**Math 120 Final Exam, Fall 2014**

1. (10 pts.) Mark each statement as either true [T] or false [F].

   a. If $\lim_{x \to 5} f(x) = 0$ and $\lim_{x \to 5} g(x) = 0$, then $\lim_{x \to 5} \frac{f(x)}{g(x)}$ does not exist. ..................... ☐

   b. If $p$ is a polynomial, then $\lim_{x \to b} p(x) = p(b)$. ..................................... ☐

   c. If $f$ is continuous at $c$, then $f$ is differentiable at $c$. ................................ ☐

   d. A function can have two different horizontal asymptotes. ......................... ☐

   e. If $f$ has a local minimum at $c$, then $f'(c) = 0$. ...................................... ☐

   f. If $f''(c) = 0$, then $f$ has an inflection point at $c$. ................................... ☐

   g. If $f$ and $g$ are both continuous functions on the interval $[a, b]$, then
   $\int_a^b f(x)g(x)\,dx = \left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right)$. ................................. ☐

   h. If $\frac{d}{dx}(f(x)) = \frac{d}{dx}(g(x))$, then $f(x) = g(x)$. ..................................... ☐

   i. If $f$ is differentiable and $f(x) > 0$, then $\frac{d}{dx}\sqrt{f(x)} = \frac{f'(x)}{2\sqrt{f(x)}}$. ...................... ☐

   j. If $f'(x)$ exists and is nonzero for all $x$, then $f(1) \neq f(0)$. .......................... ☐

2. (18 pts.) An engineer needs to build a rectangular box that has a square base and no top with an outside surface area of 1200 square feet. What are the dimensions of the box meeting these restrictions that has the *greatest* possible volume? (For full credit, you must indicate what function you are maximizing and on what interval.)

3. Find $\frac{dy}{dx}$ for each of the following. (You don't need to simplify your answers.)

   (a) (4 pts.) $y = e^{3x} + \sin x + \pi^{2014}$

   (b) (5 pts.) $y = \frac{10}{\sqrt[3]{x}} + 2^x - \arctan x$

   (c) (6 pts.) $y = \frac{1 + 2\sec x}{e^x - 4\tan x}$

   (d) (4 pts.) $y = (1 + \sin x)\sqrt{1 + 4\cos x}$

   (e) (3 pts.) $y = \left(\ln(9x^2 + 5x)\right)^{10}$

4. (6 pts.) Find the equation of the tangent line to the graph $y = x^3 + 2x^2 + 5x + 7$ at the point where $x = -1$.

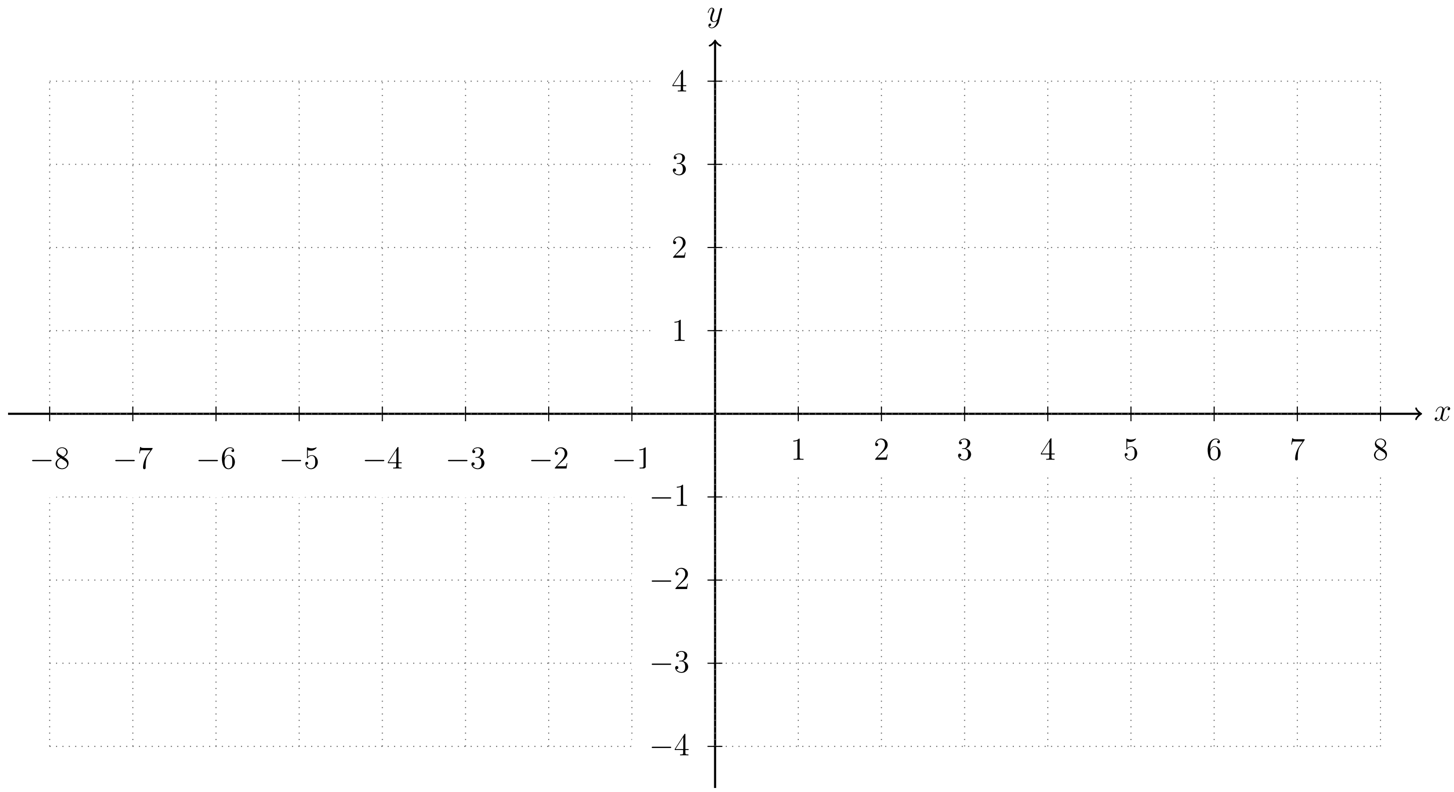
5. (7 pts.) Find $\frac{dy}{dx}$ along the curve defined by $x^4 + 2xy - 3\sin y = 5$.

