MATH 502: ADVANCED LINEAR ALGEBRA.

1. VECTOR SPACES

1.2. **Linear mappings: Exercises.**

5. (i). Recall the space $C(X)$ has as a basis $\{\delta_a\}_{a\in X}$, as was shown in class (or see the text for f_a which we call δ_a) and the function $i_X : X \to C(X)$ was defined by $i_X(a) = \delta_a$. Let $\phi_0 : \{\delta_a\} \to F$ be defined by $\phi_0(\delta_a) = f(a)$ for all $a \in X$. Then by Proposition II in the text (with $E := C(X)$) there exists a unique linear map $\phi : C(X) \to F$ satisfying $\phi\Big|_{\{\delta_a\}_{a\in X}} = \phi_0$. In other words there is a unique linear map $\phi : C(X) \to F$ satisfying

$$\begin{aligned} \phi(i_X(a)) &= \phi(\delta_a) \quad \text{by the def. of } i_X \\ &= \phi_0(\delta_a) \quad \text{as } \phi\Big|_{\{\delta_a\}_{a\in X}} = \phi_0 \\ &= f(a) \quad \text{by def. of } \phi_0, \end{aligned}$$

for all $a \in X$. Hence there exists a unique linear map $\phi : C(X) \to F$ satisfying $\phi \circ i_X = f$. This means that the diagram commutes.

(ii). Let $C(Y) = F$ and $f = i_Y \circ \alpha : X \to C(Y)$ in part (i). Then there exists a unique linear map $\alpha_* : C(X) \to C(Y)$ satisfying $\alpha_* \circ i_X = f = i_Y \circ \alpha$. If $\beta : Y \to Z$ is another function, then we get again by part (i), a (unique) linear map $\beta_* : C(Y) \to C(Z)$ satisfying $\beta_* \circ i_Y = i_Z \circ \beta$. Putting these two together we have

$$\beta_* \circ \alpha_* \circ i_X = \beta_* \circ i_Y \circ \alpha = i_Z \circ \beta \circ \alpha. \tag{1}$$

On the other hand, again by part (i), there exists a unique linear map $(\alpha \circ \beta)_* : C(X) \to C(Z)$ satisfying $(\alpha \circ \beta)_* \circ i_X = i_Z \circ (\beta \circ \alpha)$. Now as compositions of linear maps are linear, we get that $\beta_* \circ \alpha_* : C(X) \to C(Z)$ is linear and since $\beta_* \circ \alpha_*$ satisfies equation (1), we must have by uniqueness (of $(\alpha \circ \beta)_*$) that

$$(\alpha \circ \beta)_* = \beta_* \circ \alpha_*.$$

(iii). Now $i_E : E \to C(E)$ is given by $i_E(\mathbf{v}) = \delta_{\mathbf{v}}$ for all $\mathbf{v} \in E$. Using $f = \iota$ the identity on E, we get from part (i) that there exists a unique linear map $\pi_E : C(E) \to E$ satisfying $\pi_E \circ i_E = \iota$.

(iv). Assume that $\phi : E \to F$ is linear. First observe that for all $\mathbf{v} \in E$, one has

$$\begin{aligned} \pi_F(\phi_*(\delta_{\mathbf{v}})) = (\pi_F \circ \phi_*)(\delta_{\mathbf{v}}) &= \pi_F(\phi_*(i_E(\mathbf{v}))) \quad \text{by def. of } i_E \\ &= \pi_F(i_F(\phi(\mathbf{v}))) \quad \text{by (ii)}, \\ &= \iota(\phi(\mathbf{v})) \quad \text{by (iii)}, \\ &= \phi(\mathbf{v}) \quad \text{as } \iota \text{ is the identity map}, \\ &= \phi(\pi_E(\iota_E(\mathbf{v}))) \quad \text{by (iii)}, \\ &= \phi(\pi_E(\delta_{\mathbf{v}})) \quad \text{by the def. of } i_E. \end{aligned}$$

Now ϕ is linear by hypothesis, and π_E, π_F and ϕ_* are linear by parts (iii) and (ii) respectively. As a consequence the compositions $\phi \circ \pi_E$ and $\pi_F \circ \phi_*$ are linear (see §1.10). The above calculation shows that the two linear maps agree on the basis $\{\delta_{\mathbf{v}}\}$. Then by the uniqueness property of either Proposition I or Proposition II, we see that $\phi \circ \pi_E = \pi_F \circ \phi_*$.

