Instructions: Fully explain your answers and show your work. If you are not sure if you have written enough, please ask.

1. (12 pts.) Set up (but do not evaluate) integrals that give the following quantities:
   (a) The volume of the solid of revolution obtained by revolving the region enclosed by the $x$-axis and the graph $y = 2x - x^2$ about the line $x = -1$.
   (b) The work required to pump half of the water out over the top of a full swimming pool which is 8 feet wide, 6 feet deep, and 20 feet. (Assume the density of water is 62.5 pounds per cubic foot.)
   (c) The length of the part of the graph $y = \frac{1}{3}x^3$ for $0 \leq x \leq 1$.

2. (6 pts.) Let $R$ be the region between the graph $y = \sqrt{x}$ and the $x$-axis, for $0 \leq x \leq 4$. Find the coordinates of the center of mass of this region.

3. (5 pts.) Solve the initial value problem $\frac{dy}{dx} = \frac{6 - 2y}{x^2 - 1}$, $y(0) = 5$.

4. (a) (5 pts.) Find the approximate value of $\int_0^8 \ln(1+x^2) \, dx$ using Simpson’s rule, with $n = 4$ intervals.
   (b) (5 pts.) Find the approximate value of this integral, with error less than $1/100$, using a power series.

5. (16 pts.) Find the following antiderivatives:
   (a) $\int x^2 \cosh x \, dx$
   (b) $\int \sec^3 x \tan x \, dx$
   (c) $\int \frac{x - 3}{x^3 + 9x} \, dx$
   (d) $\int \frac{1}{\sqrt{x^2 + 4}} \, dx$

6. (8 pts.) Evaluate the improper integral or show that it diverges:
7. (10 pts.) Explain why each of the following series converge, find the approximate sum of the series using the first 4 terms, and estimate the remainder.

(a) \[ \sum_{k=1}^{\infty} \frac{1}{k^3} \]

(b) \[ \sum_{n=1}^{\infty} \frac{\sqrt{2n - 1}}{n^2} \]

8. (10 pts.) Determine if the following series converge absolutely, converge conditionally, or diverge.

(a) \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2 + 1}} \]

(b) \[ \sum_{n=1}^{\infty} (-1)^n \left( \frac{n}{2n - 1} \right)^n \]

9. (a) (5 pts.) Calculate the Taylor polynomial \( T_3(x) \) of degree 3, centered at 2, for \( f(x) = \frac{1}{x^2} \).

(b) (6 pts.) Use Taylor’s Inequality to find an upper bound for the remainder for \( T_3(x) \) when \( 1 \leq x \leq 3 \).

10. (6 pts.) Determine the radius and interval of convergence for the power series \[ \sum_{n=1}^{\infty} \frac{2^n(x + 1)^n}{n^2} \]

11. (6 pts.) Find the Maclaurin series for \( f(x) = \arcsin(x) \); express your answer in summation notation, and find the first 4 nonzero terms.

12. (10 pts.) For the parametrized curve defined by \( x = t^2 + t, \ y = t^3 - t \), find

(a) the slope of the tangent when the curve passes through the origin;
(b) the values of \( t \) for which the curve is concave up;

(c) an integral that computes the length of the loop formed by the curve. (Do not evaluate this integral.)

