\rangle$. Since

$$\psi(3,2,1,0) = (-3+2\cdot 2-1, -2+2\cdot 1) = (0,0,0,0) = (-2+2\cdot 1+0, -1+2\cdot 0+1) = \psi(-2,-1,0,1)$$

we get that

$$\psi(\lambda_1(3,2,1,0) + \lambda_2(-2,-1,0,1)) = (0,0).$$

Hence $\ker\psi \supset \langle(3,2,1,0), (-2,-1,0,1)\rangle$. From (1) note that $(3,2,1,0), (-2,-1,0,1)$ are linearly independent and hence form a basis for $\ker\psi$.

d) Is the following sequence exact? Explain why or why not.

$$0 \longrightarrow \mathbb{R}^2 \xrightarrow{\ \phi\ } \mathbb{R}^4 \xrightarrow{\ \psi\ } \mathbb{R}^2 \longrightarrow 0.$$

It is exact at $\mathbb{R}^2$ by part a) and b. Note that by part c)

$$\begin{aligned}\operatorname{im}\phi = \{\phi(\chi^1, \chi^2) &= \{(3\chi^1 - 2\chi^2, 2\chi^1 - \chi^2, \chi^1, \chi^2) \,|\, \chi^1, \chi^2 \in \mathbb{R}\}\\ &= \{\chi^1(3,2,1,0) + \chi^2(-2,-1,0,1), \,|\, \chi^1, \chi^2 \in \mathbb{R}\}\\ &= \langle(3,2,1,0), (-2,-1,0,1)\rangle = \ker\psi.\end{aligned}$$

2. Define $B : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, by

$$B((\chi^1, \chi^2), (\xi^1, \xi^2)) := 2\chi^1\xi^1 - \chi^2\xi^1 - \chi^1\xi^2 + 2\chi^2\xi^2$$

a) Show B is bilinear.

Proof;

$$\begin{aligned}
B(\lambda_1(\chi_1^1, \chi_1^2) + \lambda_2(\chi_2^1, \chi_2^2), (\xi^1, \xi^2)) &= B((\lambda_1\chi_1^1 + \lambda_2\chi_2^1, \lambda_1\chi_1^2 + \lambda_2\chi_2^2), (\xi^1, \xi^2)) \\
&= 2(\lambda_1\chi_1^1 + \lambda_2\chi_2^1)\xi^1 - (\lambda_1\chi_1^2 + \lambda_2\chi_2^2)\xi^1 \\
&\quad - (\lambda_1\chi_1^1 + \lambda_2\chi_2^1)\xi^2 + 2(\lambda_1\chi_1^2 + \lambda_2\chi_2^2)\xi^2 \\
&= 2\lambda_1\chi_1^1\xi^1 + 2\lambda_2\chi_2^1\xi^1 - \lambda_1\chi_1^2\xi^1 - \lambda_2\chi_2^2\xi^1 \\
&\quad - \lambda_1\chi_1^1\xi^2 - \lambda_2\chi_2^1\xi^2 + 2\lambda_1\chi_1^2\xi^2 + 2\lambda_2\chi_2^2\xi^2 \\
&= \lambda_1 B((\chi_1^1, \chi_1^2), (\xi^1, \xi^2)) + \lambda_2 B((\chi_2^1, \chi_2^2), (\xi^1, \xi^2))
\end{aligned}$$

Note that $B((\chi^1, \chi^2), (\xi^1, \xi^2)) = B((\xi^1, \xi^2), (\chi^1, \chi^2))$ so that knowing that B is linear in the first entry, means that it is also linear in the second.

b) Is B non-degenerate?

Yes it is non-degenerate: For fixed $(\chi^1, \chi^2) \in \mathbb{R}^2$,

$$0 = B((\chi^1, \chi^2), (\xi^1, \xi^2)) := 2\chi^1\xi^1 - \chi^2\xi^1 - \chi^1\xi^2 + 2\chi^2\xi^2$$

for all $\xi^1, \xi^2 \in \mathbb{R}$, we have in particular

$$0 = B((\chi^1, \chi^2), (1, 0)) := 2\chi^1 - \chi^2$$

and

$$0 = B((\chi^1, \chi^2), (0, 1)) := -\chi^1 + 2\chi^2.$$

This means that $\chi^2 = 2\chi^1 = 2(2\chi^2) = 4\chi^2$. Hence $\chi^1 = \chi^2 = 0$. Similarly if for fixed (ξ^1, ξ^2), we have

$$0 = B((\chi^1, \chi^2), (\xi^1, \xi^2)) = 2\chi^1\xi^1 - \chi^2\xi^1 - \chi^1\xi^2 + 2\chi^2\xi^2$$

for all $\xi^1, \xi^2 \in \mathbb{R}$, then since $B((\chi^1, \chi^2), (\xi^1, \xi^2)) = B((\xi^1, \xi^2), (\chi^1, \chi^2))$ we use the previous calculation to see that $(\xi^1, \xi^2) = (0, 0)$.

