Math 221–002, Final Exam. Name: ____________________

200 pts, 3 hours May 2, 2005

No calculators, notes, books, or any other outside materials. **Supporting work will be required on every problem** (unless otherwise indicated). You are expected to know the values of all trigonometric functions at multiples of $\pi/4$ and of $\pi/6$.

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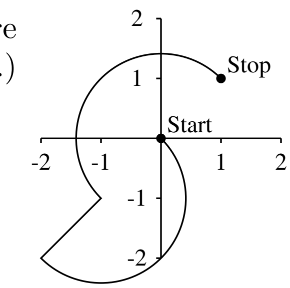
1 (14 pts). Let R be the rectangle $[1,3] \times [-1,3]$ in the xy-plane, and let $f(x,y) = x + 3y$. Find a Riemann sum for $f(x,y)$ on R using $n = m = 2$ and choosing the sample points to be the point in each subrectangle that is closest to the origin.

2 (14 pts). Find the equation of the plane tangent to the surface $16\sqrt{x} - y^2 z = 23$ at the point $(4, -3, 1)$.

3 (16 pts). Evaluate the line integral $\int_C (y-2)\,dx + (x-2)\,dy$ where C is the closed path pictured at right. (Each arc in C is a semicircle.)



4 (16 pts). Let B be the region $\{(x,y,z) \in \mathbb{R}^3 : 1 \le x^2 + y^2 + z^2 \le 9\}$. Find $\int\int\int_B z^2\,dV$. Warning: the answer is not "pretty." Don't bother trying to simplify.

5 (18 pts). Let U be the solid in $\mathbb{R}^3$ which lies above the square $[0,1] \times [0,1]$ in the xy-plane and below the plane $z + y = 1$. Find the flux of $\mathbf{F} = x^2\mathbf{i} + 4xy\mathbf{j} - xy\mathbf{k}$ across the boundary of U in the outward orientation.

6 (16 pts). Let G be the portion of the surface $z = 9 - x^2 - y^2$ which lies above the xy-plane, and let $\mathbf{F} = 2y\mathbf{i} - x\mathbf{j} + e^{xy}\mathbf{k}$. Find $\displaystyle\int\int_G \operatorname{curl}\mathbf{F} \cdot \mathbf{n}\,dS$, where G is oriented upward.

7 (16 pts). Find an equation of the plane containing the point $(0, -1, 1)$ and the line $x = 1 + 2t$, $y = 2 - t$, $z = 4t$.

8 (12 pts). Sketch the surface given by $x^2 - y^2 - 4z^2 = 1$. What is the name for such a surface?

Note: It is possible to earn full credit on the later parts of the next question even if you get an earlier part wrong.

9. Suppose that a particle traces out a curve in space, and that at one moment, the particle's velocity and acceleration are $\mathbf{v} = \langle 2, 0, -1 \rangle$ and $\mathbf{a} = \langle 1, 1, 1 \rangle$.

a (7 pts). Find the cosine and sine of the angle between $\mathbf{v}$ and $\mathbf{a}$.

b (7 pts). Find a_T and a_N, the tangential and normal componants of $\mathbf{a}$. You are not required to find $\mathbf{T}$ or $\mathbf{N}$.

c (4 pts). At the moment in question, is the particle speeding up or slowing down? Explain briefly.

d (7 pts). Find the curvature κ of the particle's path at the point in question.

10 (10 pts). The curve C is parametrized by $\mathbf{r}(u) = \langle u^2, u - u^3, 2u \rangle$. Find the equation of the line tangent to C at the point $(1, 0, -2)$.

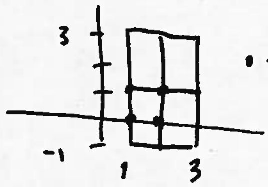
11 (10 pts). Suppose all you know about the vector $\mathbf{w}$ is that its projection onto $\langle 2, 1\rangle$ is $\langle 1, \frac{1}{2}\rangle$. Can you say for certain whether $\mathbf{w} \cdot \langle 2, 1\rangle$ is positive, negative, or zero? If so, which is it and why? If not, explain.