6. (8 pts.) (a) Find the linearization $L(x)$ of $f(x) = \sqrt{1 + 2x}$ at $a = 12$.

   (b) Use your answer to part (a) to approximate $\sqrt{23}$.

7. (14 pts.) Using the axes below, sketch the graph of a function $f(x)$ with *all* the following properties:

- $\lim_{x \to -2^-} f(x) = -\infty$ and $\lim_{x \to -2^+} f(x) = +\infty$.
- $\lim_{x \to +\infty} f(x) = -\infty$ and $\lim_{x \to -\infty} f(x) = 0$.
- $f'(1)$ does not exist.
- $f'(x) > 0$ on $(-\infty, -4)$ and $(1, 4)$, while $f'(x) < 0$ on $(-4, -2)$, $(-2, 1)$ and $(4, \infty)$.
- $f''(x) > 0$ on $(-\infty, -6)$ and $(-2, 1)$, while $f''(x) < 0$ on $(-6, -2)$ and $(1, \infty)$.




8. Using the following table of values for $f, g, f'$ and $g'$, answer the questions below.

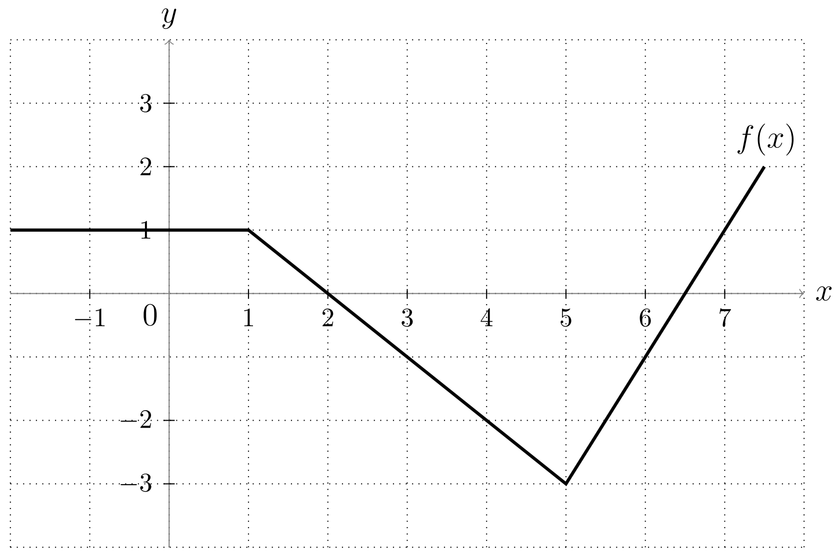
| $x$ | $f(x)$ | $g(x)$ | $f'(x)$ | $g'(x)$ |
|---|---|---|---|---|
| $-4$ | $-1$ | $9$ | $3$ | $10$ |
| $3$ | $-4$ | $7$ | $5$ | $9$ |
| $7$ | $5$ | $-1/3$ | $10$ | $2$ |

(a) (2 pts.) If $F(x) = f(x)g(x)$, find $F'(3)$.

(b) (3 pts.) If $G(x) = f(g(x))$, find $G'(3)$.

9. (5 pts.) Given that the function $f(x)$ is as in the graph below, and that $g(x) = \int_0^x f(t)\,dx$, determine the following five values:

$g(2) = \underline{\hspace{2cm}}$ $g(5) = \underline{\hspace{2cm}}$ $g(-1) = \underline{\hspace{2cm}}$ $g'(4) = \underline{\hspace{2cm}}$ $g''(6) = \underline{\hspace{2cm}}$




10. Suppose the *derivative* of $f(x)$ is $f'(x) = \dfrac{x^2 - 3}{x^3}$ and the domain of $f$ is $x \neq 0$.
    (a) (3 pts.) Find all critical numbers for $f$.