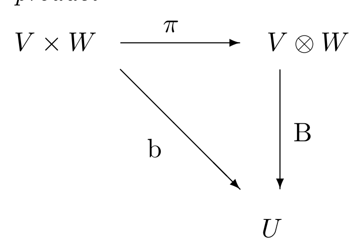
3. Let V and W over Γ and let $\mathcal{C}(V \times W)$ denote the free vector space on the set $V \times W$ with canonical inclusion $\iota_{V\times W} : V \times W \to \mathcal{C}(V \times W)$ given by $\iota_{V\times W}(v, w) := \delta_{(v,w)}$. Consider the subspace $\mathcal{B}$ spanned by elements of the form

$$\delta_{(\mathbf{v}+\mathbf{v}',\mathbf{w})} - \delta_{(\mathbf{v},\mathbf{w})} - \delta_{(\mathbf{v}',\mathbf{w})} \tag{2}$$
$$\delta_{(\mathbf{v},\mathbf{w}+\mathbf{w}')} - \delta_{(\mathbf{v},\mathbf{w})} - \delta_{(\mathbf{v},\mathbf{w}')} \tag{3}$$
$$a\delta_{(\mathbf{v},\mathbf{w})} - \delta_{(a\mathbf{v},\mathbf{w})} \tag{4}$$
$$\delta_{(a\mathbf{v},\mathbf{w})} - \delta_{(\mathbf{v},a\mathbf{w})} \tag{5}$$

for $a \in \Gamma$, $\mathbf{v}, \mathbf{v}' \in V$, and $\mathbf{w}, \mathbf{w}' \in W$.

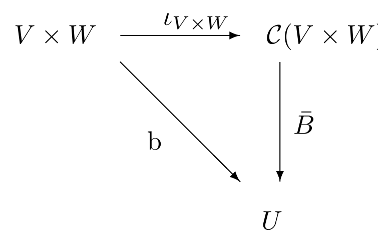
The quotient space $\mathcal{C}(V \times W)/\mathcal{B}$ is be denoted by $V \otimes W$ and is called the *tensor product* of V and W. The equivalence class of an element $\delta_{(\mathbf{v},\mathbf{w})}$ in $V \otimes W$ is denoted by $\mathbf{v} \otimes \mathbf{w}$. In other words $v \otimes w = \delta_{(v,w)} + \mathcal{B}$. Let $\pi : V \times W \to V \otimes W$ denote the inclusion $\iota_{V\times W}$ followed by the canonical projection so that $(\mathbf{v}, \mathbf{w}) \mapsto \mathbf{v} \otimes \mathbf{w}$. Show that if $b : V \times W \to U$ is a bilinear map, then there exists a unique linear map $B : V \otimes W \to U$ satisfying $B \circ \pi = b$. In other words the diagram below is commutative:

tensor product



Sln: Now by the universal mapping property of $\mathcal{C}(V \times W)$ there exists a unique linear map $\bar{B} : C(V \times W) \to U$ satisfying $\bar{B} \circ i_{V\times W} = b$. Since b is bilinear we get

$$\begin{aligned}
\bar{B}(\delta_{(\mathbf{v}+\mathbf{v}',\mathbf{w})} - \delta_{(\mathbf{v},\mathbf{w})} - \delta_{(\mathbf{v}',\mathbf{w})}) &= \bar{B}(\delta_{(\mathbf{v}+\mathbf{v}',\mathbf{w})}) - \bar{B}(\delta_{(\mathbf{v},\mathbf{w})}) - \bar{B}(\delta_{(\mathbf{v}',\mathbf{w})}) \\
&= b(\mathbf{v} + \mathbf{v}', \mathbf{w}) - b(\mathbf{v}, \mathbf{w}) - b(\mathbf{v}', \mathbf{w}) = 0, \\
\bar{B}(\delta_{(\mathbf{v},\mathbf{w}+\mathbf{w}')} - \delta_{(\mathbf{v},\mathbf{w})} - \delta_{(\mathbf{v},\mathbf{w}')}) &= \bar{B}(\delta_{(\mathbf{v},\mathbf{w}+\mathbf{w}')}) - \bar{B}(\delta_{(\mathbf{v},\mathbf{w})}) - \bar{B}(\delta_{(\mathbf{v},\mathbf{w}')}) \\
&= b(\mathbf{v}, \mathbf{w} + \mathbf{w}') - b(\mathbf{v}, \mathbf{w}) - b(\mathbf{v}, \mathbf{w}') = 0, \\
\bar{B}(a\delta_{(\mathbf{v},\mathbf{w})} - a\delta_{(\mathbf{v},\mathbf{w})}) &= \bar{B}(a\delta_{(\mathbf{v},\mathbf{w})}) - \bar{B}(\delta_{(a\mathbf{v},\mathbf{w})}) \\
&= b(a(\mathbf{v}, \mathbf{w})) - b(a\mathbf{v}, \mathbf{w})) = 0, \\
\bar{B}(a\delta_{(\mathbf{v},\mathbf{w})} - \delta_{(\mathbf{v},a\mathbf{w})}) &= \bar{B}(a\delta_{(\mathbf{v},\mathbf{w})}) - \bar{B}(\delta_{(\mathbf{v},a\mathbf{w})}) \\
&= b(a\mathbf{v}, \mathbf{w}) - b(\mathbf{v}, a\mathbf{w}) = 0.
\end{aligned}$$