13. (10 pts.) Sketch the curve defined by the polar equation \( r = 1 + 2 \sin \theta \), for \( 0 \leq \theta \leq 2\pi \) on the given grid, and find the area enclosed by the curve inside the first quadrant.
<table>
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<td><strong>Divergence Test</strong></td>
<td>If ( \lim_{k \to \infty} a_k \neq 0 ), then ( \sum_{k=1}^{\infty} a_k ) diverges. (If ( \lim_{k \to \infty} a_k = 0 ), there is no conclusion.)</td>
</tr>
<tr>
<td><strong>Integral Test</strong></td>
<td>Let ( \sum_{k=1}^{\infty} a_k ) be a series with positive terms, such that ( a_k = f(k) ) for a continuous function ( f(x) ) which is decreasing for ( x ) sufficiently large. Then ( \sum_{k=1}^{\infty} a_k ) and ( \int f(x) , dx ) either both converge or both diverge.</td>
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<tr>
<td><strong>Comparison Test</strong></td>
<td>Let ( \sum_{k=1}^{\infty} a_k ) and ( \sum_{k=1}^{\infty} b_k ) be series with non-negative terms such that ( a_k \leq b_k ). If ( \sum_{k=1}^{\infty} b_k ) converges, then ( \sum_{k=1}^{\infty} a_k ) converges. If ( \sum_{k=1}^{\infty} a_k ) diverges, then ( \sum_{k=1}^{\infty} b_k ) diverges.</td>
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<tr>
<td><strong>Limit Comparison Test</strong></td>
<td>Let ( \sum_{k=1}^{\infty} a_k ) and ( \sum_{k=1}^{\infty} b_k ) be two series with positive terms, and let ( L = \lim_{k \to \infty} a_k/b_k ). If ( L ) is positive and finite, then either both series converge or both diverge.</td>
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<tr>
<td><strong>Alternating Series Test</strong></td>
<td>If ( \sum_{k=1}^{\infty} u_k ) is an alternating series with (</td>
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<tr>
<td><strong>Ratio &amp; Root Test</strong></td>
<td>Let ( \sum_{k=1}^{\infty} u_k ) be any series, and let ( L = \lim_{k \to \infty}</td>
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\[
\begin{align*}
\sin^2 \theta + \cos^2 \theta &= 1 \\
1 + \cot^2 \theta &= \csc^2 \theta \\
\cos^2 x &= \frac{1}{2}(1 + \cos 2x) \\
\sin^2 x &= \frac{1}{2}(1 - \cos 2x) \\
\sinh x &= \frac{1}{2}(e^x - e^{-x}) \\
\cosh x &= \frac{1}{2}(e^x + e^{-x}) \\
(sin x)' &= \cos x \\
(cos x)' &= -\sin x \\
(tan x)' &= \sec^2 x \\
(sec x)' &= \sec x \tan x \\
(csc x)' &= -\csc x \cot x \\
(cot x)' &= -\csc^2 x \\
(arcsin x)' &= \frac{1}{\sqrt{1-x^2}} \\
(arctan x)' &= \frac{1}{1+x^2} \\
\int \tan x \, dx &= \ln |\sec x| + C \\
\int \sec^n x \, dx &= \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx \\
\int u \, dv &= uv - \int v \, du \\
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for all } x \\
(1+x)^k &= \sum_{n=0}^{\infty} \binom{k}{n} x^n \quad \text{for } -1 < x < 1 \\
\sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{for all } x \\
\cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad \text{for all } x \\
arctan(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{for } -1 < x < 1 \\
(1+x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n} \quad \text{for } -1 < x < 1
\end{align*}
\]
Math 220 (Fall 2009)  Final Exam

Instructions: Fully explain your answers and show your work. If you are not sure if you have written enough, please ask.

1. (12 pts.) Set up (but do not evaluate) integrals that give the following quantities:

(a) The volume of the solid of revolution obtained by revolving the region enclosed by the $x$-axis and the graph $y = 2x - x^2$ about the line $x = -1$.

\[ V = 2\pi \int_{0}^{2} (x+1)(2x - x^2) \, dx \]

(b) The work required to pump half of the water out over the top of a full swimming pool which is 8 feet wide, 6 feet deep, and 20 feet long. (Assume the density of water is 62.5 pounds per cubic foot.)

\[ W = \int_{0}^{3} \chi \cdot 160 \cdot \rho \, d\chi \quad \text{foot-pounds} \]
\[ \rho = 62.5 \]

(c) The length of the part of the graph $y = \frac{1}{3}x^3$ for $0 \leq x \leq 1$.

\[ L = \int_{0}^{1} \sqrt{1 + x^4} \, dx \]

\[ \frac{dy}{dx} = x^2 \]
2. (6 pts.) Let $R$ be the region between the graph $y = \sqrt{x}$ and the $x$-axis, for $0 \leq x \leq 4$. Find the coordinates of the center of mass of this region.

$$A = \int_{0}^{4} \sqrt{x} \ dx = \frac{16}{3}$$

$$\bar{x} = \frac{1}{A} \int_{0}^{4} x \sqrt{x} \ dx = \frac{12}{5}$$

$$\bar{y} = \frac{1}{2A} \int_{0}^{4} x \ dx = \frac{3}{4}$$

Center of mass is $\left( \frac{12}{5}, \frac{3}{4} \right)$

3. (5 pts.) Solve the initial value problem $\frac{dy}{dx} = \frac{6 - 2y}{x^2 - 1}, y(0) = 5$.

$$\int \frac{1}{3-y} \ dy = \int \frac{2}{x^2-1} \ dx$$

$$\int \frac{2}{x^2-1} \ dx = \int \left( \frac{1}{x-1} - \frac{1}{x+1} \right) \ dx$$

$$= \ln |x-1| - \ln |x+1| + C_1$$

$$\int \frac{1}{3-y} \ dy = -\ln |3-y| + C_2$$

$$3-y = \left( \frac{x+1}{x-1} \right) \cdot C_3$$

$$x=0, \ y=5 \Rightarrow C_3 = 2$$

$$3-5 = \frac{1}{-1} \cdot C_3$$

$$y = 3 - 2 \cdot \left( \frac{x+1}{x-1} \right)$$
4. (a) (5 pts.) Find the approximate value of \( \int_0^8 \ln(1+x^2) \, dx \) using Simpson’s rule, with \( n = 4 \) intervals.

