Math 221 Final / Spring 2002 / Alex Kasman

1. (a) For the function $p(x,y) = \frac{x}{x+y}$ find the requested derivatives:

   $p_{xx}(x,y) =$ ______________ $p_{xy}(x,y) =$ ______________.

   (b) Find the curl and divergence of the vector field $\mathbf{F}(x,y,z) = \langle x+y, x-y, z^2+xy \rangle$:

   $\operatorname{curl}\mathbf{F}(x,y,z) =$ ______________________ $\operatorname{div}\mathbf{F}(x,y,z) =$ ______________________

   (c) Write an equation for the tangent plane to the graph of the function $f(x,y) = x^2 + xy + y^3$ at the point $(1,1,3)$.

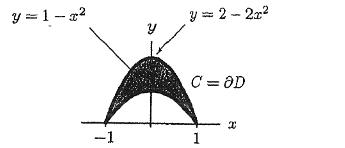
2. (a) Sketch the region of integration for

   $$\int_0^1 \int_{\sqrt{x}}^1 \sin(y^3)\,dy\,dx$$

   and evaluate by reversing the order of integration.

   (b) Evaluate the double integral $\int_{-3}^{3} \int_0^{\sqrt{9-x^2}} \frac{2}{1+x^2+y^2}\,dy\,dx$.

3. Let D be the region in the plane bounded by the curves $y = 2-2x^2$ and $y = 1-x^2$ (see figure) and let $C = \partial D$ be the curve which is the boundary of this region oriented counter-clockwise.



   Use Green's Theorem to evaluate the line integral

   $$\oint_C \mathbf{F}\cdot d\mathbf{r} \quad \text{for the vector field} \quad \mathbf{F}(x,y) = \langle \sin(x), 2x + \cos(y) \rangle.$$

4. Find the absolute maximum and minimum values that the function $f(x,y) = x^3 - xy$ takes on the square $0 \le x \le 2$ and $1 \le y \le 3$.

5. Throughout this question, we will consider the vector field $\mathbf{F}(x,y,z) = \langle y, -x, 1+z^2 \rangle$.

   (a) What is the curl of $\mathbf{F}$? $\operatorname{curl}\mathbf{F}(x,y,z) =$ ______________________.

   (b) Let S be the portion of the unit sphere $x^2+y^2+z^2 = 1$ for $x \ge 0$, $y \ge 0$, $z \ge 0$, oriented with the outward facing normal. Find the flux of the vector field $\operatorname{curl}\mathbf{F}$ over S.

   (c) If $\mathbf{F}$ is a force field, how much work does it do in moving an object around the boundary of the surface S in the clockwise direction when viewed from above? (Hint: Stokes' Theorem will save you effort here.)

6. In all parts of this question we will be considering the function $f(x,y) = xy - y^2$ and its gradient vector field.

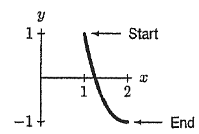
   (a) Find the gradient vector field of f

   $$\mathbf{F}(x,y) = \nabla f(x,y) = \text{\_\_\_\_\_\_\_\_\_\_\_\_}$$

   (b) What is the directional derivative of f at the point $(2,2)$ in the direction of the unit vector $\mathbf{u} = \langle \frac{3}{5}, \frac{4}{5} \rangle$?

   (c) What unit vector gives the direction in which you should travel from the point $(2,2)$ in order to get the greatest increase in the value of the function f?

   (d) Let C be the curve in the plane parametrized as $x(t) = 1+t^2$, $y(t) = \cos(\pi t)$, for $0 \le t \le 1$ (see figure):



   As you can see from the graph, this curve starts at the point $(1,1)$ and ends up at the point $(2,-1)$.

   Evaluate the line integral of the vector field $\mathbf{F}$ (from part a above) along this curve.

   $$\int_C \mathbf{F}\cdot d\mathbf{r} = \text{\_\_\_\_\_\_\_\_\_\_\_\_}$$

7. A particle is travelling through space so that at time t seconds it is at the position given by the function

   $$\mathbf{r}(t) = \langle 9t, t^2 - 3t, t^2 - t \rangle \qquad 0 \le t \le \infty.$$

   (a) Find the velocity vector $\mathbf{v}(t)$ and the acceleration vector $\mathbf{a}(t)$.

   (b) Write the equation(s) of the tangent line to the path followed by the particle at the point $(18,-2,2)$. (You may write the equation(s) for the line in any form: symmetric, parametric, vector.)

   (c) At what times is the acceleration vector orthogonal to the velocity vector?

   (d) What is the physical significance of your answer to the previous question? (In particular, what is true about the speed when the acceleration and velocity vectors are orthogonal?)

