

COLLEGE OF CHARLESTON  
DEPARTMENT OF MATHEMATICS



Name:

Examination in:

Math Course Number	
Examination Date	
Examination Time	

Total number of problems:

Professor:

Proctor:

Results available by:

Permitted aids:

Phone number:

in:

Show all work to receive full credit.

	Score
1	
2	
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<b>Total</b>	

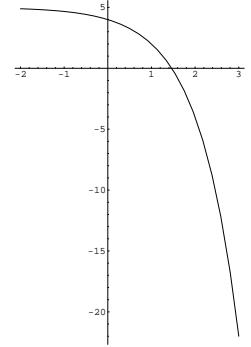
1. Write down a table that gives values of

$$f(x) = \frac{\tan((\pi/3) + x) - \tan(\pi/3)}{x}$$

for  $x$  near zero. Use this table to guess the value of

$$\lim_{x \rightarrow 0} \frac{\tan((\pi/3) + x) - \tan(\pi/3)}{x}.$$

The table on the right indicates that the limit should be approximately equal to 4. You can also see this in the graph on the left (the point  $(0, 4)$  should be omitted from the graph).



$x$	$(\tan((\pi/3) + x) - \tan(\pi/3))/x$
-0.001	3.993085104748051
-0.0009	3.9937753986054836
-0.0008	3.9944659577623165
-0.0007	3.995156782370391
-0.0006	3.995847872581004
-0.0005	3.9965392285461476
-0.0004	3.997230850417144
-0.0003	3.997922738346501
-0.0002	3.9986148924853193
-0.0001	3.9993073129873666
0.0001	4.00069295367844
0.0002	4.001386174180554
0.0003	4.002079661653557
0.0004	4.002773416251214
0.0005	4.003467438125963
0.0006	4.004161727430381
0.0007	4.004856284318681
0.0008	4.005551108943184
0.0009	4.006246201457348
0.001	4.006941562014754

2. Find

a)  $\lim_{t \rightarrow 2} \frac{t^2 - 5t + 6}{t - 2}$ , **Solution:**  $\lim_{t \rightarrow 2} \frac{t^2 - 5t + 6}{t - 2} = \lim_{t \rightarrow 2} t - 3 = -1$

b)  $\lim_{z \rightarrow 0} \frac{\sqrt{z^2 + 16} - 4}{z}$ ,

**Solution:**

$$\begin{aligned} \lim_{z \rightarrow 0} \frac{\sqrt{z^2 + 16} - 4}{z} &= \lim_{z \rightarrow 0} \left( \frac{\sqrt{z^2 + 16} - 4}{z} \right) \left( \frac{\sqrt{z^2 + 16} + 4}{\sqrt{z^2 + 16} + 4} \right) \\ &= \lim_{z \rightarrow 0} \frac{z^2 + 16 - 16}{z(\sqrt{z^2 + 16} + 4)} \\ &= \lim_{z \rightarrow 0} \frac{z}{\sqrt{z^2 + 16} + 4} \\ &= \frac{\lim_{z \rightarrow 0} z}{\sqrt{\lim_{z \rightarrow 0} z^2 + 16} + 4} = 0 \end{aligned}$$

c)  $\lim_{w \rightarrow 0} \left( w^4 + \frac{\cos(4w)}{10,000} \right)$ .

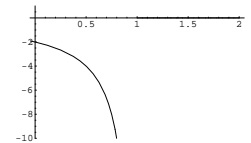
**Solution:**  $\lim_{w \rightarrow 0} \left( w^4 + \frac{\cos(4w)}{10,000} \right) = \lim_{w \rightarrow 0} w^4 + \lim_{w \rightarrow 0} \frac{\cos(4w)}{10,000} = 0 + 10^{-4} = 10^{-4}$ , as  $\cos w$  is continuous and  $\lim_{w \rightarrow 0} \cos(4w) = \cos(\lim_{w \rightarrow 0} 4w) = \cos 0 = 1$ .

d)  $\lim_{x \rightarrow 1^-} \left( \frac{1}{x-1} - \frac{1}{|x-1|} \right)$ ,

**Solution:** Since  $|x - 1| = -(x - 1)$  for  $x < 1$  we get

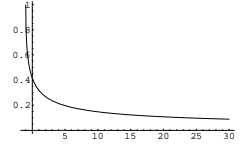
$$\begin{aligned} \lim_{x \rightarrow 1^-} \left( \frac{1}{x-1} - \frac{1}{|x-1|} \right) &= \lim_{x \rightarrow 1^-} \left( \frac{1}{x-1} - \frac{1}{-(x-1)} \right) \\ &= \lim_{x \rightarrow 1^-} \frac{2}{x-1} = -\infty. \end{aligned}$$

e)  $\lim_{x \rightarrow \infty} (\sqrt{2+x} - \sqrt{1+x})$ ,



**Solution:**

$$\begin{aligned} \lim_{x \rightarrow \infty} (\sqrt{2+x} - \sqrt{1+x}) &= \lim_{x \rightarrow \infty} (\sqrt{2+x} - \sqrt{1+x}) \frac{(\sqrt{2+x} + \sqrt{1+x})}{(\sqrt{2+x} + \sqrt{1+x})} \\ &= \lim_{x \rightarrow \infty} \frac{2+x - (1+x)}{\sqrt{2+x} + \sqrt{1+x}} \\ &= \lim_{x \rightarrow \infty} \frac{1}{\sqrt{2+x} + \sqrt{1+x}} = 0. \end{aligned}$$



**3.** Let

$$f(x) = \begin{cases} x/(x^2 - x - 2) & \text{for } x < 1 \\ 3 & \text{for } x \geq 1. \end{cases}$$

- a) Find  $\lim_{x \rightarrow 1^-} f(x)$  and  $\lim_{x \rightarrow 1^+} f(x)$ .  
 b) Is  $f(x)$  continuous at  $x = 1$ ?

