

COLLEGE OF CHARLESTON
DEPARTMENT OF MATHEMATICS



Name:

Examination in: Advanced Linear Algebra

Math Course Number	Math 502
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Total number of problems: 5

Professor: Ben Cox

Proctor:

Phone number: 953-5715

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Permitted aids: Proctor, No use of calculators are permitted on this exam.

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1. Define a linear map $\psi : \mathbb{R}^4 \rightarrow \mathbb{R}^2$ by

$$\psi(\xi^1, \xi^2, \xi^3, \xi^4) := (-\xi^1 + 7\xi^2 + 5\xi^3, -9\xi^2 + 7\xi^3 - \xi^4).$$

a) Is ψ surjective?

b) Let $\iota : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the identity map. Find a linear map $\chi : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ such that $\psi \circ \chi = \iota$.

Sln: a). (This was a variation of the 1. b) of the practice exam.) Lets show that it is surjective. Observe that

$$\begin{aligned}\psi(-1, 0, 0, 0) &= (1, 0) = \mathbf{e}_1 \\ \psi(0, 0, 0, -1) &= (0, 1) = \mathbf{e}_2.\end{aligned}$$

Hence $\mathbf{e}_1, \mathbf{e}_2 \in \text{im } \psi$, which implies that $\mathbb{R}^2 = \text{im } \psi$ so that ψ is surjective.

b). We know that

$$\psi(-1, 0, 0, 0) := (1, 0) = \mathbf{e}_1, \quad \psi(0, 0, 0, -1) := (0, 1) = \mathbf{e}_2$$

Define $\chi(\mathbf{e}_1) := (-1, 0, 0, 0)$ and $\chi(\mathbf{e}_2) := (0, 0, 0, -1)$ and we extend this linearly so

$$\chi(\lambda_1, \lambda_2) = \chi(\lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2) = \lambda_1 \chi(\mathbf{e}_1) + \lambda_2 \chi(\mathbf{e}_2) = (-\lambda_1, 0, 0, -\lambda_2).$$

Then

$$\psi \chi(\lambda_1, \lambda_2) = \psi(-\lambda_1, 0, 0, -\lambda_2) = (\lambda_1, \lambda_2).$$

Thus $\psi \circ \chi = \iota$. There are numerous other ways to define a χ that also satisfies $\psi \circ \chi$.

2. Let V and W over Γ and let $\mathcal{C}(V \times W)$ denote the free vector space on the set $V \times W$ with canonical inclusion $\iota_{V \times W} : V \times W \rightarrow \mathcal{C}(V \times W)$ given by $\iota_{V \times W}(v, w) := \delta_{(v, w)}$. Consider the subspace \mathcal{B} spanned by elements of the form

$$\delta_{(\mathbf{v}+\mathbf{v}', \mathbf{w})} - \delta_{(\mathbf{v}, \mathbf{w})} - \delta_{(\mathbf{v}', \mathbf{w})} \quad (1)$$

$$\delta_{(\mathbf{v}, \mathbf{w}+\mathbf{w}')} - \delta_{(\mathbf{v}, \mathbf{w})} - \delta_{(\mathbf{v}, \mathbf{w}')} \quad (2)$$

$$a\delta_{(\mathbf{v}, \mathbf{w})} - \delta_{(a\mathbf{v}, \mathbf{w})} \quad (3)$$

$$\delta_{(a\mathbf{v}, \mathbf{w})} - \delta_{(\mathbf{v}, a\mathbf{w})} \quad (4)$$

for $a \in \Gamma$, $\mathbf{v}, \mathbf{v}' \in V$, and $\mathbf{w}, \mathbf{w}' \in W$.

The quotient space $\mathcal{C}(V \times W)/\mathcal{B}$ is denoted by $V \otimes W$ and is called the *tensor product* of V and W . The equivalence class of an element $\delta_{(\mathbf{v}, \mathbf{w})}$ in $V \otimes W$ is denoted by $\mathbf{v} \otimes \mathbf{w}$. In other words $v \otimes w = \delta_{(v, w)} + \mathcal{B}$. Let $\pi : V \times W \rightarrow V \otimes W$ denote the inclusion $\iota_{V \times W}$ followed by the canonical projection so that $(\mathbf{v}, \mathbf{w}) \mapsto \mathbf{v} \otimes \mathbf{w}$. Show that if $b : V \times W \rightarrow U$ is a bilinear map, then there exists a unique linear map $B : V \otimes W \rightarrow U$ satisfying $B \circ \pi = b$. In other words the diagram to the right is commutative.