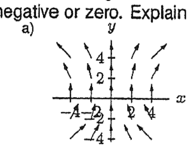
8. Let S be the surface described parametrically by

   $$\mathbf{r}(u,v) = \langle u^2+v, v^2+u, u-v \rangle \quad \text{for} \quad 1 \le u \le 2 \;\; -1 \le v \le 1.$$

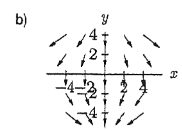
   Write but don't evaluate an iterated integral which will give the value of the surface integral of the function $f(x,y,z) = y - z$ over this surface.

9. Let C be the piecewise smooth curve that starts at the point $(0,-4)$ in the plane and travels up the y-axis to the origin and then follows the x-axis out to the point $(4,0)$. For each of the vector fields shown in the figures below, determine whether the line integral of the field along C would be positive, negative or zero. Explain how you know.

a)



b)



10. Let S be the part of the graph of $g(x,y) = 4 - x^2 - 4y^2$ which is above the xy-plane with the usual (upwards) orientation. Let $\mathbf{F}$ be the vector field

   $$\mathbf{F}(x,y,z) = \langle y+z^2, -x-z^2, z^3 \rangle \quad \text{which has curl} \quad \operatorname{curl}\mathbf{F}(x,y,z) = \langle 2z, 2z, -2 \rangle.$$

   (a) Write (but **do not evaluate**) an **iterated** integral that will give the flux of the vector field $\operatorname{curl}\mathbf{F}$ across the surface S.

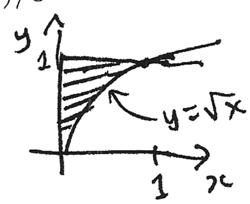
   (b) The boundary of S can be parametrized as $x(t) = 2\cos(t)$, $y(t) = \sin(t)$ and $z(t) = 0$ for $0 \le t \le 2\pi$. (This goes around in the counter-clockwise direction when viewed from above.) Write **and evaluate** the line integral of $\mathbf{F}$ around this curve using this parametrization.

   (c) According to Stokes' Theorem, what is the relationship between the integral you wrote in (a) and the value of the line integral you found in (b)?

Math 221 Final Exam Answers (Spring 2002)

1. a. $p_{xx} = -2y/(x+y)^3$, $p_{xy} = (x-y)/(x+y)^3$
   b. $\operatorname{curl} \mathbf{F} = \langle x, -y, 0 \rangle$, $\div \mathbf{F} = 2z$
   c. $3x + 4y - z = 4$

2. $\int_0^1 \int_0^{y^2} \sin(y^3)\, dx dy = (1 - \cos 1)/3$



3. $\int_{-1}^1 \int_{1-x^2}^{2-2x^2} 2\, dy dx = 8/3$

4. $f(2,1) = 6$, $f(1,3) = -2$

5. a. $\operatorname{curl} \mathbf{F} = \langle 0, 0, -2 \rangle$
   b. $\int_0^{\pi/2} \int_0^{\pi/2} (\operatorname{curl} \mathbf{F} \cdot \mathbf{n}) \sin^2\phi \, d\phi \, d\theta = -\pi/3$
   c. $+\pi/3$

6. a. $\mathbf{F} = \langle y, x - 2y \rangle$
   b. $-8/5$
   c. $\mathbf{u} = \langle 1, -1 \rangle / \sqrt{2}$
   d. $f(2,-1) - f(1,1) = -3$

7. a. $\mathbf{v}(t) = \langle 9, 2t-3, 2t-1 \rangle$, $\mathbf{a}(t) = \langle 0, 2, 2 \rangle$
   b. $\dfrac{x-18}{9} = \dfrac{y+2}{1} = \dfrac{z-2}{3}$, or $\{x = 9t + 18, y = t - 2, z = 3t + 2\}$
   c. $t = 1$
   d. When the velocity and acceleration are orthogonal, the time derivative of the speed is zero.

8. $\int_{-1}^1 \int_1^2 (y(u,v) - z(u,v)) |\mathbf{r}_u \times \mathbf{r}_v| du dv$
   $= \int_{-1}^1 \int_1^2 (v^2 + v) \sqrt{(1+2u)^2 + (1+2v)^2 + (1-4uv)^2} du dv$

9. a. Along the y-axis portion of C, the vector field points along C's tangent vector, while along the x-axis portion the vector field is perpendicular to C. So, the line integral is positive.
   b. Similarly, the line integral is negative.

10. a. $\int_{-1}^1 \int_{-2\sqrt{1-y^2}}^{2\sqrt{1-y^2}} (4x + 16y)(4 - x^2 - 4y^2) - 2\, dx dy$
    b. $\int_0^{2\pi} \langle y(t), -x(t), 0 \rangle \cdot \langle -2\sin t, \cos t, 0 \rangle dt = -4\pi$
    c. They are equal.