Since elements in (2), (3), (4) and (5) form a spanning set for $\mathcal{B}$, have $\bar{B}(\mathcal{B}) = 0$. If we let $\bar{\pi} : \mathcal{C}(V \times W) \to V \otimes W = C(V \times W)/\mathcal{B}$ denote the canonical projection, then by the First Isomorphism Theorem there exists a unique linear map $B : V \otimes W \to U$ such that $B \circ \bar{\pi} = \bar{B}$. Since $\pi = \bar{\pi} \circ \iota_{V\times W}$, we get $B \circ \pi = B \circ \bar{\pi} \circ \iota_{V\times W} = \bar{B} \circ \iota_{V\times W} = b$..

Note that since $\{\delta_{(\mathbf{v},\mathbf{w})} \,|\, \mathbf{v} \in V, \mathbf{w} \in W\}$ spans $\mathcal{C}(V \times W)$, their image $\{\mathbf{v} \otimes \mathbf{w} \,|\, \mathbf{v} \in V, \mathbf{w} \in W\}$ span $V \otimes W$. Observe also that for any $\mathbf{v} \in V$, $\mathbf{v} \otimes \mathbf{0} = \mathbf{v} \otimes 0 \cdot \mathbf{w} = 0 \cdot (\mathbf{v} \otimes \mathbf{w}) = \mathbf{0}$. Moreover

$$\begin{aligned}
(\mathbf{v} + \mathbf{v}') \otimes \mathbf{w} &= (\mathbf{v} \otimes \mathbf{w}) + (\mathbf{v}' \otimes \mathbf{w}) \\
\mathbf{v} \otimes (\mathbf{w} + \mathbf{w}') &= \mathbf{v} \otimes \mathbf{w} + \mathbf{v} \otimes \mathbf{w}' \\
a(\mathbf{v} \otimes \mathbf{w}) &= (a\mathbf{v}) \otimes \mathbf{w} = \mathbf{v} \otimes (a\mathbf{w})
\end{aligned}$$

4. The space $E_1^* := \mathbb{R}^2$ and $E_2^* := \mathbb{R}^2$ are dual space to $E = \mathbb{R}^2$ under the pairing

$$\langle (\chi^1, \chi^2), (\xi^1, \xi^2) \rangle_1 := 2\chi^1\xi^1 - \chi^2\xi^1 - \chi^1\xi^2 + 2\chi^2\xi^2$$

and

$$\langle (\chi^1, \chi^2), (\xi^1, \xi^2) \rangle_2 := \chi^1\xi^1 + \chi^2\xi^2.$$

Determine the linear mapping $\phi : E_1^* \to E_2^*$ such that

$$\langle \phi(\chi^1, \chi^2), (\xi^1, \xi^2) \rangle_1 = \langle (\chi^1, \chi^2), (\xi^1, \xi^2) \rangle_2 \tag{6}$$