    (b) (4 pts.) Find all intervals where $f$ is increasing, and all intervals where $f$ is decreasing.

    (c) (2 pts.) For each critical number, determine if it gives a local maximum, local minimum, or neither.

    (d) (6 pts.) Find all intervals where $f$ is concave up, and all intervals where $f$ is concave down.

    (e) (1 pt.) Find the $x$-coordinates of any inflection points.

11. Find the following limits, find the infinite limit, or state that the limit does not exist.

  (a) (4 pts.) $\lim_{x\to 6} \dfrac{x^2-36}{x^2-6x}$

  (b) (4 pts.) $\lim_{x\to -3^+} \dfrac{x^2+2x}{x^2+x-6}$

  (c) (5 pts.) $\lim_{x\to 0} \dfrac{x^2}{1-\cos(3x)}$

  (d) (7 pts.) $\lim_{x\to -\infty} \dfrac{2x+1}{\sqrt{x^2+1}}$

  (e) (5 pts.) $\lim_{x\to +\infty} x\tan\left(\dfrac{5}{x}\right)$

12. (7 pts.) Suppose that $f(0)=-1$ and $f'(x)\le 3$ for all $x$. How large can $f(4)$ possibly be?

13. (7 pts.) Use the definition of derivative to find $f'(x)$ if $f(x)=\dfrac{1}{\sqrt{x}}$.
(Use only algebraic techniques to evaluate the limit; do not use L'Hôpital's Rule.)

14. (9 pts.) Find the absolute maximum and minimum values for $f(x)=\cos x-\sin x$ on the interval $[0,\pi]$.

15. (8 pts.) Suppose we know that $\int_0^1 f(x)\,dx = 5$ and $\int_0^4 f(x)\,dx = 2$.

Use this information to evaluate the following:

(a) $\int_1^4 f(x)\,dx$

(b) $\int_0^4 (2x - 3f(x))\ dx$

16. (5 pts.) Use a Riemann sum to approximate $\int_2^{10} \ln x\,dx$. Use $n = 4$ equal-length subintervals and take the sample points to be the right-hand endpoints. (Do not simplify your answer.)

17. An elevator is moving up and down in a vertical shaft, and we measure its height (in meters) above ground level as a function of time (in minutes). Suppose the elevator's acceleration is $a(t) = 2t - 6$.

(a) (4 pts.) Assume that the elevator is traveling upward at 8 meters per minute at time $t = 0$. What is its velocity 3 minutes later?

(b) (4 pts.) Assume that the elevator is 10 meters above ground level at $t = 0$. What is its height 3 minutes later?

(c) (5 pts.) What is the total distance travelled by the elevator during the time interval $0 \leq t \leq 3$?

18. Find the following indefinite integrals:

    (a) (7 pts.) $\displaystyle\int \left(5x^{2/3} + 3\sec x \tan x - \frac{4}{1+x^2}\right) dx$

    (b) (7 pts.) $\displaystyle\int \frac{3x - 5x^4}{x^3}\, dx$

    (c) (7 pts.) $\displaystyle\int \frac{\cos x}{\sin x - 7}\, dx$

19. Evaluate the following definite integrals **(simplify your answers)**:

    (a) (5 pts.) $\displaystyle\int_0^{\pi/3} \sin x \, dx$

    (b) (9 pts.) $\displaystyle\int_0^{2} x^2\sqrt{1+x^3}\, dx$

**Math 120 Final Exam, Fall 2014, with Solutions**

1. (10 pts.) Mark each statement as either true [T] or false [F].

   a. If $\lim_{x\to 5} f(x) = 0$ and $\lim_{x\to 5} g(x) = 0$, then $\lim_{x\to 5} \dfrac{f(x)}{g(x)}$ does not exist. ........................ F

   b. If $p$ is a polynomial, then $\lim_{x\to b} p(x) = p(b)$. ........................................ T

   c. If $f$ is continuous at $c$, then $f$ is differentiable at $c$. .................................. F

   d. A function can have two different horizontal asymptotes. ........................... T

   e. If $f$ has a local minimum at $c$, then $f'(c) = 0$. ....................................... F

   f. If $f''(c) = 0$, then $f$ has an inflection point at $c$. ...................................... F

   g. If $f$ and $g$ are both continuous functions on the interval $[a, b]$, then $\int_a^b f(x)g(x)\,dx = \left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right)$. ........................................ F

   h. If $\dfrac{d}{dx}(f(x)) = \dfrac{d}{dx}(g(x))$, then $f(x) = g(x)$. ........................................ F

   i. If $f$ is differentiable and $f(x) > 0$, then $\dfrac{d}{dx}\sqrt{f(x)} = \dfrac{f'(x)}{2\sqrt{f(x)}}$. ........................ T

   j. If $f'(x)$ exists and is nonzero for all $x$, then $f(1) \neq f(0)$. ............................ T

2. (18 pts.) An engineer needs to build a rectangular box that has a square base and no top with an outside surface area of 1200 square feet. What are the dimensions of the box meeting these restrictions that has the *greatest* possible volume? (For full credit, you must indicate what function you are maximizing and on what interval.)