\[
\Delta x = 0.2
\]

\[
\begin{align*}
\int_S &= \frac{\Delta x}{3} \left( f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4) \right) \\
&= \frac{0.2}{3} \left( 0 + 4 \times 0.03922 + 2 \times 0.14842 + 4 \times 0.30748 + 0.49469 \right) \\
&= 0.14522
\end{align*}
\]

(b) (5 pts.) Find the approximate value of this integral, with error less than 1/100, using a power series.

\[
\ln(1+x^2) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n}}{n}
\]

\[
= x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \ldots
\]

\[
\int_0^{0.8} \ln(1+x^2) \, dx = \frac{1}{3} 0.8^3 - \frac{1}{2 \times 5} 0.8^5 + \frac{1}{3 \times 7} 0.8^7 - \frac{1}{4 \times 9} 0.8^9 + \ldots
\]

\[
(\text{Alternating series}) \quad \frac{0.8^7}{3 \times 7} = 0.00998
\]

(This term is small enough.)

\[
\begin{align*}
I &\approx \frac{1}{3} \cdot 0.8^3 - \frac{1}{10} \cdot 0.8^5 \\
I &\approx 0.13789
\end{align*}
\]
5. (16 pts.) Find the following antiderivatives:

(a) \[\int x^2 \cosh x \, dx = x^2 \sinh x - \int 2x \sinh x \, dx\]

By parts:
\[= x^2 \sinh x - 2x \cosh x + \int 2 \cosh x \, dx\]

1. \(u = x^2\)
   \(du = 2x \, dx\)
   \(v = \sinh x\)

2. \(u = 2x\)
   \(du = \sinh x \, dx\)
   \(v = \cosh x\)

(b) \[\int \sec^3 x \tan x \, dx = \int u^2 \, du = \frac{1}{3} u^3 + C\]

\(u = \sec x\)

\(du = \sec x \tan x \, dx\)

\[= \frac{1}{3} \sec^3 x + C\]

(c) \[\int \frac{x - 3}{x^3 + 9x} \, dx\]

\[= \int \left(\frac{1/3}{x} + \frac{1/3x + 1}{x^2 + 9}\right) \, dx\]

\[= -\frac{1}{3} \ln |x| + \frac{1}{6} \ln (x^2 + 9) + \frac{1}{3} \arctan \left(\frac{x}{3}\right) + C\]

Partial Fractions:

\[\frac{x - 3}{x^3 + 9x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 9}\]

\[x - 3 = A \cdot (x^2 + 9) + (Bx + C) \cdot x\]

\[x - 3 = (A + B)x^2 + Cx + 9A\]

\[x^0: \quad 9A = -3 \quad \Rightarrow \quad A = -\frac{1}{3}\]

\[x^1: \quad C = 1\]

\[x^2: \quad A + B = 0 \quad \Rightarrow \quad B = \frac{1}{3}\]
(d) \( \int \frac{1}{\sqrt{x^2 + 4}} \, dx = \int \frac{2 \sec^2 \theta}{2 \sec \theta} \, d\theta = \int \sec \theta \, d\theta \)

\( x = 2 \tan \theta \)

\( dx = 2 \sec^2 \theta \, d\theta \)

\( \tan \theta = \frac{x}{2} \)

\( \sec \theta = \frac{\sqrt{x^2 + 4}}{2} \)

\( = \ln | \sec \theta + \tan \theta | + C_1 \)

\( = \ln \left( \frac{\sqrt{x^2 + 4}}{2} + \frac{x}{2} \right) + C_1 \)

\( = \ln \left( \sqrt{x^2 + 4} + x \right) + C_2 \)

6. (8 pts.) Evaluate the improper integral or show that it diverges:

(a) \( \int_1^2 \frac{1}{x-1} \, dx \)

\( = \lim_{a \to 1^+} \int_a^2 \frac{1}{x-1} \, dx \)

\( = \lim_{a \to 1^+} [\ln |x-1|]_a^2 \)

\( = \lim_{a \to 1^+} (\ln(1) - \ln(a-1)) \)

\( = +\infty \)

Diverges.

