1 (14 pts). Let \( R \) be the rectangle \([1, 3] \times [-1, 3]\) in the xy-plane, and let \( f(x, y) = x + 3y \). Find a Riemann sum for \( f(x, y) \) on \( R \) using \( n = m = 2 \) and choosing the sample points to be the point in each subrectangle that is closest to the origin.

2 (14 pts). Find the equation of the plane tangent to the surface \( 16\sqrt{x} - y^2z = 23 \) at the point \((4, -3, 1)\).

3 (16 pts). Evaluate the line integral \( \int_C (y - 2) \, dx + (x - 2) \, dy \) where \( C \) is the closed path pictured at right. (Each arc in \( C \) is a semicircle.)

4 (16 pts). Let \( B \) be the region \( \{(x, y, z) \in \mathbb{R}^3 : 1 \leq x^2 + y^2 + z^2 \leq 9 \} \). Find \( \iiint_B z^2 \, dV \). Warning: the answer is not “pretty.” Don’t bother trying to simplify.

5 (18 pts). Let \( U \) be the solid in \( \mathbb{R}^3 \) which lies above the square \([0, 1] \times [0, 1]\) in the xy-plane and below the plane \( z + y = 1 \). Find the flux of \( \mathbf{F} = x^2\mathbf{i} + 4xy\mathbf{j} - xy\mathbf{k} \) across the boundary of \( U \) in the outward orientation.

6 (16 pts). Let \( G \) be the portion of the surface \( z = 9 - x^2 - y^2 \) which lies above the xy-plane, and let \( \mathbf{F} = 2y\mathbf{i} - x\mathbf{j} + e^{xy}\mathbf{k} \). Find \( \int\int_G \text{curl} \ \mathbf{F} \cdot \mathbf{n} \, dS \), where \( G \) is oriented upward.

7 (16 pts). Find an equation of the plane containing the point \((0, -1, 1)\) and the line \( x = 1 + 2t, \ y = 2 - t, \ z = 4t \).

8 (12 pts). Sketch the surface given by \( x^2 - y^2 - 4z^2 = 1 \). What is the name for such a surface?

Note: It is possible to earn full credit on the later parts of the next question even if you get an earlier part wrong.

9. Suppose that a particle traces out a curve in space, and that at one moment, the particle’s velocity and acceleration are \( \mathbf{v} = \langle 2, 0, -1 \rangle \) and \( \mathbf{a} = \langle 1, 1, 1 \rangle \).

a (7 pts). Find the cosine and sine of the angle between \( \mathbf{v} \) and \( \mathbf{a} \).

b (7 pts). Find \( a_T \) and \( a_N \), the tangential and normal components of \( \mathbf{a} \). You are not required to find \( \mathbf{T} \) or \( \mathbf{N} \).

c (4 pts). At the moment in question, is the particle speeding up or slowing down? Explain briefly.

d (7 pts). Find the curvature \( \kappa \) of the particle’s path at the point in question.

10 (10 pts). The curve \( C \) is parametrized by \( \mathbf{r}(u) = \langle u^2, u - u^3, 2u \rangle \). Find the equation of the line tangent to \( C \) at the point \((1, 0, -2)\).
11 (10 pts). Suppose all you know about the vector $\mathbf{w}$ is that its projection onto $\langle 2, 1 \rangle$ is $\langle 1, \frac{1}{2} \rangle$. Can you say for certain whether $\mathbf{w} \cdot \langle 2, 1 \rangle$ is positive, negative, or zero? If so, which is it and why? If not, explain.

12. Suppose that the temperature at the point $(x, y, z)$ in $\mathbb{R}^3$ is given by the function $T(x, y, z) = 3xy - yz$. Here $x, y, z$ are each measured in centimeters (cm) and $T$ is measured in degrees Celsius ($^\circ$C).

a (5 pts). A particle is moving along a line in direction parallel to $\mathbf{j}$. At what rate is the surrounding temperature changing when the particle passes through the point $(3, 2, -1)$? What are the units in your answer?

b (10 pts). If another particle is at the point $(3, 2, -1)$, in what direction should the particle move in order to see the greatest increase in temperature? Express your answer as a unit vector. What will be the rate of increase in temperature moving in that direction at that point?

