Math 502: Advanced Linear Algebra.

1. Linear Mappings

2.4. **Direct Sums of Vector Spaces.**

1. We show that

$$\ker\phi = E_1 \ominus E_2 := \{(\mathbf{v}, -\mathbf{v}) \mid \mathbf{v} \in E_1 \cap E_2\}.$$

$$\begin{aligned}(\mathbf{v}, \mathbf{w}) \in \ker\phi &\Longrightarrow \mathbf{0} = \phi(\mathbf{v}, \mathbf{w}) = \mathbf{v} + \mathbf{w} \\ &\Longrightarrow \mathbf{v} = -\mathbf{w} \in E_1 \cap E_2 \text{ as } \mathbf{v} \in E_1 \text{ and } \mathbf{w} \in E_2.\end{aligned}$$

Thus $\ker\phi \subset E_1 \ominus E_2 = \{(\mathbf{v}, -\mathbf{v}) \mid \mathbf{v} \in E_1 \cap E_2\}$.

$$\begin{aligned}(\mathbf{v}, \mathbf{w}) \in E_1 \ominus E_2 = \{(\mathbf{v}, -\mathbf{v}) \mid \mathbf{v} \in E_1 \cap E_2\} &\Longrightarrow \mathbf{w} = -\mathbf{v} \\ &\Longrightarrow \phi(\mathbf{v}, \mathbf{w}) = \mathbf{v} + \mathbf{w} = \mathbf{v} - \mathbf{v} = \mathbf{0} \\ &\Longrightarrow (\mathbf{v}, \mathbf{w}) \in \ker\phi\end{aligned}$$

This proves the other equality.

Now ϕ is an isomorphism if and only if $\ker\phi = 0$. But then if $\mathbf{v} \in E_1 \cap E_2$, then $(\mathbf{v}, -\mathbf{v}) \in \ker\phi = 0$ by the above. Hence $\mathbf{v} = \mathbf{0}$. This proves $E_1 \cap E_2 = \{\mathbf{0}\}$. If on the other hand $E_1 \cap E_2 = \{\mathbf{0}\}$, then $\ker\phi = 0$ by the above equality of sets. Since $E_1 + E_2 = \operatorname{im}\phi$ it is clear then that ϕ must be an isomorphism.

4. Suppose π is a projection of E. Then $\pi^2 = \pi \circ \pi = \pi$. Set $\omega = 2\pi - \iota$ where ι is the identity map on E. Then

$$\omega^2 = (2\pi - \iota) \circ (2\pi - \iota) = 4\pi^2 - 2\pi \circ \iota - 2\iota \circ \pi + \iota^2 = \iota$$

as $\pi^2 = \pi$ and $\iota^2 = \iota$. Since π and ι are linear so is ω. From the above calculation ω must then be an involution.

Now suppose Ω is an involution defined on E. Set $\Pi = \frac{1}{2}(\Omega + \iota)$ where ι is the identity map on E. Then

$$\Pi^2 = \frac{1}{4}(\Omega + \iota) \circ (\Omega + \iota) = \frac{1}{4}(\Omega^2 + \iota \circ \Omega + \Omega \circ \iota + \iota^2) = \frac{1}{4}(2\Omega + 2\iota) = \Pi,$$

as $\iota \circ \Omega = \Omega \circ \iota = \Omega$. Hence Π is a projection operator.

7. This can be proved in more than one way. We plan to use Proposition I in §4.

Define functions $\phi_1 : \{\delta_a \mid a \in X\} \to C(X \cup Y)$ and $\phi_2 : \{\delta_b \mid b \in Y\}$ by

$$\phi_1(\delta_a) = \delta_a, \quad \phi_2(\delta_b) = \delta_b, \quad \forall a \in X, b \in Y.$$

Then by problem 5(i) in §2, we get linear maps $\bar{\phi}_1 : C(X) \to C(X \cup Y)$ and $\bar{\phi}_2 : C(Y) \to C(X \cup Y)$ satisfying $\bar{\phi}_1|_X = \phi_X$ and $\bar{\phi}_1|_Y = \phi_Y$. Similarly there exist linear maps $\bar{\pi}_1 : C(X \cup Y) \to C(X)$ and $\bar{\pi}_2 : C(X \cup Y) \to C(Y)$ satisfying

$$\bar{\pi}_1(\delta_z) = \begin{cases} \delta_a & \text{if } z = a \in X \\ 0 & \text{otherwise} \end{cases} \qquad \bar{\pi}_2(\delta_z) = \begin{cases} \delta_b & \text{if } z = b \in Y \\ 0 & \text{otherwise} \end{cases}$$

Now we check

$$\begin{aligned}\bar{\pi}_1\bar{\phi}_1(\delta_a) = \delta_a, \quad \bar{\pi}_2\bar{\phi}_2(\delta_b) = \delta_b, \quad a \in X, b \in Y. \\ \bar{\pi}_1\bar{\phi}_2(\delta_b) = 0, \quad \bar{\pi}_2\bar{\phi}_1(\delta_a) = 0, \quad a \in X, b \in Y.\end{aligned}$$

and

$$\begin{aligned}(\bar{\phi}_1\bar{\pi}_1 + \bar{\phi}_2\bar{\pi}_2)(\delta_z) &= \begin{cases} (\bar{\phi}_1\bar{\pi}_1)(\delta_z) & \text{if } z = a \in X \\ (\bar{\phi}_2\bar{\pi}_2)(\delta_z) & \text{if } z = b \in Y \end{cases} \\ &= \delta_z\end{aligned}$$

By Proposition I, we must have $C(X) \oplus C(Y) \cong C(X \cup Y)$.