**Solution:** Calling the base width $x$ and the height $y$, the volume is $V = x^2y$ and the surface area is $S = x^2 + 4xy$.

$S = 1200$ so $1200 = x^2 + 4xy$ and solving for $y$ gives $y = (1200 - x^2)/(4x)$.

Substituting into the volume formula to eliminate $y$ and produce a function of one variable, $V(x) = x^2(1200 - x^2)/(4x) = (1200x - x^3)/4$

The physical constraints are $x \geq 0$¡ $y \geq 0$, $y = (1200 - x^2)/(4x) > 0$ and thus $x \leq 20\sqrt{3}$.

Differentiating $V(x)$ gives $V'(x) = (1200 - 3x^2)/4$ with only one critical number in the interval: $x = 20$.

The values of the function at the endpoints and at the critical point inside the interval are $V(0) = 0$, $V(20) = 4000$ and $V(20\sqrt{3}) = 0$, so the maximum volume is 4000, for base width $x = 20$ feet and thus height $y = 10$ feet.

3. Find $\dfrac{dy}{dx}$ for each of the following. (You don't need to simplify your answers.)

(a) (4 pts.) $y = e^{3x} + \sin x + \pi^{2014}$

**Solution:** $y' = 3e^{3x} + \cos x$,

(b) (5 pts.) $y = \dfrac{10}{\sqrt[3]{x}} + 2^x - \arctan x$

**Solution:** $y' = 10\left(-\dfrac{1}{3}x^{-4/3}\right) + 2^x(\ln 2) - \dfrac{1}{1+x^2}$

(c) (6 pts.) $y = \dfrac{1 + 2\sec x}{e^x - 4\tan x}$

**Solution:** $y' = \dfrac{(e^x - 4\tan x)(2\sec x \tan x) - (1 + 2\sec x)(e^x - 4\sec^2 x)}{(e^x - 4\tan x)^2}$

(d) (4 pts.) $y = (1 + \sin x)\sqrt{1 + 4\cos x}$

**Solution:** $y' = \cos x\sqrt{1 + 4\cos x} + (1 + \sin x) \cdot \dfrac{1}{2\sqrt{1 + 4\cos x}}(-4\sin x)$

(e) (3 pts.) $y = (\ln(9x^2 + 5x))^{10}$

**Solution:** $y' = 10\ln(9x^2 + 5x)^9 \cdot \dfrac{1}{9x^2 + 5x} \cdot (18x + 5)$

4. (6 pts.) Find the equation of the tangent line to the graph $y = x^3 + 2x^2 + 5x + 7$ at the point where $x = -1$.

**Solution:** $y'(x) = 3x^2 + 4x + 5$ , $y'(-1) = 3 - 4 + 5 = 4$ , $y(-1) = -1 + 2 - 5 + 7 = 3$
tangent line equation is $y - 3 = 4(x + 1)$ or $y = 4(x + 1) + 3 = 4x + 7$

5. (7 pts.) Find $\frac{dy}{dx}$ along the curve defined by $x^4 + 2xy - 3\sin y = 5$.

**Solution:** Implicit diff. gives $4x^3 + 2(y + x\frac{dy}{dx}) - 3\cos y\frac{dy}{dx} = 0$.

Gathering terms on each side: $(2x - 3\cos y)\frac{dy}{dx} = -4x^3 - 2y$

Solving for the derivative: $\frac{dy}{dx} = \frac{-4x^3 - 2y}{2x - 3\cos y}$

6. (8 pts.) (a) Find the linearization $L(x)$ of $f(x) = \sqrt{1+2x}$ at $a = 12$.

**Solution:** $f'(x) = \frac{2}{2\sqrt{1+2x}} = \frac{1}{\sqrt{1+2x}}$

$f'(12) = \frac{1}{\sqrt{1+2\cdot 12}} = \frac{1}{5}$ ; $f(12) = \sqrt{1+2\cdot 12} = 5$

$y - 5 = \frac{1}{5}(x - 12)$

$L(x) = y = \frac{x}{5} + \frac{13}{5}$

(b) Use your answer to part (a) to approximate $\sqrt{23}$.