Now suppose that $\phi \circ \pi_E = \pi_F \circ \phi_*$ and we want to show that ϕ is linear. This follows from the following calculation for $\lambda_{\mathbf{v}} \in \Gamma$:

$$\begin{aligned}
\phi(\sum_{\mathbf{v}\in E} \lambda_{\mathbf{v}}\mathbf{v}) &= \phi\left(\sum_{\mathbf{v}\in E} \lambda_{\mathbf{v}}\pi_E \circ i_E(\mathbf{v})\right) \quad \text{as } \pi_E \circ i_E = \iota \text{ by (iii)} \\
&= \phi\left(\sum_{\mathbf{v}\in E} \lambda_{\mathbf{v}}\pi_E(\delta_{\mathbf{v}})\right) \quad \text{by def. of } i_E \\
&= \phi\left(\pi_E\left(\sum_{\mathbf{v}\in E} \lambda_{\mathbf{v}}\delta_{\mathbf{v}}\right)\right) \quad \text{as } \pi_E \text{ is linear by (iii),} \\
&= (\pi_F \circ \phi_*)\left(\sum_{\mathbf{v}\in E} \lambda_{\mathbf{v}}\delta_{\mathbf{v}}\right) \quad \text{by hypothesis} \\
&= \sum_{\mathbf{v}\in E} \lambda_{\mathbf{v}}(\pi_F \circ \phi_*)\,(\delta_{\mathbf{v}}) \quad \text{as } \pi_F \circ \phi_* \text{ is linear} \\
&= \sum_{\mathbf{v}\in E} \lambda_{\mathbf{v}}(\phi \circ \pi_E)\,(\delta_{\mathbf{v}}) \quad \text{by hypothesis} \\
&= \sum_{\mathbf{v}\in E} \lambda_{\mathbf{v}}(\phi \circ \pi_E)\,(i_E(\mathbf{v})) \quad \text{by def. of } i_E \\
&= \sum_{\mathbf{v}\in E} \lambda_{\mathbf{v}}\phi(\mathbf{v}) \quad \text{as } \pi_E \circ i_E = \iota \text{ by (iii)}
\end{aligned}$$

(v). Let us show that the generators of $N(E)$ are in $\ker \pi_E$:

$$\begin{aligned}
\pi_E(\delta_{\lambda a+\mu b} - \lambda\delta_a - \mu\delta_b) &= \pi_E(\delta_{\lambda a+\mu b}) - \lambda\pi_E(\delta_a) - \mu\pi_E(\delta_b) \quad \text{as } \pi_E \text{ is linear by (iii)} \\
&= \pi_E(i_E(\lambda a+\mu b)) - \lambda\pi_E(i_E(a)) - \mu\pi_E(i_E(b)) \quad \text{by the def. of } i_E \\
&= \lambda a + \mu b - \lambda a - \mu b \quad \text{as } \pi_E \circ i_E = \iota \\
&= 0.
\end{aligned}$$

Since the generators of $N(E)$ are in $\ker \pi_E$, we get that any linear combination of the

$$\delta_{\lambda a+\mu b} - \lambda\delta_a - \mu\delta_b$$

are in $\ker \pi_E$. Since $N(E)$ is generated these elements we get $N(E) \subset \ker \pi_E$. On the other hand suppose $\sum_{a\in E} \lambda_a\delta_a \in \ker \pi_E$. Observe that $\delta_{\mathbf{0}} \in N(E)$ by taking $\lambda = 0 = \mu$ in generators above and that by induction elements of the form

$$\sum_{i=0^n} \lambda_i\delta_{a_i} - \delta_{\sum_{i=0}^n \lambda_i a_i}$$

are in $N(E)$ for $\lambda_i \in \Gamma$ and $a_i \in E$. Then

$$\begin{aligned}
\mathbf{0} = \pi_E\left(\sum_{a\in E} \lambda_a\delta_a\right) &\quad \text{by hypothesis and def. of } \ker \phi_E \\
= \sum_{a\in E} \lambda_a\pi_E i_E(a) &\quad \text{as } \pi_E \text{ is linear and by the def. of } i_E \\
= \sum_{a\in E} \lambda_a a &\quad \text{as } \pi_E \circ i_E = \iota \text{ from (iii)}
\end{aligned}$$