**Solution:** a).

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} x/(x^2 - x - 2) = 1/(1 - 1 - 2) = -1/2.$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} 3 = 3.$$

b) Since  $\lim_{x \rightarrow 1^-} f(x) = -1/2 \neq 3 = \lim_{x \rightarrow 1^+} f(x)$  we get that  $\lim_{x \rightarrow 1^-} f(x)$  does not exist and hence  $f(x)$  is not continuous at  $x = 1$ .

**4.** Find a  $\delta > 0$  such that

$$\left| \frac{x^2 + x - 2}{2x - 2} - \frac{3}{2} \right| < 0.2$$

whenever  $|x - 1| < \delta$ .

**Solution:** We rewrite the left hand side of the above inequality as

$$\left| \frac{x^2 + x - 2}{2x - 2} - \frac{3}{2} \right| = \left| \frac{(x-1)(x+2)}{2(x-1)} - \frac{3}{2} \right| = \left| \frac{x+2}{2} - \frac{3}{2} \right| = \frac{1}{2} |x-1|.$$

We would like this to be less than 0.2, so we should take  $\delta = 2 \cdot 0.2 = 0.4$ . Let  $\implies$  mean “implies”. Then

$$\begin{aligned} |x-1| < \delta = 0.4 &\implies \frac{1}{2} |x-1| < 0.2 \\ &\implies \left| \frac{x+2}{2} - \frac{3}{2} \right| < 0.2 \\ &\implies \left| \frac{(x-1)(x+2)}{2(x-1)} - \frac{3}{2} \right| < 0.2 \\ &\implies \left| \frac{x^2 + x - 2}{2x - 2} - \frac{3}{2} \right| < 0.2 \end{aligned}$$

which was to be proven.

5. Where are the following functions continuous?

a)  $f(t) = \frac{1}{t^2 - 1}$ ,

**Solution:** This rational function is continuous where ever it is defined. Since it is defined everywhere except at  $x = \pm 1$ , it is continuous everywhere except at  $x = \pm 1$ .

b)  $f(x) = \begin{cases} 1 - x^2 & \text{if } x \leq 2, \\ x^2 - 2x & \text{if } x > 2. \end{cases}$

**Solution:** Since polynomials are continuous, the only place that this function might not be continuous at is at  $x = 2$ . We check

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (1 - x^2) = 1 - 2^2 = -3 \quad \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (x^2 - 2x) = 2^2 - 2^2 = 0.$$

Since  $\lim_{x \rightarrow 2^-} f(x) \neq \lim_{x \rightarrow 2^+} f(x)$  the two sided limit does not exist and  $f(x)$  is not continuous at  $x = 2$ .

6. Show that there exists a number  $c$  such that

$$c^3 = \sqrt{c + 1}.$$

Hint: Consider the function  $f(x) = x^3 - \sqrt{x + 1}$ .

**Solution:** Here we use the Intermediate Value Theorem. We evaluate

$$f(-1) = (-1)^3 - \sqrt{-1 + 1} = -1, \quad f(8) = 8^3 - \sqrt{8 + 1} = 512 - 3 = 509$$

and observe that  $f(x)$  is continuous on the interval  $[-1, 8]$ . Since  $f(-1) < 0 < f(8)$ , the Intermediate Value Theorem tells us that there exists a  $c$  in the interval  $[-1, 8]$  such that  $f(c) = 0$ .

7. The point  $P(-1, 1)$  lies on the curve  $y = 1/(x + 2)$ .

a) The point  $Q(2, 1/4)$  is also on the curve. Find the slope of the secant line through these two points.

**Solution:** The slope of the secant line is

$$\frac{(1/4) - 1}{2 - (-1)} = \frac{-3/4}{3} = -1/4$$

b) Suppose now  $R(x, 1/(x + 2))$  is another point on the curve. Find the slope of the secant line through  $R$  and  $P$ .

**Solution:** The slope of the secant line is

$$\frac{\frac{1}{x+2} - 1}{x - (-1)} = \frac{\frac{-x-1}{x+2}}{x+1} = -\frac{1}{x+2}$$

c) Find the slope of the tangent line to the curve  $y = 1/(x + 2)$  at the point  $P(-1, 1)$ .

The slope of the tangent line at  $P(-1, 1)$  is

$$\lim_{x \rightarrow -1} \frac{\frac{1}{x+2} - 1}{x - (-1)} = -\lim_{x \rightarrow -1} \frac{1}{x+2} = -1$$