12. Suppose that the temperature at the point (x, y, z) in $\mathbb{R}^3$ is given by the function $T(x, y, z) = 3xy - yz$. Here x, y, z are each measured in centimeters (cm) and T is measured in degrees Celcius (°C).

a (5 pts). A particle is moving along a line in direction parallel to $\mathbf{j}$. At what rate is the surrounding temperature changing when the particle passes throught the point $(3, 2, -1)$? What are the units in your answer?

b (10 pts). If another particle is at the point $(3, 2, -1)$, in what direction should the particle move in order to see the greatest increase in temperature? Express your answer as a unit vector. What will be the rate of increase in temperature moving in that direction at that point?

13 (18 pts). Find the point on the surface $x - 2y - 3z = 28$ closest to the origin.

1. 

$\bullet$ = sample point R.S. $= \Delta x \Delta y \, \Sigma f(\bullet)$

$= 1 \cdot 2 \cdot \big(f(1,0) + f(2,0) + f(1,1) + f(2,1)\big)$

$= 2(1+2+4+5) = 24.$

2. $f(x,y,z) = 16x^{1/2} - y^2 z$. $\vec{n} = \nabla f \,@\, (4,-3,1)$. $\nabla f = \langle 8x^{-1/2}, -2yz, -y^2 \rangle$

$\vec{n} = \langle 4, 6, -9 \rangle$. plane is $4(x-4) + 6(y+3) - 9(z-1) = 0.$

3. Potential $= xy - 2x - 2y$. $\int = xy - 2x - 2y \Big|_{(0,0)}^{(1,1)} = 1 - 2 - 2 - 0 = -3.$

4. $\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \int_{\rho=1}^{3} \rho^2 \cos^2\phi \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = 2\pi \cdot \int_1^3 \rho^4 d\rho \cdot \int_0^\pi \cos^2\phi \sin\phi \, d\phi$

$= 2\pi \cdot \frac{1}{5}(3^5 - 1)\left(-\frac{1}{3}\cos^3\phi \Big|_0^\pi\right) = 2\pi \cdot \frac{1}{5}(242) \cdot \frac{1}{3} \cdot 2$

5. $\iiint_U \nabla \cdot \vec{F} \, dV = \int_0^1 \int_0^1 \int_0^{1-y} (2x + 4x) \, dz \, dx \, dy = \int_0^1 (6x) dx \int_0^1 (1-y) dy = \frac{3}{2}.$

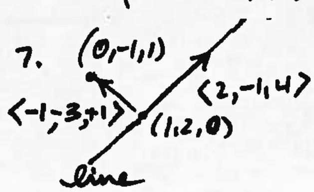
6. $\oint_C \vec{F} \cdot d\vec{r}$, where C is circle $x^2 + y^2 = 9$ in xy plane

$x = 3\cos\theta$ $y = 3\sin\theta$

$= \int 2y \, dx - x \, dy + e^{xy} dz$ $dx = -3\sin\theta d\theta$ $dy = 3\cos\theta d\theta$ $dz = 0$

$= \int_0^{2\pi} 2 \cdot 3\sin\theta(-3\sin\theta) d\theta - 3\cos\theta \cdot 3\cos\theta \, d\theta$

$= -9 \int_0^{2\pi} (2\sin^2\theta + \cos^2\theta) d\theta = -9 \int_0^{2\pi} (\sin^2\theta + 1) d\theta = -9 \int_0^{2\pi} \left(\frac{1}{2}(1 - \cos 2\theta) + 1\right) d\theta = -27\pi$

7. 