Now since $\mathbf{0} = \sum_{a\in E} \lambda_a a$

$$\sum_{a\in E} \lambda_a\delta_a = \sum_{a\in E} \lambda_a\delta_a - \delta_{\sum_{a\in E} \lambda_a a} + \delta_{\mathbf{0}} \in N(E)$$

as the difference of the first two summands is in $N(E)$ and so is $\delta_{\mathbf{0}}$.

1.3. **Subspaces and Factor Spaces: Exercises.**

1. a) This is a subspace. Indeed let

$$F_a := \{(\xi^1, \xi^2, \xi^3) \,|\, \xi^i \in \Gamma, \xi^1 = \xi^2 = \xi^3\}.$$

Then

$$\begin{aligned}\mathbf{v}, \mathbf{w} \in F_a \Longrightarrow\ & \mathbf{v} = (\xi^1, \xi^2, \xi^3),\ \text{and}\ \mathbf{w} = (\chi^1, \chi^2, \chi^3)\\ & \text{for some}\ \xi^i, \chi^j \in \Gamma,\ \ \xi^1 = \xi^2 = \xi^3, \chi^1 = \chi^2 = \chi^3\\ \Longrightarrow\ & \mathbf{v} + \mathbf{w} = (\xi^1 + \chi^1, \xi^2 + \chi^2, \xi^3 + \chi^3),\ \text{with}\ \xi^1 + \chi^1 = \xi^2 + \chi^2 = \xi^3 + \chi^3\\ \Longrightarrow\ & \mathbf{v} + \mathbf{w} \in F_a.\end{aligned}$$

Similarly

$$\begin{aligned}\mathbf{v} \in F_a, \lambda \in \Gamma \Longrightarrow\ & \mathbf{v} = (\xi^1, \xi^2, \xi^3),\\ & \text{for some}\ \xi^i \in \Gamma,\ \ \xi^1 = \xi^2 = \xi^3,\\ \Longrightarrow\ & \lambda \cdot \mathbf{v} = (\lambda\xi^1, \lambda\xi^2, \lambda\xi^3),\ \text{with}\ \lambda\xi^1 = \lambda\xi^2 = \lambda\xi^3\\ \Longrightarrow\ & \lambda\mathbf{v} \in F_a.\end{aligned}$$

c) This is a subspace. Indeed let

$$F_c := \{(\xi^1, \xi^2, \xi^3) \,|\, \xi^i \in \Gamma, \xi^1 = \xi^2 - \xi^3\}.$$

Then

$$\begin{aligned}\mathbf{v}, \mathbf{w} \in F_c \Longrightarrow\ & \mathbf{v} = (\xi^1, \xi^2, \xi^3),\ \text{and}\ \mathbf{w} = (\chi^1, \chi^2, \chi^3)\\ & \text{for some}\ \xi^i, \chi^j \in \Gamma,\ \ \xi^1 = \xi^2 - \xi^3, \chi^1 = \chi^2 - \chi^3\\ \Longrightarrow\ & \mathbf{v} + \mathbf{w} = (\xi^1 + \chi^1, \xi^2 + \chi^2, \xi^3 + \chi^3),\ \text{with}\ \xi^1 + \chi^1 = \xi^2 + \chi^2 - (\xi^3 + \chi^3)\\ \Longrightarrow\ & \mathbf{v} + \mathbf{w} \in F_c.\end{aligned}$$