Sln: (This was problem 3 on the practice exam.) Now by the universal mapping property of $\mathcal{C}(V \times W)$ there exists a unique linear map $\bar{B} : \mathcal{C}(V \times W) \rightarrow U$ satisfying $\bar{B} \circ \iota_{V \times W} = b$. Since b is bilinear we get

$$\begin{aligned}\bar{B}(\delta_{(\mathbf{v}+\mathbf{v}', \mathbf{w})} - \delta_{(\mathbf{v}, \mathbf{w})} - \delta_{(\mathbf{v}', \mathbf{w})}) &= \bar{B}(\delta_{(\mathbf{v}+\mathbf{v}', \mathbf{w})}) - \bar{B}(\delta_{(\mathbf{v}, \mathbf{w})}) - \bar{B}(\delta_{(\mathbf{v}', \mathbf{w})}) \\ &= b(\mathbf{v} + \mathbf{v}', \mathbf{w}) - b(\mathbf{v}, \mathbf{w}) - b(\mathbf{v}', \mathbf{w}) = 0, \\ \bar{B}(\delta_{(\mathbf{v}, \mathbf{w}+\mathbf{w}')} - \delta_{(\mathbf{v}, \mathbf{w})} - \delta_{(\mathbf{v}, \mathbf{w}')} &= \bar{B}(\delta_{(\mathbf{v}, \mathbf{w}+\mathbf{w}')} - \bar{B}(\delta_{(\mathbf{v}, \mathbf{w})}) - \bar{B}(\delta_{(\mathbf{v}, \mathbf{w}')} \\ &= b(\mathbf{v}, \mathbf{w} + \mathbf{w}') - b(\mathbf{v}, \mathbf{w}) - b(\mathbf{v}, \mathbf{w}') = 0, \\ \bar{B}(a\delta_{(\mathbf{v}, \mathbf{w})} - \delta_{(a\mathbf{v}, \mathbf{w})}) &= \bar{B}(a\delta_{(\mathbf{v}, \mathbf{w})}) - \bar{B}(\delta_{(a\mathbf{v}, \mathbf{w})}) \\ &= b(a\mathbf{v}, \mathbf{w}) - b(a\mathbf{v}, \mathbf{w}) = 0, \\ \bar{B}(a\delta_{(\mathbf{v}, \mathbf{w})} - \delta_{(\mathbf{v}, a\mathbf{w})}) &= \bar{B}(a\delta_{(\mathbf{v}, \mathbf{w})}) - \bar{B}(\delta_{(\mathbf{v}, a\mathbf{w})}) \\ &= b(a\mathbf{v}, \mathbf{w}) - b(\mathbf{v}, a\mathbf{w}) = 0.\end{aligned}$$

$$\begin{array}{ccc} V \times W & \xrightarrow{\pi} & V \otimes W \\ & \searrow b & \downarrow B \\ & & U \end{array}$$

$$\begin{array}{ccc} V \times W & \xrightarrow{\iota_{V \times W}} & \mathcal{C}(V \times W) \\ & \searrow b & \downarrow \bar{B} \\ & & U \end{array}$$

Since elements in (1), (2), (3) and (4) form a spanning set for \mathcal{B} , have $\bar{B}(\mathcal{B}) = 0$. If we let $\bar{\pi} : \mathcal{C}(V \times W) \rightarrow V \otimes W = \mathcal{C}(V \times W)/\mathcal{B}$ denote the canonical projection, then by the First Isomorphism Theorem there exists a unique linear map $B : V \otimes W \rightarrow U$ such that $B \circ \bar{\pi} = \bar{B}$. Since $\pi = \bar{\pi} \circ \iota_{V \times W}$, we get $B \circ \pi = B \circ \bar{\pi} \circ \iota_{V \times W} = \bar{B} \circ \iota_{V \times W} = b..$

3. Let ι of a vector space E to itself.

- Prove that if $\pi : E \rightarrow E$ is a projection operator, then $\omega := 2\pi - \iota$ is an involution.
- Prove that if $\Omega : E \rightarrow E$ is an involution, then $\Omega = 2\Pi - \iota$ for some projection operator $\Pi : E \rightarrow E$.

Sln: (This was problem 4 from section 2.4 whose solution was written up for you earlier.) Suppose π is a projection of E . Then $\pi^2 = \pi \circ \pi = \pi$. Set $\omega = 2\pi - \iota$ where ι is the identity map on E . Then

$$\omega^2 = (2\pi - \iota) \circ (2\pi - \iota) = 4\pi^2 - 2\pi \circ \iota - 2\iota \circ \pi + \iota^2 = \iota$$

as $\pi^2 = \pi$ and $\iota^2 = \iota$. Since π and ι are linear so is ω . From the above calculation ω must then be an involution.