Use $\vec{n} = \langle -1, -3, 1 \rangle \times \langle 2, -1, 4 \rangle = \langle -11, 6, 7 \rangle$

plane is $-11(x-1) + 6(y-2) + 7z = 0$

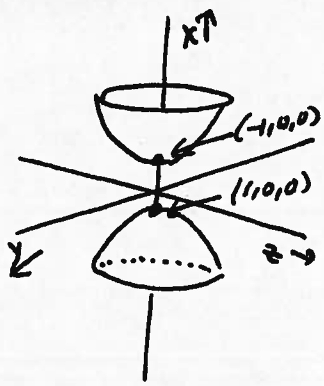
8. cross sections: $y = 0$, $x^2 - 4z^2 = 1$ hyperbola

$z = 0$, $x^2 - y^2 = 1$ ״

$x = 0$ $-y^2 - 4z^2 = 1$ DNE

if x = const > 1 or < -1, const $- 1 = y^2 + 4z^2$ ellipse

surface is a hyperboloid of two sheets.



9.a. $v \cdot a = 1$. $|v| = \sqrt{5}$ $|a| = \sqrt{3}$ $1 = \sqrt{15}\cos\theta$; $\theta = \cos^{-1}\left(\frac{1}{\sqrt{15}}\right)$

$v \cdot a > 0$ so θ is acute. $\sin\theta = +\sqrt{1-\cos^2\theta} = \sqrt{1-\frac{1}{15}} = \sqrt{\frac{14}{15}}$.

b. $a_T = \frac{v \cdot a}{|v|} = \frac{1}{\sqrt{5}}$. $a_N = \sqrt{|a|^2 - a_T^2} = \sqrt{3 - \frac{1}{5}} = \sqrt{\frac{14}{5}}$.

c. $a_T = \frac{d^2s}{dt^2} = +\frac{1}{\sqrt{5}}$, so speeding up.

d. $\kappa = \frac{|v \times a|}{|v|^3} = \frac{|\langle 1,-3,2\rangle|}{5^{3/2}} = \frac{\sqrt{14}}{\sqrt{5}^3}$.

10. $\vec{r} = (1,0,-2)$ @ $u = -1$. $\frac{d\vec{r}}{dt} = \langle 2u, 1-3u^2, 2\rangle = \langle -2,-2,2\rangle^*$ @ $u=-1$.

line: $x = 1-2t$, $y = -2t$, $z = -2+2t$. (*or use $\langle -1,-1,1\rangle$)

11. $\langle 1, 1/2\rangle = \frac{w \cdot \langle 2,1\rangle}{5}\langle 2,1\rangle$. $\frac{w \cdot \langle 2,1\rangle}{5} = +\frac{1}{2}$, $w \cdot \langle 2,1\rangle = +5/2$. positive

12. a. $T_y = 3x - z = 10$ °/cm

b. $\nabla T = \langle 3y, 3x-z, -y\rangle = \langle 6, 10, -2\rangle$ @ $(3,2,-1)$.

direction of greatest increase is $\frac{\langle 6,10,-2\rangle}{|\quad " \quad|} = \left\langle \frac{3}{\sqrt{35}}, \frac{5}{\sqrt{35}}, \frac{-1}{\sqrt{35}}\right\rangle$

D_uT in that direction is $|\nabla T| = \sqrt{140}$

13. Minimize $x^2+y^2+z^2$ subject to constraint $x-2y-3z=28$.

Lagrange multipliers: $\nabla f = \langle 2x, 2y, 2z\rangle$ $\nabla g = \langle 1,-2,-3\rangle$.

Solve to find critical points: $2x = \lambda$, $2y = -2\lambda$, $2z = -3\lambda$. $x-2y-3z=28$

$x = \frac{\lambda}{2}$, $y = -\lambda$, $z = -\frac{3}{2}\lambda$. $\frac{\lambda}{2} + 2\lambda + \frac{9}{2}\lambda = 28$, $\lambda = 4$.

$x = 2, y = -4, z = -6$. closest point $= (2,-4,-6)$.

(surface in question is a plane and has a unique point closest to $(0,0,0)$. It has no farthest point, so critical point $(2,-4,-6)$ <u>must</u> be a minimum.)