Similarly

$$\begin{aligned}\mathbf{v} \in F_c, \lambda \in \Gamma \Longrightarrow\ & \mathbf{v} = (\xi^1, \xi^2, \xi^3),\\ & \text{for some}\ \xi^i \in \Gamma,\ \ \xi^1 = \xi^2 - \xi^3,\\ \Longrightarrow\ & \lambda \cdot \mathbf{v} = (\lambda\xi^1, \lambda\xi^2, \lambda\xi^3),\ \text{with}\ \lambda\xi^1 = \lambda(\xi^2 - \xi^3) = \lambda\xi^2 - \lambda\xi^3\\ \Longrightarrow\ & \lambda\mathbf{v} \in F_c.\end{aligned}$$

Hence it is a subspace.

d) This is not a subspace. Indeed let

$$S_d := \{(\xi^1, \xi^2, \xi^3) \,|\, \xi^i \in \Gamma, \xi^1 = 1\}.$$

then $(1, 0, 0) \in S_d$, but $(1, 0, 0) + (1, 0, 0) = (2, 0, 0) \notin S_d$.

3. a) The subspace

$$F_a := \{(\xi^1, \xi^2, \xi^3) \,|\, \xi^i \in \Gamma, \xi^1 = \xi^2 = \xi^3\}.$$

has as a basis $B_a = \{(1, 1, 1)\}$. Indeed if

$$\begin{aligned}\mathbf{v} \in F_a \Longrightarrow\ & \mathbf{v} = (\xi^1, \xi^2, \xi^3)\\ & \text{for some}\ \xi^i \in \Gamma,\ \text{with}\ \xi^1 = \xi^2 = \xi^3\\ \Longrightarrow\ & \mathbf{v} = (\xi^1, \xi^2, \xi^3) = \xi^1(1, 1, 1).\end{aligned}$$

(so $(1, 1, 1)$ spans F_a) and

$$(0, 0, 0) = \lambda(1, 1, 1) = (\lambda, \lambda, \lambda)\ \text{for some}\ \lambda \in \Gamma \Longrightarrow \lambda = 0,$$

(so $(1,1,1)$ is linearly independent). To find a basis of the quotient space we need to enlarge B_a by adding two addition vectors to form a basis of Γ^3. For such vectors one might take $\mathbf{e}_1 = (1,0,0), \mathbf{e}_2 = (0,1,0)$ (there are infinitely many other pairs of vectors that will also work). We need to show that $B_a \cup \{\mathbf{e}_1, \mathbf{e}_2\}$ forms a basis of Γ^3. Indeed

$$\begin{aligned}
\mathbf{v} \in \Gamma^3 \implies & \mathbf{v} = (\xi^1, \xi^2, \xi^3) \text{ for some } \xi^i, \\
& \implies \mathbf{v} = (\xi^1, \xi^2, \xi^3) = \xi^1\mathbf{e}_1 + \xi^2\mathbf{e}_2 + \xi^3\mathbf{e}_3 = \xi^1\mathbf{e}_1 + \xi^2\mathbf{e}_2 + \xi^3((1,1,1) - \mathbf{e}_1 - \mathbf{e}_2) \\
& \quad = (\xi^1 - \xi^3)\mathbf{e}_1 + (\xi^2 - \xi^3)\mathbf{e}_2 + \xi^3(1,1,1)
\end{aligned}$$

(so $B_a \cup \{\mathbf{e}_1, \mathbf{e}_2\}$ spans Γ^3) and

$$\begin{aligned}
(0,0,0) &= \lambda^1\mathbf{e}_1 + \lambda^2\mathbf{e}_2 + \lambda^3(1,1,1) \text{ for some } \lambda^i \in \Gamma \\
&= (\lambda^1 + \lambda^3, \lambda^2 + \lambda^3, \lambda^3) \\
&\implies \lambda^1 + \lambda^3 = \lambda^2 + \lambda^3 = \lambda^3 = 0 \\
&\implies \lambda^1 = \lambda^2 = \lambda^3 = 0
\end{aligned}$$