Sln: We first write

$$\phi(\mathbf{e}_1) = a\mathbf{e}_1 + c\mathbf{e}_2, \quad \phi(\mathbf{e}_2) = b\mathbf{e}_1 + d\mathbf{e}_2$$

where $a, b, c, d \in \mathbb{R}$ are to be determined. We plug the above into the equation (6) to get

$$\begin{aligned} 1 = \langle \mathbf{e}_1, \mathbf{e}_1 \rangle_2 &= \langle \phi(\mathbf{e}_1), \mathbf{e}_1 \rangle_1 \\ &= \langle a\mathbf{e}_1 + c\mathbf{e}_2, \mathbf{e}_1 \rangle_1 \\ &= 2a - c \end{aligned}$$

$$\begin{aligned} 0 = \langle \mathbf{e}_1, \mathbf{e}_2 \rangle_2 &= \langle \phi(\mathbf{e}_1), \mathbf{e}_1 \rangle_1 \\ &= \langle a\mathbf{e}_1 + c\mathbf{e}_2, \mathbf{e}_2 \rangle_1 \\ &= -a + 2c \\ &\text{so } c = 1/3, \;\; a = 2/3, \end{aligned}$$

$$\begin{aligned} 1 = \langle \mathbf{e}_2, \mathbf{e}_2 \rangle_2 &= \langle \phi(\mathbf{e}_2), \mathbf{e}_2 \rangle_1 \\ &= \langle b\mathbf{e}_1 + d\mathbf{e}_2, \mathbf{e}_2 \rangle_1 \\ &= -b + 2d \end{aligned}$$

$$\begin{aligned} 0 = \langle \mathbf{e}_2, \mathbf{e}_1 \rangle_2 &= \langle \phi(\mathbf{e}_2), \mathbf{e}_1 \rangle_1 \\ &= \langle b\mathbf{e}_1 + d\mathbf{e}_2, \mathbf{e}_1 \rangle_1 \\ &= 2b - d \\ \\ &\text{so } d = 2/3, \;\; b = 1/3. \end{aligned}$$

5. Let $a < b$ be two real numbers and let $t_1, \dots, t_n$ be n distinct real numbers in this interval.

a) Show that the assignment l
$$p(t) \mapsto \int_{t=a}^{b} p(t)\, dt$$
is a linear functional from the space of all polynomials of degree less than n with real coefficients, to $\mathbb{R}$.

b) Show that the assignment l_k for fixed $1 \le k \le n$, given by
$$l_k(p(t)) = p(t_k)$$
is a linear functional from the space of all polynomials of degree less than n with real coefficients, to $\mathbb{R}$.

c) Show that $l_1, \dots, l_n$ are linearly independent.

d) Show that there exists fixed real numbers $m_1, \dots, m_n$ such that
$$\int_{t=a}^{b} p(t)\, dt = m_1 p(t_1) + \cdots + m_n p(t_n)$$
for all polynomials $p(t)$ of degree less than n.

Proof: (From Lax's text on Linear Algebra). Let P_n denote the space of all polynomials of degree less than n with real coefficients. Then $l : P_n \to \mathbb{R}$ and $l_k : P_n \to \mathbb{R}$. For parts a) and b) want to show that $l, l_k \in L(P_n)$.

a) and b): From Calculus we know that for any $\lambda, \gamma \in \mathbb{R}$ and $p(t), q(t) \in P_n$,
$$l(\lambda p(t) + \gamma q(t)) = \int_{t=a}^{b} \lambda p(t) + \gamma q(t)\, dt = \lambda \int_{t=a}^{b} p(t)\, dt + \gamma \int_{t=a}^{b} q(t)\, dt = \lambda l(p(t)) + \gamma l(q(t)).$$
Moreover
$$l_k(\lambda p(t) + \gamma q(t)) = \lambda p(t_k) + \gamma q(t_k)) = \lambda l_k(p(t)) + \gamma l_k(q(t)).$$
Hence $l, l_1, \dots, l_n$ are linearly functionals.

c). We need to show that if for some $\lambda_k \in \mathbb{R}$
$$0 = \sum_{k=1}^{n} \lambda_k l_k,$$
then $\lambda_k = 0$ for all k. For this we define polynomials
$$p_j(t) := \prod_{j \neq k} (t - t_k).$$
Then $p_j(t_k) = 0$ if $j \neq k$ and $p_j(t_j) \neq 0$ as $t_k \neq t_j$ for all $j \neq k$.

Now we get
$$0 = \sum_{k=1}^{n} \lambda_k l_k(p_j(t)) = \sum_{k=1}^{n} \lambda_k (p_j(t_k)) = \lambda_j p_j(t_j)$$
so that $\lambda_j = 0$. This proves that the l_j are linearly independent.

d). Since $\dim L(P_n) = \dim P_n = n$ we must have that $l_1, \dots, l_n$ form a basis of $L(P_n)$. This means that there exist real numbers $m_1, \dots, m_n$ such that
$$l = m_1 l_1 + \cdots + m_n l_n.$$
Thus there exist real numbers $m_1, \dots, m_n$ such that
$$\int_{t=a}^{b} p(t)\, dt = m_1 p(t_1) + \cdots + m_n p(t_n)$$
for all $p(t) \in P_n$.