

COLLEGE OF CHARLESTON
DEPARTMENT OF MATHEMATICS



Name:

Examination in: Introductory Calculus

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|---------------------------|-------------|
| Math Course Number | Math 120 |
| Examination Date | 9-16-2008 |
| Examination Time | 10:50-12:05 |

Total number of problems: 7

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Proctor: Ben Cox

Phone number: 953-5715

Results available by: Sept. 23

in: Maybank 117

Permitted aids: Proctor, TI-83 or TI-86 or equivalent calculator.

Show all work to receive full credit.

| | Score |
|--------------|-------|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| 7 | |
| Total | |

1. Find

a) $\lim_{x \rightarrow 2} x^3 - x^2 + 1$, **Solution:** $\lim_{x \rightarrow 2} x^3 - x^2 + 1 = 2^3 - 2^2 + 1 = 8 - 4 + 1 = 5$.

b) $\lim_{z \rightarrow -7} \frac{2z^2 + 11z - 21}{z + 7}$, **Solution:** $\lim_{z \rightarrow -7} \frac{2z^2 + 11z - 21}{z + 7} = \lim_{z \rightarrow -7} \frac{(2z - 3)(z + 7)}{z + 7} = \lim_{z \rightarrow -7} (2z - 3) = 2(-7) - 3 = -17$.

c) $\lim_{x \rightarrow \infty} (\sqrt{x + x^2} - x)$.

Solution:

$$\begin{aligned} \lim_{x \rightarrow \infty} (\sqrt{x + x^2} - x) &= \lim_{x \rightarrow \infty} (\sqrt{x + x^2} - x) \frac{(\sqrt{x + x^2} + x)}{(\sqrt{x + x^2} + x)} = \lim_{x \rightarrow \infty} \frac{x + x^2 - x^2}{\sqrt{x + x^2} + x} \\ &= \lim_{x \rightarrow \infty} \frac{1}{(\sqrt{x + x^2/x} + 1)} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{(1/x) + 1} + 1} = 1/2. \end{aligned}$$

2. Prove using the ϵ, δ definition of the limit that

$$\lim_{x \rightarrow 14} \left(\frac{3x}{7} - 2 \right) = 4.$$

Solution: We need to show that for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left| \frac{3x}{7} - 2 - 4 \right| < \epsilon \quad \text{whenever} \quad |x - 14| < \delta.$$

We rewrite the left hand side of the above as

$$\left| \frac{3x}{7} - 2 - 4 \right| = \left| \frac{3x}{7} - 6 \right| = \frac{3}{7} |x - 14|.$$

We would like this to be less than ϵ , so we should take $\delta = \frac{7\epsilon}{3}$. Then

$$\begin{aligned} |x - 14| < \delta = \frac{7\epsilon}{3} &\implies \frac{3}{7} |x - 14| < \epsilon \\ &\implies \left| \frac{3x}{7} - 6 \right| < \epsilon \\ &\implies \left| \frac{3x}{7} - 2 - 4 \right| < \epsilon \end{aligned}$$

which was to be proven.

3. Show that there exists a number c such that

$$\cos c = c^3.$$

Hint: Consider the function $f(x) = x^3 - \cos x$, on the interval $[0, \pi]$.

We use the Intermediate Value Theorem to prove this result. Observe that

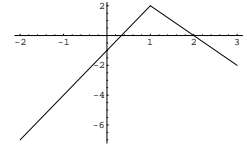
$$f(0) = \cos 0 - 0^3 = 1, \quad \text{and} \quad f(\pi) = \cos \pi - \pi^3 = -1 - \pi^3.$$

Since $f(0) = 1 > 0 > -1 - \pi^3 = f(\pi)$ and $f(x)$ is continuous on the interval $[0, \pi]$, we get by the Intermediate Value Theorem that there exists a number c between 0 and π such that $f(c) = 0$ i.e. $\cos c = c^3$.

4. Let

$$f(x) = \begin{cases} 3x - 1 & \text{for } x \leq 1 \\ -2x + 4 & \text{for } x > 1 \end{cases}$$

- a) Find $\lim_{x \rightarrow 1^-} f(x)$ and $\lim_{x \rightarrow 1^+} f(x)$. **Solution:** $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} 3x - 1 = 2$ and $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (-2x + 4) = 2$.
- b) Determine $\lim_{x \rightarrow 1} f(x)$ if it exists. **Solution:** Yes, it exists and $\lim_{x \rightarrow 1} f(x) = 2$.
- c) Is $f(x)$ continuous at $x = 1$? **Solution:** Yes, since $\lim_{x \rightarrow 1} f(x) = 2 = f(1)$
- d) Sketch a graph of $f(x)$.



5. Let

$$f(x) = \frac{x}{x-1}$$

- a) Using limits find the slope of the tangent line to the curve $f(x)$ at $x = 2$.
- b) Find the equation for the tangent line at $x = 2$.
- (a). The slope of the tangent line at $x = 2$ is the limiting value of the slopes of the secant lines i.e.

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2} &= \lim_{x \rightarrow 2} \frac{\left(\frac{x}{x-1}\right) - \left(\frac{2}{2-1}\right)}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{x - 2(x-1)}{(x-1)(x-2)} \\ &= \lim_{x \rightarrow 2} \frac{-x + 2}{(x-1)(x-2)} = \lim_{x \rightarrow 2} \frac{-1}{x-1} = -1. \end{aligned}$$

(b). The equation of the tangent line is

$$y - f(2) = m(x - 2), \quad \text{in other words} \quad y = -(x - 2) + 2 = -x + 4.$$

6. Find any vertical and/or horizontal asymptotes of the curve. Give reasons for your answer.

$$g(x) = 1 + \frac{1}{x - 3000}$$

We suspect there is a vertical asymptote at $x = 3000$, check this by computing

$$\lim_{x \rightarrow 3000^-} 1 + \frac{1}{x - 3000} = -\infty \quad \lim_{x \rightarrow 3000^+} 1 + \frac{1}{x - 3000} = \infty.$$

To find the horizontal asymptote, compute

$$\lim_{x \rightarrow \infty} 1 + \frac{1}{x - 3000} = 1$$

Similarly,

$$\lim_{x \rightarrow -\infty} 1 + \frac{1}{x - 3000} = 1$$

So $y = 1$ is the only horizontal asymptote.

7. Write down a table that gives values of

$$f(x) = \frac{3^x - 1}{x}$$

for x near zero. Use this table to guess the approximate value of

$$\lim_{x \rightarrow 0} \frac{3^x - 1}{x}.$$

The table on the right indicates that the limit should be approximately equal to 1.09. You can also see this on the graph. The exact answer is $\ln 3$ as we will see later.

| x | $(3^x - 1)/x$ |
|-------|---------------|
| -0.09 | 1.046046 |
| -0.08 | 1.051718 |
| -0.07 | 1.057431 |
| -0.06 | 1.063186 |
| -0.05 | 1.068983 |
| -0.04 | 1.074823 |
| -0.03 | 1.080705 |
| -0.02 | 1.086630 |
| -0.01 | 1.092599 |
| 0.01 | 1.104669 |
| 0.02 | 1.110770 |
| 0.03 | 1.116917 |
| 0.04 | 1.123108 |
| 0.05 | 1.129346 |
| 0.06 | 1.135629 |
| 0.07 | 1.141959 |
| 0.08 | 1.148336 |
| 0.09 | 1.154760 |