Now suppose Ω is an involution defined on E . Set $\Pi = \frac{1}{2}(\Omega + \iota)$ where ι is the identity map on E . Then

$$\Pi^2 = \frac{1}{4}(\Omega + \iota) \circ (\Omega + \iota) = \frac{1}{4}(\Omega^2 + \iota \circ \Omega + \Omega \circ \iota + \iota^2) = \frac{1}{4}(2\Omega + 2\iota) = \Pi,$$

as $\iota \circ \Omega = \Omega \circ \iota = \Omega$ and $\Omega^2 = \iota$. Since Ω and ι are linear, so is Π . Hence Π is a projection operator.

4. Given a pair of dual basis $\{\mathbf{v}_k^* \mid 1 \leq k \leq n\}$ and $\{\mathbf{v}_k \mid 1 \leq k \leq n\}$, of E^* and E respectively, show that the bases

$$\{\mathbf{v}_1^* + \sum_{k=2}^n \lambda_k \mathbf{v}_k^*, \mathbf{v}_2^*, \dots, \mathbf{v}_n^*\}$$

and

$$\{\mathbf{v}_1, \mathbf{v}_2 - \lambda_2 \mathbf{v}_1, \dots, \mathbf{v}_n - \lambda_n \mathbf{v}_1\}$$

are again dual bases.

Sln: (This was chapter 2 section 6's problem number 4 whose solution was written up for you earlier.)

$$\langle \mathbf{x}^{*1} + \sum_{\nu=2}^n \lambda_\nu \mathbf{x}^{*\nu}, \mathbf{x}_1 \rangle = \langle \mathbf{x}^{*1}, \mathbf{x}_1 \rangle + \sum_{\nu=2}^n \lambda_\nu \langle \mathbf{x}^{*\nu}, \mathbf{x}_1 \rangle = 1$$

$$\begin{aligned} \langle \mathbf{x}^{*1} + \sum_{\nu=2}^n \lambda_\nu \mathbf{x}^{*\nu}, \mathbf{x}_k - \lambda_k \mathbf{x}_1 \rangle &= -\lambda_k \langle \mathbf{x}^{*1}, \mathbf{x}_1 \rangle + \sum_{\nu=2}^n \lambda_\nu \langle \mathbf{x}^{*\nu}, \mathbf{x}_k \rangle \\ &= -\lambda_k + \lambda_k = 0 \text{ for } 2 \leq k \leq n. \end{aligned}$$

Moreover for $2 \leq \nu \leq n$,

$$\begin{aligned} \langle \mathbf{x}^{*\nu}, \mathbf{x}_1 \rangle &= 0, \\ \langle \mathbf{x}^{*\nu}, \mathbf{x}_k - \lambda_k \mathbf{x}_1 \rangle &= \langle \mathbf{x}^{*\nu}, \mathbf{x}_k \rangle = \delta_{\nu,k}. \end{aligned}$$

5. Let E^* and E be dual spaces with respect to the non-degenerate bilinear form $\langle \cdot, \cdot \rangle$. Suppose $F \subset E$ and $H^* \subset E^*$ are subspaces. Consider the assignment

$$\langle \cdot, \cdot \rangle_\pi : H^*/H^* \cap F^\perp \times F/F \cap (H^*)^\perp$$

given by

$$\langle \bar{\mathbf{v}}^*, \bar{\mathbf{u}} \rangle_\pi := \langle \mathbf{v}^*, \mathbf{u} \rangle$$

for all $\bar{\mathbf{v}}^* \in H^*/H^* \cap F^\perp$ and $\bar{\mathbf{u}} \in F/F \cap (H^*)^\perp$.

- a) Show $\langle \cdot, \cdot \rangle_\pi$ is well defined.
- b) Show $\langle \cdot, \cdot \rangle_\pi$ is linear in the first entry. (It is also linear in the second entry, but you don't need to show this.)
- c) Show $\langle \cdot, \cdot \rangle_\pi$ is non-degenerate.

Sln: We went over a simpler example than this problem in class where we only considered $H^* = E^*$. a):

