Math 502: Advanced Linear Algebra.

2. Linear Mappings: Exercises.

§5: 2.

$$
\begin{aligned}
\mathbf{v}^* \in (E_1+E_2)^\perp &\iff \langle \mathbf{v}^*, \mathbf{w}\rangle = 0 \text{ for all } \mathbf{w} \in E_1+E_2 \\
&\qquad \text{by the definition of } (\ \ )^\perp \\
&\implies \langle \mathbf{v}^*, \mathbf{u}_1\rangle = 0 = \langle \mathbf{v}^*, \mathbf{u}_2\rangle \text{ for all } \mathbf{u}_1 \in E_1 \text{ and } \mathbf{u}_2 \in E_2 \\
&\qquad \text{as } E_i \subset E_1+E_2 = \{\mathbf{u}_1+\mathbf{u}_2 \,|\, \mathbf{u}_i \in E_i,\ i=1,2\} \\
&\implies \mathbf{v}^* \in E_1^\perp \cap E_2^\perp \\
&\qquad \text{by the definition of } (\ \ )^\perp
\end{aligned}
$$

This proves that $(E_1+E_2)^\perp \subset E_1^\perp \cap E_2^\perp$.

$$
\begin{aligned}
\mathbf{v}^* \in E_1^\perp \cap E_2^\perp &\iff \langle \mathbf{v}^*, \mathbf{u}_1\rangle = 0 = \langle \mathbf{v}^*, \mathbf{u}_2\rangle \text{ for all } \mathbf{u}_1 \in E_1 \text{ and } \mathbf{u}_2 \in E_2 \\
&\implies \langle \mathbf{v}^*, \mathbf{w}\rangle = 0 \text{ for all } \mathbf{w} \in E_1+E_2 \\
&\qquad \text{as } E_1+E_2 := \{\mathbf{u}_1+\mathbf{u}_2 \,|\, \mathbf{u}_i \in E_i,\ i=1,2\} \\
&\qquad \text{and } \langle \mathbf{v}^*, \mathbf{u}_1+\mathbf{u}_2\rangle = \langle \mathbf{v}^*, \mathbf{u}_1\rangle + \langle \mathbf{v}^*, \mathbf{u}_2\rangle = 0 \\
&\implies \mathbf{v}^* \in (E_1+E_2)^\perp.
\end{aligned}
$$

This proves that $(E_1+E_2)^\perp \supset E_1^\perp \cap E_2^\perp$.

§6: 1.

$$
\begin{aligned}
(E_1+E_2)^\perp &= (E_1^{\perp\perp} + E_2^{\perp\perp})^\perp \quad \text{by Prop. IV, section 5} \\
&= (E_1^{\perp\perp})^\perp \cap (E_2^{\perp\perp})^\perp \quad \text{by Problem 2, section 5} \\
&= E_1^\perp \cap E_2^\perp \quad \text{by Prop. V, section 5}
\end{aligned}
$$

4.

$$
\langle \mathbf{x}^{*1} + \sum_{\nu=2}^n \lambda_\nu \mathbf{x}^{*\nu}, \mathbf{x}_1\rangle = \langle \mathbf{x}^{*1}, \mathbf{x}_1\rangle + \sum_{\nu=2}^n \lambda_\nu \langle \mathbf{x}^{*\nu}, \mathbf{x}_1\rangle = 1
$$

$$
\begin{aligned}
\langle \mathbf{x}^{*1} + \sum_{\nu=2}^n \lambda_\nu \mathbf{x}^{*\nu}, \mathbf{x}_k - \lambda_k \mathbf{x}_1\rangle &= -\lambda_k \langle \mathbf{x}^{*1}, \mathbf{x}_1\rangle + \sum_{\nu=2}^n \lambda_\nu \langle \mathbf{x}^{*\nu}, \mathbf{x}_k\rangle \\
&= -\lambda_k + \lambda_k = 0 \text{ for } 2 \le k \le n.
\end{aligned}
$$

Moreover for $2 \le \nu \le n$,

$$
\begin{aligned}
\langle \mathbf{x}^{*\nu}, \mathbf{x}_1\rangle &= 0, \\
\langle \mathbf{x}^{*\nu}, \mathbf{x}_k - \lambda_k \mathbf{x}_1\rangle &= \langle \mathbf{x}^{*\nu}, \mathbf{x}_k\rangle = \delta_{\nu,k}.
\end{aligned}
$$