(so $B_a \cup \{\mathbf{e}_1, \mathbf{e}_2\}$ consists of linearly independent vectors). If we let $\pi_a : \Gamma^3 \to \Gamma^3/F_a$ be the canonical projection, then $\{\pi_a(\mathbf{e}_1), \pi_a(\mathbf{e}_2)\}$ is a basis of the quotient space.

c) F_c has basis $B_c = \{(1,1,0), (-1,0,1)\}$ (it has infinitely many other pairs of basis elements so you may have come up with a different pair). Indeed

$$\begin{aligned}
\mathbf{v} \in F_c \implies & \mathbf{v} = (\xi^1, \xi^2, \xi^3) \text{ with } \xi^1 = \xi^2 - \xi^3, \\
& \implies \mathbf{v} = (\xi^1, \xi^2, \xi^3) = (\xi^2 - \xi^3, \xi^2, \xi^3) = (\xi^2, \xi^2, 0) + (-\xi^3, 0, \xi^3) \\
& \quad = \xi^2(1,1,0) + \xi^3(-1,0,1).
\end{aligned}$$

This proves that B_c spans F_c. Now

$$\begin{aligned}
(0,0,0) = \lambda^2(1,1,0) + \lambda^3(-1,0,1) \quad & \text{for some } \lambda^i \in \Gamma \\
& \implies (0,0,0) = \lambda^2(1,1,0) + \lambda^3(-1,0,1) = (\lambda^2 - \lambda^3, \lambda^2, \lambda^3) \\
& \implies \lambda^2 = 0 = \lambda^3.
\end{aligned}$$

Hence B_c consists of linearly independent vectors. Then B_c is a basis for F_c. To find a basis of the quotient space we need to enlarge B_c by adding an addition vector to form a basis of Γ^3. For such a vector one might take $\mathbf{e}_1 = (1,0,0)$. We need to show that $B_c \cup \{\mathbf{e}_1\}$ forms a basis of Γ^3. Indeed

$$\begin{aligned}
\mathbf{v} \in \Gamma^3 \implies & \mathbf{v} = (\xi^1, \xi^2, \xi^3) \text{ for some } \xi^i, \\
& \implies \mathbf{v} = (\xi^1, \xi^2, \xi^3) = \xi^1\mathbf{e}_1 + \xi^2\mathbf{e}_2 + \xi^3\mathbf{e}_3 = \xi^1\mathbf{e}_1 + \xi^2((1,1,0) - \mathbf{e}_1) + \xi^3((-1,0,1) + \mathbf{e}_1) \\
& \quad = (\xi^1 - \xi^2 + \xi^3)\mathbf{e}_1 + \xi^2(1,1,0) + \xi^3(-1,0,1)
\end{aligned}$$

(so $B_c \cup \{\mathbf{e}_1\}$ spans Γ^3) and

$$\begin{aligned}
(0,0,0) &= \lambda^1\mathbf{e}_1 + \lambda^2(1,1,0) + \lambda^3(-1,0,1) \text{ for some } \lambda^i \in \Gamma \\
&= (\lambda^1 + \lambda^2 - \lambda^3, \lambda^2, \lambda^3) \\
&\implies \lambda^1 - \lambda^2 + \lambda^3 = \lambda^2 = \lambda^3 = 0 \\
&\implies \lambda^1 = \lambda^2 = \lambda^3 = 0
\end{aligned}$$

(so $B_c \cup \{\mathbf{e}_1\}$ consists of linearly independent vectors). If we let $\pi_c : \Gamma^3 \to \Gamma^3/F_c$ be the canonical projection, then $\{\pi_c(\mathbf{e}_1)\}$ is a basis of the quotient space.

d) The subspace generated by S_d is

$$F_d := \{(\xi^1, \xi^2, \xi^3) \mid \xi^i \in \Gamma\} = \Gamma^3.$$

Indeed $\mathbf{e}_1 \in F_d$ and so are the standard basis vectors

$$\mathbf{e}_2 = (1,1,0) - \mathbf{e}_1 \in F_d, \quad \mathbf{e}_3 = (1,1,1) - \mathbf{e}_1 - \mathbf{e}_2 \in F_d$$

as any linear combination of elements in F_d are in F_d (this follows from the definition of the subspace generated by a set). Hence any vector $(\xi^1, \xi^3, \xi^3) = \xi^1\mathbf{e}_1 + \xi^2\mathbf{e}_2 + \xi^3\mathbf{e}_3$ is in F_d. Now the quotient space is $\Gamma^3/F_d = 0$.