13 (18 pts). Find the point on the surface $x - 2y - 3z = 28$ closest to the origin.
1. \( i = \text{sample point} \quad R.S. = \Delta x \Delta y \Sigma f(x) \)
   \( = 1.2 (f(1,0) + f(2,0) + f(1,1) + f(2,1)) \)
   \( = 2 (1 + 2 + 4 + 5) = 24 \)

2. \( f(x,y,z) = 16x^{1/2} - y^2 \), \( \bar{n} = \nabla f @ (4, -3, 1) \).
   \( \Delta f = \langle 8x^{1/2}, -2y, -y^2 \rangle \)
   \( \bar{n} = \langle 4, 6, -9 \rangle \). Plane is \( 4(x-4) + 6(y+3) - 9(z-1) = 0 \).

3. Potential \( = xy - 2x - 2y \).
   \( S = xy - 2x - 2y \left( \begin{array}{c} x \end{array} \right) \)
   \( = 1 - 2 - 0 = -3 \).

4. \( S = \int_0^{2\pi} \int_0^1 \int_0^{\pi} \rho^2 \cos \phi \cdot \sin \phi \rho^2 \sin \phi d\rho d\phi d\theta = 2\pi \cdot \int_0^1 \rho^3 d\rho \cdot \int_0^{\pi} \cos \phi \sin \phi d\phi d\theta \)
   \( = 2\pi \cdot \frac{1}{2} \left( \frac{\cos^2 \phi}{2} \right) = \pi \cdot \frac{1}{2} (2\pi) 
   \)
   \( \frac{1}{2} \cdot 2 \)

5. \( \int_S \nabla \cdot \bar{F} dV = \int_S \nabla \cdot \bar{F} (2x + 4y) dx dy dz \)
   \( = \int_0^1 \left( \frac{6x}{y} \right) dx \int_0^1 (1-y) dy = \frac{3}{2} \).

6. \( \theta \bar{\lambda} = \bar{g} \cdot \hat{N}, \) where \( C \) is circle \( x^2 + y^2 = 9 \) in \( xy \) plane.
   \( C = \int \nabla \cdot \bar{g} \cdot \hat{N} \)
   \( = \int 2y dx - x dy + e^x d^2 \)
   \( = 3 \cos \theta \) \( y = 3 \sin \theta \)
   \( dx = -3 \sin \theta d\theta \)
   \( dy = 3 \cos \theta d\theta \)
   \( d\theta = 0 \)

7. \( \int_0^{2\pi} 2 \cdot 3 \sin \theta \cdot 3 \cos \theta d\theta = 3 \cos \theta \cdot 3 \sin \theta d\theta \)
   \( = -9 \int_0^{2\pi} (\sin^2 \theta + \cos^2 \theta) d\theta = -9 \int_0^{2\pi} (\sin^2 \theta + 1) d\theta = -9 \int_0^{2\pi} (\cos^2 \theta + 1) d\theta = -27 \pi \)

8. Cross sections:
   \( y = 0, \quad x^2 - 4y^2 = 1 \) hyperbola
   \( z = 0, \quad x^2 - y^2 = 1 \)
   \( x = 0, \quad y^2 - 4z^2 = 1 \) DNE

   If \( x = \text{const} > 1 \) or \( z = \text{const} > 1 \),
   \( x^2 - y^2 = 1 \), \( y^2 - 4z^2 = 1 \).

   Surface is a hyperboloid of two sheets.
9a. \( \mathbf{v} \cdot \mathbf{a} = 1 \). \( \mathbf{v} \parallel \mathbf{a} \), \( |\mathbf{v}| = \sqrt{3} \), \( |\mathbf{a}| = \sqrt{5} \) \( |\mathbf{v}|^2 = 3 \). \( 1 = \sqrt{15} \cos \theta \); \( \theta = \cos^{-1} \left( \frac{1}{\sqrt{15}} \right) \).

