Read each problem carefully. No calculators, notes, books, or any outside materials. Unless otherwise indicated, **supporting work will be required on every problem**; one-word answers or answers which simply restate the question will receive no credit.

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1 (4 pts). Is $\left\{ \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 5 \\ -6 \end{pmatrix} \right\}$ linearly independent in $\mathbb{R}^3$?

2 (8 pts). Let H be the set $\left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 : 2a - 3b = c \right\}$. Is H a subspace of $\mathbb{R}^3$?

3a (8 pts). Let $\mathcal{D}$ denote the set $\{t^2, t(t-1), (t-1)(t-2), t(t-2)\}$ of four polynomials in $\mathbb{P}_2$. Does $\mathcal{D}$ span $\mathbb{P}_2$?

3b (4 pts). Is $\mathcal{D}$ a basis for $\mathbb{P}_2$?

4 (4 pts). If a set of five vectors $\{v_1, v_2, \ldots, v_5\}$ spans the vector space V, what can you conclude about the dimension of V?

5. Let T be the transformation from $\mathbb{P}_2$ into $\mathbb{R}^2$ is given by the rule $T\big(p(t)\big) = \begin{pmatrix} p(0) \\ p(1) \end{pmatrix}$.

a (3 pts). Find $T(t^2 + 1)$.

b (8 pts). Prove that the transformation T is linear.

c (8 pts). Let $\mathcal{B}$ denote the basis $\{1, t, t^2\}$ for $\mathbb{P}_2$ and let $\mathcal{C}$ denote the standard basis $\{e_1, e_2\}$ for $\mathbb{R}^2$. Find the matrix of T relative to the bases $\mathcal{B}$ and $\mathcal{C}$.

d (6 pts). Find a polynomial $p(t)$ (other than $\mathbf{0}$) which is in the kernel of T. Include in your answer an explanation of what it means for $p(t)$ to belong to the kernel of T.

6. Suppose $X^{-1} = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$ $Y^{-1} = \begin{pmatrix} -2 & -1 \\ 0 & 3 \end{pmatrix}$ and $Z^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.

a (3 pts). Find $(X^T)^{-1}$.

b (6 pts). Find $\det Y$.

c (6 pts). Find $(XYZ)^{-1}$.

7 (12 pts). Let $A = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 3 & 2 & 0 & 3 \\ \alpha & 1 & 1 & 1 \\ \beta & 1 & 1 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$ and $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$.

If $\det A = 4$ and $Ax = b$, find x_1.

8a (12 pts). Solve for x. Write your solution in parametric vector form.

$$\begin{array}{rcrcrcrcr} x_1 & - & 2x_2 & + & x_3 & + & 5x_4 & = & 6 \\ 2x_1 & - & 4x_2 & + & x_3 & + & 7x_4 & = & 7 \\ -10x_1 & + & 20x_2 & - & 3x_3 & - & 29x_4 & = & -25 \end{array}$$

8b (3 pts). Find a basis for the nulspace of the matrix $\begin{pmatrix} 1 & -2 & 1 & 5 \\ 2 & -4 & 1 & 7 \\ -10 & 20 & -3 & -29 \end{pmatrix}$

8c (3 pts). What is the rank of the matrix in 8b?
(No work required. A correct answer is sufficient for full credit.)

9 (12 pts). Find the inverse of the matrix $\begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

10 (8 pts). Suppose that A and B are row equivalent, where

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

a. Find a basis for Col A

b. Find a basis for Col B

c. Find a basis for Row A

d. Find a basis for Row B

11a (6 pts). Find the orthogonal projection of $\begin{pmatrix} -2 \\ -2 \\ 3 \\ 2 \end{pmatrix}$ onto span $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$.

11b (14 pts). Produce an orthogonal set of vectors by applying Gram-Schmidt to
$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} -3 \\ 5 \\ 1 \\ 3 \end{pmatrix} \right\}$

11c (4 pts). Is the set you produced in 11b a basis for $\mathbb{R}^4$?

12 (4 pts). Find a unit vector in the direction of $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

13 (10 pts). Is the matrix $\begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ diagonalizable? Explain.

14a (10 pts). Find the least-squares solution to $Cx = f$ where

$$C = \begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} 1 \\ 1 \\ -5 \end{pmatrix}$$

14b (4 pts). Find the orthogonal projection of f onto the span of the columns of C.

14c (4 pts). Find the distance from f to the span of the columns of C.

15 (12 pts). Suppose Q is a linear map from $\mathbb{R}^5$ into $\mathbb{R}^4$.
a. Could Q be one-to-one? If so, provide an example. If not, explain.

b. Could Q be onto? If so, provide an example. If not, explain.

16 (6 pts). If G is a 3×3 matrix and $G\begin{pmatrix}1\\0\\1\end{pmatrix} = G\begin{pmatrix}1\\1\\0\end{pmatrix}$, find an eigenvalue and eigenvector of G.

17 (8 pts). Find all eigenvalues of $\begin{pmatrix}-4 & 3\\-6 & 5\end{pmatrix}$.

Ans, Final exam, Math 203, Dec 2009

1. No (too many vectors) 2. yes ($H = \text{Nul}\begin{bmatrix} 2 & -3 & -1 \end{bmatrix}$)

3a. yes 3b. No 4. $\dim V \le 5$

5. a. $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ b. (Show $T(p+q) = T(p) + T(q)$. Then show $T(cp) = cT(p)$). c. Matrix $= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$. d. means $T(p) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $p(t) = t(t-1)$ is in Ker T.

6. a. $\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$ b. $-\frac{1}{6}$ c. $\begin{pmatrix} -2 & -14 \\ 0 & 6 \end{pmatrix}$

7. $x_1 = \frac{1}{4}$

8a. $x = \begin{pmatrix} 1 \\ 0 \\ 5 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ -3 \\ 1 \end{pmatrix}$ b. $\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ -3 \\ 1 \end{pmatrix} \right\}$

8c. 2 (= # basic vars)

9. $\begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 1 \\ -1 & 1 & -1 \end{pmatrix}$ 10a. $\left\{ \begin{pmatrix} a \\ d \\ g \end{pmatrix} \begin{pmatrix} c \\ f \\ i \end{pmatrix} \right\}$ 10b. $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$

10c $\{ (1 \;\; -1 \;\; 0), (0 \;\; 0 \;\; 1) \}$ = 10d.

11a. $(-3, -1, 1, 0)^T$

11b. $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -2 \\ 3 \end{pmatrix}$ 11c. yes 12. $\frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

13. No. ($\lambda = 3$ is a double root, but $\dim \text{Nul}(A - 3I) = 1$, not 2)

14a. $\hat{x} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ 14b. $\begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix}$ 14c. $\sqrt{14}$

15b. yes. $Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$ 15a. No. can't have pivot in every col.

16. $\lambda = 0$, $x = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$. 17. $\lambda = -1, 2$.