5. a)

$$\begin{aligned}\mathbf{v} \in \Gamma^3 &\implies \mathbf{v} = (\xi^1, \xi^2, \xi^3) \text{ for some } \xi^i,\\ &\implies \mathbf{v} = (\xi^1, \xi^2, \xi^3) = \xi^1\mathbf{e}_1 + \xi^2\mathbf{e}_2 + \xi^3\mathbf{e}_3 = \xi^1\mathbf{e}_1 + \xi^2\mathbf{e}_2 + \xi^3((1,1,1) - \mathbf{e}_1 - \mathbf{e}_2)\\ &= (\xi^1 - \xi^3)\mathbf{e}_1 + (\xi^2 - \xi^3)\mathbf{e}_2 + \xi^3(1,1,1)\end{aligned}$$

so that $\mathbf{v} \in F_b + F_a = F_a + F_b$.

b)

$$\begin{aligned}\mathbf{v} \in \Gamma^3 &\implies \mathbf{v} = (\xi^1, \xi^2, \xi^3) \text{ for some } \xi^i,\\ &\implies \mathbf{v} = (\xi^1, \xi^2, \xi^3) = \xi^1\mathbf{e}_1 + \xi^2\mathbf{e}_2 + \xi^3\mathbf{e}_3 = \xi^1\mathbf{e}_1 + \xi^2((1,1,0) - \mathbf{e}_1) + \xi^3((-1,0,1) + \mathbf{e}_1)\\ &= (\xi^1 - \xi^2 + \xi^3)\mathbf{e}_1 + \xi^2(1,1,0) + \xi^3(-1,0,1)\end{aligned}$$

so that $\mathbf{v} \in F_b + F_c$.

$$\begin{aligned}\mathbf{v} \in \Gamma^3 &\implies \mathbf{v} = (\xi^1, \xi^2, \xi^3) \text{ for some } \xi^i,\\ &\implies \mathbf{v} = (\xi^1, \xi^2, \xi^3) = \xi^1\mathbf{e}_1 + \xi^2\mathbf{e}_2 + \xi^3\mathbf{e}_3 = \xi^1\mathbf{e}_1 + \xi^2((1,1,0) - \mathbf{e}_1) + \xi^3((-1,0,1) + \mathbf{e}_1)\\ &= (\xi^1 - \xi^2 + \xi^3)\mathbf{e}_1 + \xi^2(1,1,0) + \xi^3(-1,0,1)\\ &= (\xi^1 - \xi^2 + \xi^3)(-(1,1,0) - (-1,0,1) + (1,1,1)) + \xi^2(1,1,0) + \xi^3(-1,0,1)\\ &= (\xi^1 - \xi^2 + \xi^3)(1,1,1) + (-\xi^1 + 2\xi^2 - \xi^3)(1,1,0) + (-\xi^1 + \xi^2)(-1,0,1)\end{aligned}$$

so that $\mathbf{v} \in F_a + F_c$.

Now if $\mathbf{v} = (\xi^1, \xi^2, \xi^3) \in F_a \cap F_b$, then $\xi^1 = \xi^2 = \xi^3 = 0$. Thus $F_a \cap F_b = \{\mathbf{0}\}$. Hence $F_a \oplus F_b = \Gamma^3$.