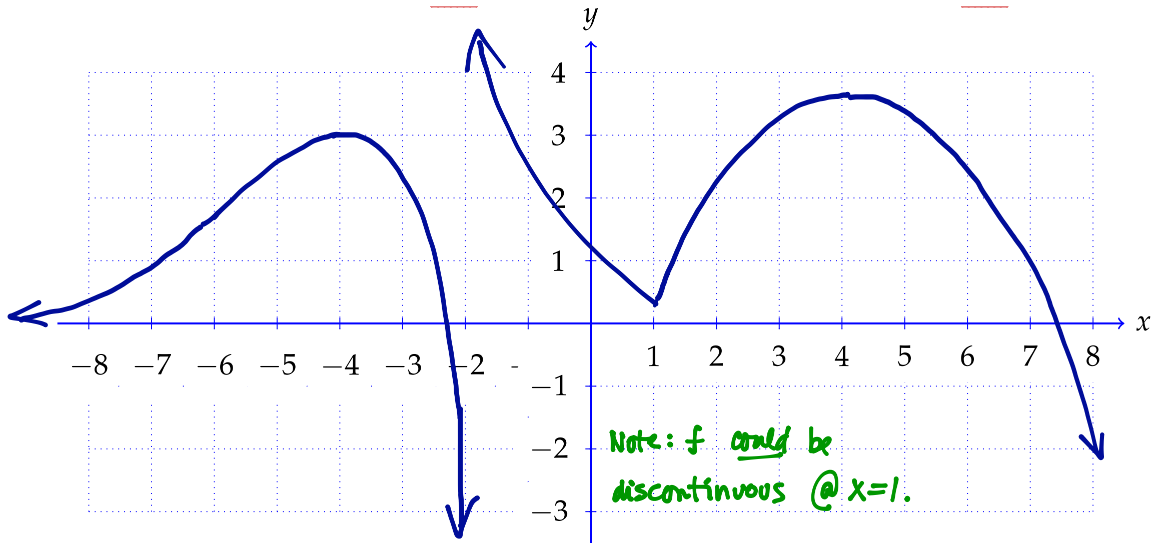
**Solution:** $\sqrt{23} \approx \frac{1}{5}(11) + \frac{13}{5} = \frac{24}{5}$

7. (14 pts.) Using the axes below, sketch the graph of a function $f(x)$ with *all* the following properties:

- $\lim_{x\to -2^-} f(x) = -\infty$ and $\lim_{x\to -2^+} f(x) = +\infty$.
- $\lim_{x\to +\infty} f(x) = -\infty$ and $\lim_{x\to -\infty} f(x) = 0$.
- $f'(1)$ does not exist.
- $f'(x) > 0$ on $(-\infty, -4)$ and $(1, 4)$, while $f'(x) < 0$ on $(-4, -2)$, $(-2, 1)$ and $(4, \infty)$.
- $f''(x) > 0$ on $(-\infty, -6)$ and $(-2, 1)$, while $f''(x) < 0$ on $(-6, -2)$ and $(1, \infty)$.

**Solution:**

- $\lim_{x\to -2^-} f(x) = -\infty$ and $\lim_{x\to -2^+} f(x) = +\infty$.
- $\lim_{x\to +\infty} f(x) = -\infty$ and $\lim_{x\to -\infty} f(x) = 0$.
- $f'(1)$ does not exist.
- $f'(x) > 0$ on $(-\infty, -4)$ and $(1, 4)$, while $f'(x) < 0$ on $(-4, -2)$, $(-2, 1)$ and $(4, \infty)$.
- $f''(x) > 0$ on $(-\infty, -6)$ and $(-2, 1)$, while $f''(x) < 0$ on $(-6, -2)$ and $(1, \infty)$.



8. Using the following table of values for $f, g, f'$ and $g'$, answer the questions below.

| $x$ | $f(x)$ | $g(x)$ | $f'(x)$ | $g'(x)$ |
|---|---|---|---|---|
| $-4$ | $-1$ | $9$ | $3$ | $10$ |
| $3$ | $-4$ | $7$ | $5$ | $9$ |
| $7$ | $5$ | $-1/3$ | $10$ | $2$ |

(a) (2 pts.) If $F(x) = f(x)g(x)$, find $F'(3)$.

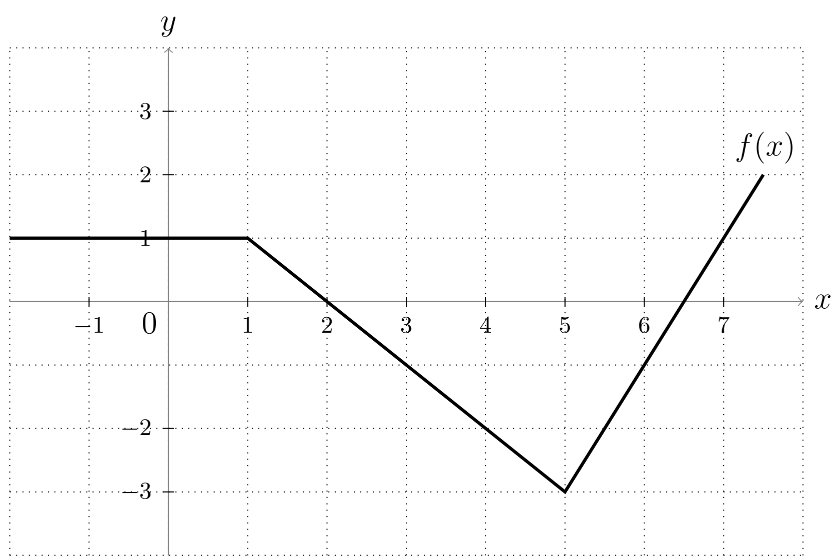
**Solution:** $F'(x) = f'(x)g(x) + f(x)g'(x)$, so $F'(3) = f'(3)g(3) + f(3)g'(3) = 5 \cdot 7 + (-4) \cdot 9 = -1$.