(b) \( \int_2^\infty \frac{1}{x(\ln x)^2} \, dx \)

\( U = \ln x \)

\( du = \frac{1}{x} \, dx \)

\( = \lim_{b \to \infty} \int_2^b \frac{1}{x(\ln x)^2} \, dx \)

\( = \lim_{b \to \infty} \int_{\ln 2}^{\ln b} u^{-2} \, du \)

\( = \lim_{b \to \infty} \left[ -\frac{1}{u} \right]_{\ln 2}^{\ln b} \)

\( = \lim_{b \to \infty} \left( -\frac{1}{\ln b} + \frac{1}{\ln 2} \right) \)

\( = \frac{1}{\ln 2} \)

Converges.
7. (10 pts.) Explain why each of the following series converge, find the approximate sum of the series using the first 4 terms, and estimate the remainder.

(a) \[ \sum_{k=1}^{\infty} \frac{1}{k\,3^k} \]

Converges by the Comparison Test:

Terms bounded by terms of convergent geometric series.

\[ S_4 = \frac{1}{3} + \frac{1}{2\cdot3^2} + \frac{1}{3\cdot3^3} + \frac{1}{4\cdot3^4} = 0.40432 \]

\[ R_4 \leq \sum_{k=5}^{\infty} \frac{1}{k\,3^k} \leq \sum_{k=5}^{\infty} \frac{1}{3^k} = \frac{1/3^5}{1-1/3} = 0.00617 \]

(b) \[ \sum_{n=1}^{\infty} \frac{\sqrt{2n-1}}{n^2} \]

Converges by the Comparison Test:

Terms bounded by convergent p series.

\[ S_4 = \frac{\sqrt{1}}{1^2} + \frac{\sqrt{3}}{2^2} + \frac{\sqrt{5}}{3^2} + \frac{\sqrt{7}}{4^2} = 1.8468 \]

\[ R_4 = \sum_{n=5}^{\infty} \frac{\sqrt{2n-1}}{n^2} \leq \sqrt{2} \sum_{n=5}^{\infty} n^{-3/2} \]

\[ \leq \sqrt{2} \int_{4}^{\infty} x^{-3/2} \, dx \]

\[ = \sqrt{2} \left[ -2 \cdot x^{-1/2} \right]_{4}^{\infty} \]

\[ = \sqrt{2} = 1.4142 \]
8. (10 pts.) Determine if the following series converge absolutely, converge conditionally, or diverge.

(a) \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2 + 1}} \]

\[ |a_n| = \frac{1}{\sqrt{n^2 + 1}} \]

\[ \lim_{n \to \infty} \frac{1}{\frac{\sqrt{n^2 + 1}}{n}} = 1 \]

So \[ \sum |a_n| \] diverges by limit comparison with a harmonic series.

But the series alternates, and \[ \lim_{n \to \infty} |a_n| = 0 \]

So the series converges conditionally.

(b) \[ \sum_{n=1}^{\infty} (-1)^n \left( \frac{n}{2n-1} \right)^n \]

\[ \lim_{n \to \infty} \left( |a_n| \right)^{1/n} = \lim_{n \to \infty} \frac{n}{2n-1} = \frac{1}{2} < 1 \]

So the series converges absolutely by the root test.
9. (a) (5 pts.) Calculate the Taylor polynomial \( T_3(x) \) of degree 3, centered at 2, for \( f(x) = \frac{1}{x^2} \).

\[
\begin{align*}
  f(x) &= x^{-2} & f(x) &= \frac{1}{4} \\
  f'(x) &= -2x^{-3} & f'(x) &= -\frac{1}{4} \\
  f''(x) &= 6x^{-4} & f''(x) &= \frac{3}{8} \\
  f'''(x) &= -24x^{-5} & f'''(x) &= -\frac{3}{4}
\end{align*}
\]

\[
T_3(x) = \frac{1}{4} - \frac{1}{4} (x-2) + \frac{3}{8} \left( \frac{x-2}{2} \right)^2 - \frac{3}{4} \left( \frac{x-2}{6} \right)^3
\]

(b) (6 pts.) Use Taylor’s Inequality to find an upper bound for the remainder for \( T_3(x) \) when \( 1 \leq x \leq 3 \).

\[
f^{(4)}(x) = 120x^{-6}
\]

\[
|f^{(4)}(x)| \leq 120 \quad \text{for} \quad x \in [1, 3]
\]

