Math 221 (Ivey) Final Exam Dec. 8, 2000

1. **Short Answer Problems** (5 pts. each)

    (a) Give parametric equations for the line through $(2, 0, -1)$ parallel to vector $< 1, 3, 2 >$.

    (b) Find the angle (in degrees) between the vectors $\vec{v} = \vec{\imath} + 2\vec{\jmath} - \vec{k}$ and $\vec{u} = \vec{\jmath} - \vec{k}$.

    (c) Find a vector that is perpendicular to $\vec{\imath} - 2\vec{\jmath} + \vec{k}$ and parallel to the plane $3x - 2z = 5$.

    (d) Find the gradient of $f(x, y) = x + 3y - x^2$ at the point $(1, 1)$.

    (e) Let f be the same and let $g(x, y) = x - y^2$. Give a unit vector $\vec{u}$ pointing along one of the coordinate axes such that $D_u f(0, 0)$ and $D_u g(0, 0)$ are both positive.

    (f) Find the equation of the tangent plane at the point $(1, 1, 1)$ to the surface defined by the equation $x^2 y + z^3 = 2$.

2. **You may complete parts (c) and (d) of this problem without doing parts (a) or (b).**

    (a) (5 pts.) Identify each quantity in the formula $\boxed{\vec{a} = v'\vec{T} + \kappa v^2 \vec{N}}$.

    (b) (5 pts.) Use the above formula to show that $|\vec{a} - (\vec{a} \cdot \vec{T})\vec{T}| = \kappa v^2$.

    (c) (7 pts.) For the cycloid defined by $\vec{r}(t) =< t - \sin t, 1 - \cos t >$, find the speed and the unit tangent vector.

    (d) (8 pts.) Use the formula in part (b) to find the curvature of the cycloid at the point where $t = \pi$.

3. (20 pts.) Find the maximum value of $f(x, y) = 3x - 8y + 1$ on the curve defined by the equation $x^2 + 4y^2 = 25$.

4. (20 pts.) Find the critical points of the function $f(x, y) = x^2 + 2y^2 + x^2 y$ and classify them as relative maxima, minima, or saddle points.

5. (15 pts.) Sketch the region of integration for the double integral $\int_0^3 \int_{y^2}^9 y \cos(x^2)\, dx\, dy$. Then, evaluate the integral by reversing the order of integration.

6. (15 pts.) Calculate $\iint_S z\, dS$, where S is the portion of the sphere $x^2 + y^2 + z^2 = 9$ inside the first octant (i.e., where $x \geq 0, y \geq 0, z \geq 0$).

7. Let $\vec{F} = (3x^2 + y)\vec{\imath} + (x - 4y)\vec{\jmath}$, and let C be the arc of the unit circle (centered at the origin) starting at $(1, 0)$ and ending at $(0, 1)$.

    (a) (5 pts.) Show that $\vec{F}$ is conservative. (Do not find a potential function.)

    (b) (10 pts.) Find a function $f(x, y)$ such that $\vec{F} = \nabla f$. Then, use this function to evaluate $\int_C \vec{F} \cdot d\vec{r}$.

    (c) (10 pts.) Evaluate the above line integral directly (i.e., without using $f(x, y)$).

8. (15 pts.) **Use Green's Theorem** to compute the work done by the force $\vec{F} = xy\vec{\imath} + (x^2 - y^2)\vec{\jmath}$ on an object that moves in a counterclockwise fashion along the boundary of the rectangle $0 \leq x \leq 1, 0 \leq y \leq 2$.

9. (13 pts.) Let $\vec{F} = yz\vec{\imath} - x^2\vec{k}$.

    (a) Compute $\textbf{curl}\, \vec{F}$.

    (b) Compute $\operatorname{div} \vec{F}$. If $\vec{F}$ is the velocity of a fluid, how do you interpret the result?

10. (22 pts.) Let S be the part of the paraboloid $z = 4 - x^2 - y^2$ above the xy plane, with upward unit normal $\vec{n}$, and let $\vec{F}$ be the same as in question 9.

    (a) Compute the flux integral $\iint_S \textbf{curl}\vec{F} \cdot \vec{n}\, dS$.

    (b) **Use Stokes' Theorem** to rewrite the integral in part (a) as a line integral, and evaluate that integral.

Answers (Math 221 Final, Dec. 8, 2000)

1. (a) $x = 2 + t,\ y = 3t,\ z = -1 + 2t$
   (b) $\theta = 30$ degrees
   (c) $< 4, 5, 6 >$
   (d) $< -1, 3 >$
   (e) $\vec{u} = \vec{\imath}$
   (f) $2x + y + 3 = 6$

2. (a) $\vec{a}$ is acceleration, v is speed, $\vec{T}$ is the unit tangent vector, $\vec{N}$ the unit normal vector, and κ is the curvature
   (b) dot with $\vec{T}$, multiply the result by $\vec{T}$, subtract from the given formula, and take the length of both sides
   (c) $v = \sqrt{2 - 2\cos t},\ \vec{T} = \dfrac{1}{v} < 1 - \cos t, \sin t >$
   (d) $\kappa = 1/4$

3. maximum is $f(3, -1) = 18$

4. local minimum at $(0, 0)$, saddle points at $(\pm 2, -1)$

5. $\sin(81)/4$

6. $3\pi/4$

7. (a) $\dfrac{\partial(x - 4y)}{\partial x} - \dfrac{\partial(3x^2 + y)}{\partial y} = 0$
   (b) Using $f(x, y) = x^3 + xy - 2y^2$, $\displaystyle\int_C \vec{F} \cdot d\vec{r} = -3.$
   (c) -3

8. 1

9. (a) $2z\vec{\jmath}$
   (b) $\text{div}\vec{F} = 0$, so the fluid is *incompressible*

10. (a) and (b): the integral is **zero**.