(b) (3 pts.) If $G(x) = f(g(x))$, find $G'(3)$.

**Solution:** $G'(x) = f'(g(x))g'(x)$, so $G'(3) = f'(g(3))g'(3) = f'(7) \cdot 9 = 10 \cdot 9 = 90$.

9. (5 pts.) Given that the function $f(x)$ is as in the graph below, and that $g(x) = \int_0^x f(t)\,dx$, determine the following five values:

$g(2) =$ 1.5 $g(5) =$ $-3$ $g(-1) =$ $-1$ $g'(4) =$ $-2$ $g''(6) =$ 2




10. Suppose the *derivative* of $f(x)$ is $f'(x) = \dfrac{x^2-3}{x^3}$ and the domain of $f$ is $x \neq 0$.

(a) (3 pts.) Find all critical numbers for $f$.

**Solution:** The derivative $f'$ is defined on the whole domain, so critical points can only occur where it vanishes, which is where its numerator $x^2 - 3$ vanishes: the critical numbers are $\pm\sqrt{3}$.

(b) (4 pts.) Find all intervals where $f$ is increasing, and all intervals where $f$ is decreasing.

**Solution:** $f'$ can change sign only at the critical numbers and at $x = 0$ where it is not defined. The sign chart

| $x$ | $(-\infty, -\sqrt{3})$ | $(-\sqrt{3}, 0)$ | $(0, \sqrt{3})$ | $(\sqrt{3}, \infty)$ |
|---|---|---|---|---|
| $x + \sqrt{3}$ | $-$ | $+$ | $+$ | $+$ |
| $x$ | $-$ | $-$ | $+$ | $+$ |
| $x - \sqrt{3}$ | $-$ | $-$ | $-$ | $+$ |
| $f'(x)$ | $-$ | $+$ | $-$ | $+$ |

shows that $f$ is increasing on the intervals $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$;
decreasing on the intervals $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$.

(c) (2 pts.) For each critical number, determine if it gives a local maximum, local minimum, or neither.

**Solution:** The increasing/decreasing behavior seen above shows (by the first derivative test) that there are local minima of $f$ at each of $x = -\sqrt{3}$ and $x = \sqrt{3}$ .

(d) (6 pts.) Find all intervals where $f$ is concave up, and all intervals where $f$ is concave down.

**Solution:**

$$f''(x) = \frac{d}{dx}\left(\frac{x^2 - 3}{x^3}\right) = \frac{d}{dx}\left(x^{-1} - 3x^{-3}\right) = -x^{-2} + 9x^{-4} = x^{-4}(-x^2 + 9) = -x^{-4}(x - 3)(x + 3)$$

This factorizes as $f''(x) = -x^{-4}(x - 3)(x + 3)$, vanishes only at the roots $\pm 3$, and can only change sign at those roots, so the sign table is

| $x$ | $(-\infty, -3)$ | $(-3, 0) \cup (0, 3)$ | $(3, \infty)$ |
|---|---|---|---|
| $x + 3$ | $-$ | $+$ | $+$ |
| $x - 3$ | $-$ | $-$ | $+$ |
| $-x^{-4}$ | $-$ | $-$ | $-$ |
| $f''(x)$ | $-$ | $+$ | $-$ |

Thus, $f$ is concave up on the two intervals $(-3, 0)$ and $(0, 3)$
and concave down on the two intervals $(-\infty, -3)$ and $(3, \infty)$.

(e) (1 pt.) Find the $x$-coordinates of any inflection points.

**Solution:** The concavity changes at each of the points $x = \pm 3$ where $f''(x) = 0$ and nowhere else, so those are the inflection points.

11. Find the following limits, find the infinite limit, or state that the limit does not exist.

(a) (4 pts.) $\lim_{x \to 6} \frac{x^2 - 36}{x^2 - 6x}$

**Solution:** $\lim_{x\to 6} \frac{x^2-36}{x^2-6x} = \lim_{x\to 6} \frac{(x-6)(x+6)}{x(x-6)} = \lim_{x\to 6} \frac{(x+6)}{x} = \frac{6+6}{6} = \frac{12}{6} = 2$

(b) (4 pts.) $\lim_{x\to -3^+} \frac{x^2+2x}{x^2+x-6}$

**Solution:**

Version 1: $\lim_{x\to -3^+} \frac{x^2+2x}{x^2+x-6} = \lim_{\epsilon\to 0^+} \frac{(-3+\epsilon)^2+2(-3+\epsilon)}{(-3+\epsilon)^2+(-3+\epsilon)-6} = \lim_{\epsilon\to 0^+} \frac{3-4\epsilon+\epsilon^2}{-5\epsilon+\epsilon^2} = -\infty$

Version 2: $\lim_{x\to -3^+} \frac{x^2+2x}{x^2+x-6} = \lim_{x\to -3^+} \frac{x^2+2x}{(x+3)(x-2)} = \lim_{x\to -3^+} \frac{\text{positive}}{(\text{going to } 0^+)(\text{negative})} = -\infty$