If $\mathbf{v} = (\xi^1, \xi^2, \xi^3) \in F_b \cap F_c$, then $\xi^1 = \xi^2 - \xi^3$ and $\xi^3 = 0$. Hence a vector like $(1,1,0) \in F_b \cap F_c$. In fact such a vector forms a basis of $F_b \cap F_c$ as it spans $F_b \cap F_c$:

$$\begin{aligned}\mathbf{v} = (\xi^1, \xi^2, \xi^3) \in F_b \cap F_c &\text{ with } \xi^1 = \xi^2 - \xi^3, \xi^3 = 0\\ &\implies \xi^1 = \xi^2 \text{ and } \mathbf{v} = (\xi^1, \xi^2, 0) = \xi^1(1,1,0).\end{aligned}$$

Thus $F_b + F_c$ is not a direct sum.

If $\mathbf{v} = (\xi^1, \xi^2, \xi^3) \in F_a \cap F_c$, then $\xi^1 = \xi^2 = \xi^3$ and $\xi^1 = \xi^2 - \xi^3$, so that $\xi^1 = \xi^2 = \xi^3 = 0$. Thus $F_a \cap F_c = \{\mathbf{0}\}$. Hence $F_a \oplus F_c = \Gamma^3$.

10. We need to show that $E_+ + E_- = E$ and $E_+ \cap E_1 = 0$. We have that if $\mathbf{v} \in E_- \cap E_+$, then $-\mathbf{v} = \omega(\mathbf{v})$ as $\mathbf{v} \in E_-$ and $\mathbf{v} = \omega(\mathbf{v})$ as $\mathbf{v} \in E_+$. Hence $\mathbf{v} = -\mathbf{v}$ and thus $2\mathbf{v} = \mathbf{0}$. Since 2 is invertible in any field of characteristic zero we can multiply through by $1/2$ to get $\mathbf{v} = \mathbf{0}$. Hence $E_+ \cap E_1 = 0$.

Let $\mathbf{u} \in E$ be an arbitrary element. Then

$$\mathbf{u} = \frac{\mathbf{u} + \omega(\mathbf{u})}{2} + \frac{\mathbf{u} - \omega(\mathbf{u})}{2} \tag{2}$$

where

$$\omega\left(\frac{\mathbf{u} + \omega(\mathbf{u})}{2}\right) = \frac{\omega(\mathbf{u}) + \omega^2(\mathbf{u})}{2} = \frac{\mathbf{u} + \omega(\mathbf{u})}{2}$$

and

$$\omega\left(\frac{\mathbf{u} - \omega(\mathbf{u})}{2}\right) = \frac{\omega(\mathbf{u}) - \omega^2(\mathbf{u})}{2} = -\frac{\mathbf{u} - \omega(\mathbf{u})}{2}$$

as $\omega^2 = I$. Hence

$$\frac{\mathbf{u} + \omega(\mathbf{u})}{2} \in E_+, \quad \frac{\mathbf{u} - \omega(\mathbf{u})}{2} \in E_-.$$

Then equation (2) tells us that $E = E_- + E_+$.

12. Take any distinct two lines E_1, E_2 and a plane E_3 not containing the two lines. This should give us $E_i \cap E_j = 0$ and $E_1 + E_2 + E_3 = \Gamma^3$, but $E_i \cap (E_j + E_k) \neq \{\mathbf{0}\}$ for certain i, j, k distinct. For example take $E_1 = \Gamma(1, 0, 0) = \{(\gamma, 0, 0) \,|, \gamma \in \Gamma\}$, $E_2 = \Gamma(1, 1, 0) = \{(\gamma, \gamma, 0) \,|, \gamma \in \Gamma\}$ and $E_3 = \{(0, \xi^2, \xi^3) \,|\, \xi^i \in \Gamma\}$. It is straightforward to see that $E_1 \cap E_2 = \{\mathbf{0}\}$, $E_1 \cap E_3 = \{\mathbf{0}\}$ and $E_2 \cap E_3 = \{\mathbf{0}\}$. Note $E_1 \cap (E_2 + E_3) \neq \{\mathbf{0}\}$ as $(1, 0, 0) = (1, 1, 0) - (0, 1, 0) \in E_1 \cap (E_2 + E_3)$. Moreover

$$\mathbf{e}_i \in E_1 + E_2 + E_3, \quad i = 1, 2, 3,$$

so that $\Gamma^3 = E_1 + E_2 + E_3$ as $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ span Γ^3.