\( \mathbf{v} \cdot \mathbf{a} > 0 \) so \( \theta \) is acute. \( \sin \theta = \sqrt{1 - \cos^2 \theta} = \sqrt{1 - \frac{1}{15}} = \frac{\sqrt{14}}{\sqrt{15}} \).

\( a_\tau = \frac{\mathbf{v} \cdot \mathbf{a}}{|\mathbf{v}|} = \frac{1}{\sqrt{15}} \). \( a_N = \sqrt{|\mathbf{a}|^2 - a_\tau^2} = \sqrt{3 - \frac{1}{15}} = \frac{\sqrt{44}}{\sqrt{15}} \).

b. \( \mathbf{a} = \frac{\mathbf{v} \times \mathbf{a}}{|\mathbf{v}|^3} = \frac{1}{5} \). \( a_N = \sqrt{|\mathbf{a}|^2 - a_\tau^2} = \sqrt{3 - \frac{1}{5}} = \frac{\sqrt{14}}{\sqrt{5}} \).

c. \( \frac{d^2 s}{dt^2} = \frac{1}{5} \), so speeding up.

a. \( \mathbf{v} = \frac{\mathbf{v} \times \mathbf{a}}{|\mathbf{v}|^3} = \frac{1}{5} \begin{pmatrix} 1 \end{pmatrix} = \frac{\sqrt{14}}{\sqrt{15}} \).

10. \( \mathbf{a} = (1, 0, -2) \); \( \mathbf{u} = -1 \). \( \frac{d\mathbf{r}}{dt} = \langle 2u, 1 - 3u^2, 2 \rangle = \langle -2, -2, 2 \rangle \); \( \mathbf{u} = -1 \).

Line: \( \mathbf{x} = 1 - 2t \), \( \mathbf{y} = -2t \), \( \mathbf{z} = -2 + 2t. \) \((\mathbf{x}, \mathbf{y}, \mathbf{z}) \text{ or } \langle -1, -1, 1 \rangle\)

11. \( \langle 1, 1 \rangle = \frac{\mathbf{\omega} \cdot \langle 2, 1 \rangle}{5} \). \( \mathbf{\omega} \cdot \langle 2, 1 \rangle = \frac{1}{2} \), \( \mathbf{\omega} \cdot \langle 2, 1 \rangle = \frac{5}{2} \). Positive

12. a. \( T_y = 3x - 2 = 10 \) \( \text{g} \)/cm

b. \( \nabla T = \langle 3y, 3x - 2, -y \rangle = \langle 0, 10, -2 \rangle \); \( \mathbf{u} = (3, -2, 1) \).

Direction of greatest increase \( \mathbf{1} \): \( \langle 6, 10, -2 \rangle \). \( \langle 3 \sqrt{19} \rangle \) \( \langle \frac{5}{\sqrt{19}} \rangle \) \( \langle \frac{1}{\sqrt{19}} \rangle \).

Direction in that direction is \( 1 \mathbf{T} \parallel \sqrt{19} \).

13. Minimize \( x^2 + y^2 + z^2 \) subject to constraint \( x - 2y - 3z = 28 \).

Lagrangian multiplier: \( \nabla T = \langle 2x, 2y, 2z \rangle \); \( \nabla g = \langle 1, -2, -3 \rangle \).

Solve to find critical points: \( 2x = \lambda \), \( 2y = -2 \lambda \), \( 2z = -3 \lambda \). \( x - 2y - 3z = 28 \).

\( x = 2, y = -4, z = -6 \). closest point is \( (2, -4, -6) \).

(surface in question is a plane and has a unique point closest to \( (0,0,0) \). It has no fathored point, so critical point \( (2, -4, -6) \) must be a minimum.)