(c) (5 pts.) $\lim_{x\to 0} \frac{x^2}{1-\cos(3x)}$

**Solution:** $\lim_{x\to 0} \frac{x^2}{1-\cos(3x)} \overset{\hat{H}}{=} \lim_{x\to 0} \frac{2x}{3\sin(3x)} \overset{\hat{H}}{=} \lim_{x\to 0} \frac{2}{3\cdot 3\cos(3x)} = \frac{2}{9\cos(0)} = \frac{2}{9}$

(d) (7 pts.) $\lim_{x\to -\infty} \frac{2x+1}{\sqrt{x^2+1}}$

**Solution:** $= \lim_{x\to -\infty} \frac{-\sqrt{(2x+1)^2}}{\sqrt{x^2+1}} = \lim_{x\to -\infty} -\sqrt{\frac{4x^2+4x+1}{x^2+1}}$

$= \lim_{x\to -\infty} -\sqrt{\frac{4+\frac{4}{x}+\frac{1}{x^2}}{1+\frac{1}{x^2}}} = -\sqrt{\frac{4+0+0}{1+0}} = -2$

Alternate route: $= \lim_{x\to -\infty} \frac{2x+1}{\sqrt{x^2}\sqrt{1+\frac{1}{x^2}}} = \lim_{x\to -\infty} \frac{2x+1}{-x\sqrt{1+\frac{1}{x^2}}}$

$= \lim_{x\to -\infty} -\frac{2+\frac{1}{x}}{\sqrt{1+\frac{1}{x^2}}} = -\frac{2+0}{\sqrt{1+0}} = -2$

(e) (5 pts.) $\lim_{x\to +\infty} x\tan\left(\frac{5}{x}\right)$

**Solution:**

$$\lim_{x\to +\infty} x\tan\left(\frac{5}{x}\right) = \lim_{x\to +\infty} \frac{\tan\left(\frac{5}{x}\right)}{1/x} \overset{\hat{H}}{=} \lim_{x\to +\infty} \frac{\sec^2\left(\frac{5}{x}\right)(-5/x^2)}{-1/x^2}$$

$$= 5\lim_{x\to +\infty} \sec^2\left(\frac{5}{x}\right) = 5\sec^2(0) = 5\cdot 1 = 5$$

12. (7 pts.) Suppose that $f(0) = -1$ and $f'(x) \leq 3$ for all $x$. How large can $f(4)$ possibly be?

**Solution:** Using the MVT with $a = 0$, $b = 4$ $\dfrac{f(b) - f(a)}{b - a} = f'(c) \leq 3,$ so $f(b) - f(a) \leq 3(b - a)$, and $a = 0$, $b = 4$ gives $f(4) - f(0) \leq 3(4 - 0)$.
Thus $f(4) \leq f(0) + 12,$, and as $f(0) = -1$, $f(4) \leq -1 + 12 \leq 11$ : the largest that $f(4)$ can be is 11.

13. (7 pts.) Use the definition of derivative to find $f'(x)$ if $f(x) = \dfrac{1}{\sqrt{x}}$.
(Use only algebraic techniques to evaluate the limit; do not use L'Hôpital's Rule.)

**Solution:**

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{1}{h} \left( \frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}} \right) \\
&= \lim_{h \to 0} \frac{1}{h} \left( \frac{\sqrt{x}}{\sqrt{x}\sqrt{x+h}} - \frac{\sqrt{x+h}}{\sqrt{x}\sqrt{x+h}} \right) = \lim_{h \to 0} \frac{1}{h} \left( \frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x}\sqrt{x+h}} \right) \\
&= \lim_{h \to 0} \frac{1}{h} \left( \frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x}\sqrt{x+h}} \cdot \frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}} \right) \\
&= \lim_{h \to 0} \frac{1}{h} \left( \frac{x - (x+h)}{\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})} \right) = \lim_{h \to 0} \frac{1}{h} \left( \frac{-h}{\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})} \right) \\
&= \lim_{h \to 0} \left( \frac{-1}{\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})} \right) \\
&= \frac{-1}{\sqrt{x}\sqrt{x}(\sqrt{x} + \sqrt{x})} = \frac{1}{2x^{3/2}}
\end{aligned}$$

14. (9 pts.) Find the absolute maximum and minimum values for $f(x) = \cos x - \sin x$ on the interval $[0, \pi]$.

**Solution:** $f'(x) = -\sin x - \cos x$

$f'(x) = 0 \Rightarrow x = 3\pi/4$

$f(0) = \cos(0) - \sin(0) = 1 - 0 = 1$

$f(1) = \cos(\pi) - \sin(\pi) = -1 - 0 = -1$

$f(3\pi/4) = \cos(3\pi/4) - \sin(3\pi/4) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = -\sqrt{2}$

Maximum is 1; minimum is $-\sqrt{2}$

15. (8 pts.) Suppose we know that $\int_0^1 f(x)\,dx = 5$ and $\int_0^4 f(x)\,dx = 2$.