So \( M = 120 \):

\[
|R_3(x)| \leq M \cdot \frac{(x-2)^4}{4!}
\]

\[
|R_3(x)| \leq 120 \cdot \frac{1^4}{4!} = 5
\]
10. (6 pts.) Determine the radius and interval of convergence for the power series \( \sum_{n=1}^{\infty} \frac{2^n(x+1)^n}{n^2} \)

\[
L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \\
= \lim_{n \to \infty} \left| \frac{2^{n+1}(x+1)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{2^n(x+1)^n} \right| \\
= \left| 2 \cdot \frac{(x+1)}{(n+1)} \right| \\
L < 1 \iff |x+1| < \frac{1}{2} \quad \text{So} \quad R = \frac{1}{2} \\
\text{So end points are } -\frac{1}{2} \text{ and } -\frac{3}{2} \\
\text{So include end points} \\
\text{Interval of convergence is } \left[-\frac{3}{2}, \frac{1}{2}\right]
\]

11. (6 pts.) Find the Maclaurin series for \( f(x) = \arcsin(x) \); express your answer in summation notation, and find the first 4 nonzero terms.

\[
f'(x) = \frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n} (-x^2)^n \quad \text{Binomial series}
\]

\[
f(x) = \int_0^x \frac{1}{\sqrt{1-x^2}} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n} \frac{x^{2n+1}}{2n+1}
\]

\[
f(0) = 0
\]

\[
f(x) = 1 + \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots
\]

\[
f(x) = x + \frac{x^3}{6} + \frac{x^5}{8} - \frac{5x^7}{112} + \cdots
\]

\[
* \left( \frac{-1/2}{n} \right) = \frac{(-1/2)(-1/2-1)(-1/2-(n-1))}{n!}
\]
12. (10 pts.) For the parametrized curve defined by \( x = t^2 + t, \ y = t^3 - t \), find

(a) the slope of the tangent when the curve passes through the origin;

\[
\frac{dy}{dt} = 3t^2 - 1 \\
\frac{dx}{dt} = 2t + 1
\]

\( x=0 \Rightarrow t(4t+1)=0 \Rightarrow t=0 \text{ or } t=-\frac{1}{4} \)
\( y=0 \Rightarrow t(4t^2-1)=0 \Rightarrow t=0 \text{ or } t=\pm 1 \) or \( t=-\frac{1}{4} \)

So the curve passes through \( (0,0) \) \( a=4, t=-1 \) and \( t=0 \).

\[
\left. \frac{dy}{dx} \right|_{t=-1} = -2 \\
\left. \frac{dy}{dx} \right|_{t=0} = -1
\]

(b) the values of \( t \) for which the curve is concave up;

\[
\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dt} \right) = \frac{d}{dt} \left( \frac{dy}{dx} \right) \cdot \frac{dt}{dx}
\]

\[
= \frac{d}{dt} \left( \frac{3t^2-1}{2t+1} \right) \left( \frac{1}{24+1} \right)
\]

\[
= \frac{(24+1)(6t) - (3t^2-1)(2)}{(24+1)^3}
\]

\[
= \frac{2(3t^2+3t+1)}{(24+1)^3}
\]

\[\text{Note:} \quad 3t^2+3t+1 = 0 \] \[\text{complex} \]
\[-3 \pm \sqrt{9-4 \cdot 3} \]
\[\text{So numerator is always positive} \]
\[\frac{d^2y}{dx^2} < 0 \text{ when } 24+1 < 0, \]
\[\text{which is } t > -\frac{1}{2} \]

(c) an integral that computes the length of the loop formed by the curve. (Do not evaluate this integral.)

\[
L = \int_{-1}^{0} \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \ dt
\]

\[
L = \int_{-1}^{0} \sqrt{(24+1)^2 + (3t^2-1)^2} \ dt
\]

\[
L = \int_{-1}^{0} \sqrt{9t^4 - 2t^2 + 2} \ dt
\]
13. (10 pts.) Sketch the curve defined by the polar equation \( r = 1 + 2 \sin \theta \), for \( 0 \leq \theta \leq 2\pi \) on the given grid, and find the area enclosed by the curve inside the first quadrant.

\[
A = \int_0^{\pi/2} \frac{1}{2} (1 + 2 \sin \theta)^2 \, d\theta
\]

\[
= \frac{1}{2} \int_0^{\pi/2} (1 + 4 \sin \theta + 4 \sin^2 \theta) \, d\theta
\]

\[
= \frac{1}{2} \left[ \int_0^{\pi/2} (1 + 4 \sin \theta + 2 - 2 \cos 2\theta) \, d\theta \right]
\]

\[
= \frac{1}{2} \left[ 3\theta - 4 \cos \theta - \sin 2\theta \right]_0^{\pi/2}
\]

\[
= \frac{1}{2} \left( \frac{3\pi}{2} - 0 - 0 - (0 - 4) \right)
\]

\[
= \frac{3\pi}{4} + 2 = 4.35
\]