$$\begin{aligned}
 \bar{\mathbf{v}}^* = \bar{\mathbf{w}}^* \text{ and } \bar{\mathbf{u}} = \bar{\mathbf{z}} &\implies \mathbf{v}^* - \mathbf{w}^* \in H^* \cap F^\perp, \mathbf{u} - \mathbf{z} \in F \cap (H^*)^\perp \\
 &\text{and } \mathbf{v}^*, \mathbf{w}^* \in H^*, \mathbf{u}, \mathbf{z} \in F \\
 &\implies \langle \mathbf{v}^* - \mathbf{w}^*, \mathbf{u} \rangle = 0 = \langle \mathbf{v}^* - \mathbf{w}^*, \mathbf{z} \rangle \\
 &\text{as } \mathbf{u}, \mathbf{z} \in F, \mathbf{v}^* - \mathbf{w}^* \in H^* \cap F^\perp \\
 &\text{and } \langle \mathbf{v}^*, \mathbf{u} - \mathbf{z} \rangle = 0 = \langle \mathbf{w}^*, \mathbf{u} - \mathbf{z} \rangle \\
 &\text{as } \mathbf{u} - \mathbf{z} \in F \cap (H^*)^\perp, \mathbf{v}^*, \mathbf{w}^* \in H^* \\
 &\implies \langle \mathbf{v}^*, \mathbf{u} \rangle = \langle \mathbf{w}^*, \mathbf{u} \rangle, \quad \langle \mathbf{v}^*, \mathbf{z} \rangle = \langle \mathbf{w}^*, \mathbf{z} \rangle \\
 &\text{and } \langle \mathbf{v}^*, \mathbf{u} \rangle = \langle \mathbf{v}^*, \mathbf{z} \rangle, \quad \langle \mathbf{w}^*, \mathbf{u} \rangle = \langle \mathbf{w}^*, \mathbf{z} \rangle \\
 &\implies \langle \bar{\mathbf{v}}^*, \bar{\mathbf{u}} \rangle_\pi := \langle \mathbf{v}^*, \mathbf{u} \rangle = \langle \mathbf{w}^*, \mathbf{u} \rangle = \langle \mathbf{w}^*, \mathbf{z} \rangle = \langle \mathbf{v}^*, \mathbf{z} \rangle = \langle \bar{\mathbf{w}}^*, \bar{\mathbf{z}} \rangle_\pi
 \end{aligned}$$

This proves that $\langle \cdot, \cdot \rangle_\pi$ is well defined.

b).

$$\begin{aligned}
 \langle \bar{\mathbf{v}}^* + \bar{\mathbf{w}}^*, \bar{\mathbf{u}} \rangle_\pi &= \langle \overline{\mathbf{v}^* + \mathbf{w}^*}, \bar{\mathbf{u}} \rangle_\pi \quad \text{by the definition of addition of vectors in quotient spaces} \\
 &= \langle \mathbf{v}^* + \mathbf{w}^*, \mathbf{u} \rangle \quad \text{by the definition of } \langle \cdot, \cdot \rangle_\pi \\
 &= \langle \mathbf{v}^*, \mathbf{u} \rangle + \langle \mathbf{w}^*, \mathbf{u} \rangle \quad \text{as } \langle \cdot, \cdot \rangle \text{ is bilinear} \\
 &= \langle \bar{\mathbf{v}}^*, \bar{\mathbf{u}} \rangle_\pi + \langle \bar{\mathbf{w}}^*, \bar{\mathbf{u}} \rangle_\pi \quad \text{by the definition of } \langle \cdot, \cdot \rangle_\pi
 \end{aligned}$$

Similarly

$$\begin{aligned}
 \langle \lambda \bar{\mathbf{v}}^*, \bar{\mathbf{u}} \rangle_\pi &= \langle \overline{\lambda \mathbf{v}^*}, \bar{\mathbf{u}} \rangle_\pi \quad \text{by the definition of scalar multiplication in quotient spaces} \\
 &= \langle \lambda \mathbf{v}^*, \mathbf{u} \rangle \quad \text{by the definition of } \langle \cdot, \cdot \rangle_\pi \\
 &= \lambda \langle \mathbf{v}^*, \mathbf{u} \rangle \quad \text{as } \langle \cdot, \cdot \rangle \text{ is bilinear} \\
 &= \lambda \langle \bar{\mathbf{v}}^*, \bar{\mathbf{u}} \rangle_\pi \quad \text{by the definition of } \langle \cdot, \cdot \rangle_\pi
 \end{aligned}$$

c). Suppose for fixed $\bar{\mathbf{v}}^* \in H^*$ one has $\langle \bar{\mathbf{v}}^*, \bar{\mathbf{u}} \rangle_\pi = 0$ for all $\bar{\mathbf{u}} \in F/F \cap (H^*)^\perp$. Then $\langle \mathbf{v}^*, \mathbf{u} \rangle = 0$ for all $\mathbf{u} \in F$. Hence $\mathbf{v}^* \in H^* \cap F^\perp$. Hence $\bar{\mathbf{v}}^* = \mathbf{0}$. Similarly suppose for fixed $\bar{\mathbf{u}} \in F/F \cap (H^*)^\perp$ one has $\langle \bar{\mathbf{v}}^*, \bar{\mathbf{u}} \rangle_\pi = 0$ for all $\bar{\mathbf{v}}^* \in H^*/H^* \cap F^\perp$. Then $\langle \mathbf{v}^*, \mathbf{u} \rangle = 0$ for all $\mathbf{v}^* \in H^*$. Hence $\mathbf{u} \in F \cap (H^*)^\perp$. Hence $\bar{\mathbf{u}} = \mathbf{0}$.