Use this information to evaluate the following:

(a) $\int_1^4 f(x)\,dx$ = -3

(b) $\int_0^4 (2x - 3f(x))\,dx$ = 10

16. (5 pts.) Use a Riemann sum to approximate $\int_2^{10} \ln x\,dx$. Use $n = 4$ equal-length subintervals and take the sample points to be the right-hand endpoints. (Do not simplify your answer.)

**Solution:**
Evaluate $\ln x$ ...at $x = 4, 6, 8, 10$, ...add these, ...and multiply by 2.
$R_4 = 2\,(\ln 4 + \ln 6 + \ln 8 + \ln(10))$.

17. An elevator is moving up and down in a vertical shaft, and we measure its height (in meters) above ground level as a function of time (in minutes). Suppose the elevator's acceleration is $a(t) = 2t - 6$.

(a) (4 pts.) Assume that the elevator is traveling upward at 8 meters per minute at time $t = 0$. What is its velocity 3 minutes later?

**Solution:** $v = t^2 - 6t + C$ $v(0) = 8$ gives $C = 8$, so $v = t^2 - 6t + 8$ At $t = 3$, $v = 3^3 - 6 \cdot 3 + 8 = -1$

(b) (4 pts.) Assume that the elevator is 10 meters above ground level at $t = 0$. What is its height 3 minutes later?

**Solution:** $s = \frac{1}{3}t^3 - 3t^2 + 8t + C$ $s(0) = 10$ gives $C = 10$ and so $s = \frac{1}{3}t^3 - 3t^2 + 8t + 10$ At $t = 3$, $s = \frac{1}{3}3^3 - 3 \cdot 3^2 + 8 \cdot 3 + 10 = 16$

(c) (5 pts.) What is the total distance travelled by the elevator during the time interval $0 \le t \le 3$?

**Solution:** $v = 0$ at $t = 2$ (Also at $t = 4$, but unnecessary.)
Distance = $|s(0) - s(2)| + |s(2) - s(3)| = |10 - 16\frac{2}{3}| + |16\frac{2}{3} - 16| = 6\frac{2}{3} + \frac{2}{3} = 7\frac{1}{3}$ or $\frac{22}{3}$.

18. Find the following indefinite integrals:

(a) (7 pts.) $\int \left(5x^{2/3} + 3\sec x \tan x - \frac{4}{1+x^2}\right) dx$

**Solution:** $\int \left(5x^{2/3} + 3\sec x \tan x - \frac{4}{1+x^2}\right) dx = 3 \quad x^{5/3} \;+\; 3\sec x \;-\; 4\tan^{-1} x \;+\; C$

(b) (7 pts.) $\displaystyle\int \frac{3x-5x^4}{x^3}\,dx$

**Solution:** $\displaystyle\int \frac{3x-5x^4}{x^3}\,dx = \int \left(3x^{-2}-5x\right)\,dx = -3\ x^{-1} - \frac{5}{2}\ x^2 + C$

(c) (7 pts.) $\displaystyle\int \frac{\cos x}{\sin x - 7}\,dx$

**Solution:** $u = \sin(x) - 7$ , $du = \cos(x)dx$

integral $= \displaystyle\int \frac{1}{u}\,du = \ln|u| + C = \ln|\sin(x) - 7| + C$

19. Evaluate the following definite integrals **(simplify your answers)**:

**Solution:** $\displaystyle\int_1^4 \left(\frac{1}{\sqrt{x}} + 2x\right)dx = \left(2\sqrt{x} + x^2\right)\Big|_1^4 = (2\sqrt{4} + 4^2) - (2+1) = 17$

(a) (5 pts.) $\displaystyle\int_0^{\pi/3} \sin x\,dx$

**Solution:** $\displaystyle\int_0^{\pi/3} \sin x\,dx = -\cos x\Big|_0^{\pi/3} = -(\cos\pi/3 - \cos 0) = -\left(\frac{1}{2} - 1\right) = \frac{1}{2}$

(b) (9 pts.) $\displaystyle\int_0^2 x^2\sqrt{1+x^3}\,dx$

**Solution:** $u = 1 + x^3,\ \ du = 3x^2dx$

$$\int_0^2 x^2\sqrt{1+x^3}\,dx = \int_1^9 \frac{1}{3}\sqrt{u}\,du = \frac{2}{9}\ u^{3/2}\Big|_1^9 = \frac{2}{9}\left(9^{3/2}-1\right) = \frac{2}{9}\,26 = \frac{52}{9}$$

Alternate solution:

$u = 1 + x^3,\ \ du = 3x^2dx$

$$\int x^2\sqrt{1+x^3}\,dx = \int \frac{1}{3}\sqrt{u}\,du = \frac{2}{9}\ u^{3/2} + C = \frac{2}{9}\left(1+x^3\right)^{3/2} + C$$

$$\int_0^2 x^2\sqrt{1+x^3}\,dx = \frac{2}{9}\left(1+x^3\right)^{3/2}\Big|_0^2 = \frac{2}{9}\left(9^{3/2}-1\right) = \frac{2}{9}\,26 = \frac{52}{9}$$