TWISTED VERTEX ALGEBRAS, BICHARACTER CONSTRUCTION AND BOSON-FERMION CORRESPONDENCES

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Abstract. The boson-fermion correspondences are an important phenomena on the intersection of several areas in mathematical physics: representation theory, vertex algebras and conformal field theory, integrable systems, number theory, cohomology. Two such correspondences are well known: the types A and B (and their super extensions). As a main result of this paper we present a new boson-fermion correspondence, of type D-A. Further, we define the new concept of twisted vertex algebra of order N, which generalizes super vertex algebras. We develop the bicharacter construction which we use for constructing classes of examples of twisted vertex algebras, as well as for deriving formulas for the operator product expansions (OPEs), analytic continuations and normal ordered products. By using the underlying Hopf algebra structure we prove general bicharacter formulas for the vacuum expectation values for two important groups of examples. We show that the correspondences of type B, C and D-A are isomorphisms of twisted vertex algebras.

1. Introduction

1.1. **Motivation.** In 2 dimensions the bosons and fermions are related by the boson-fermion correspondences. The best known boson-fermion correspondence is that of type A (see e.g. [18], [8], [19]). There are many properties and applications of this correspondence, often called the charged free fermion-boson correspondence, and an exposition of some of them is given in [19], [30]. [8] and [18] discovered its connection to the theory of integrable systems, namely to the KP and KdV hierarchies, to the theory of symmetric polynomials and representation theory of infinite dimensional Lie algebras, namely the a_∞ algebra, (whence the name "type A" derives), as well as to the $\hat{sl}_n$ and other affine Lie algebras. As this boson-fermion correspondence has so many applications and connections to various mathematical areas (including number theory and geometry, as well as random matrix theory and random processes), a deeper understanding of the types of mathematical structures that were being equated by it was sought. A partial answer early on was given by Igor Frenkel in [18], but the complete answer had to wait for the development of the theory of vertex algebras. Vertex operators were introduced in string theory and now play an important role in many areas such as quantum field theory, integrable models, representation theory, random matrix theory, statistical physics, and many others. The theory of super vertex algebras axiomatizes the properties of some, simplest, "algebras" of vertex operators (see for instance [5], [16], [15], [21], [26], [14]). Thus, as an application of the theory of super vertex algebras the question "what is the boson-fermion correspondence of type A?" can be answered precisely as follows: the boson-fermion correspondence of type A is an isomorphism of two super vertex algebras ([21]).

There are other examples of boson-fermion correspondences, for instance the boson-fermion correspondence of type B (e.g. [10], [37], [33]), the super extensions of the boson-fermion correspondences of type A and B ([20], [21]), and others ([9], [32]). The first of those, the type B, was introduced in [10], who discovered it in connection to the theory of integrable systems, namely to the BKP hierarchy, and to the representation theory of the b_∞ algebra (whence the name "type B" derives). Connections to the theory of symmetric polynomials and the symmetric group are shown in [37]. There is also ongoing work on other boson-fermion and boson-boson correspondences, e.g. the CKP correspondence (see [9], [32]).

Our motivation thus was twofold: first, we considered the questions "Are all boson-fermion correspondences isomorphisms of super vertex algebras? And if not, what are the boson-fermion correspondences, i.e., what mathematical structures do they equate?". And second, we considered the question "Are there other boson-fermion correspondences?".

The first part of the first question is readily answered by the fact that in all other cases except the type A, the boson-fermion correspondences cannot be isomorphisms between two super vertex algebras, since the associated operator product expansions have singularities at both $z = w$ and $z = -w$. Hence, in order to answer the question, "what mathematical structures do these correspondences equate", we introduce the concept of a twisted vertex algebra of order N. It generalizes the concept of a super vertex algebra in that a super vertex algebra can be considered a twisted vertex algebra of order 1. The boson-fermion correspondence of type B is then an example of an isomorphism between two twisted vertex algebras of order 2 ([1]). We show that there are other examples of isomorphisms of twisted vertex algebras in the literature: the correspondence of type CKP, while strictly speaking not a boson-fermion correspondence since the generating operator product expansions are bosonic on both sides, is another such example, as is the super extension of the boson-fermion correspondence of type B.

As one of the main results of this paper, and as an answer to the second question posed, we introduce the new example of the boson-fermion correspondence of type D-A. This boson-fermion correspondence of type D-A completes the bosonisation of the 4 double-infinite rank Lie algebras (a_∞, b_∞, c_∞, d_∞). Although the super vertex algebra of the neutral fermion of type D is well known (see [22], [21], [24]) as it gives the basic representation of d_∞), this neutral fermion super vertex algebra does not itself have a bosonic equivalent, but as we show in this paper its twisted double cover does. We further show that it is another example of an isomorphism of twisted vertex algebras. Hence all the correspondences of type B, C and D-A are shown to be isomorphisms of twisted vertex algebras.

1.2. **Overview of the paper.** After detailing the notation and some background that we will use throughout the paper, we proceed with the definition of a twisted vertex algebra (Section 3). Twisted vertex algebras generalize super vertex algebras in that they have finitely many points of locality at $z = \epsilon^i w$, where ϵ is a primitive N-th root of unity. As such, twisted vertex algebras are similar to the Γ-vertex algebras of [29], but are more general as they were designed to incorporate normal ordered products and operator product expansion coefficients as descendant fields (for a more detailed discussion on the comparison see [2]). A main difference between twisted vertex algebras and super vertex algebras (and Γ-vertex algebras of [29]) is that in most of the examples of twisted vertex algebras the space of fields strictly contains the space of states. In fact a projection map from the space of fields to the space of states is part of the definition (see Definition 3.1). In that property twisted vertex algebras more closely resemble the deformed chiral algebras defined in [13]. We do not discuss many properties of twisted vertex algebras in this paper due to length, instead see the follow-up paper [2]. In Section 3 we continue with the description of the examples of two pairs of twisted vertex algebras which constitute two examples of boson-fermion correspondences: the boson-fermion correspondence of type B (reformulated from [10] and [37] in the language of vertex algebras) and the new boson-fermion correspondence of type D-A. We list only one property out of many for each correspondence, a representative identity which is a direct corollary of the equality between the vacuum expectation values of the two sides of the correspondence. For the correspondence of type B, this equality is actually the Schur Pfaffian identity (Lemma 3.12), and for the type D-A it is a different Pfaffian identity (Lemma 3.18). As the definition of a twisted vertex algebra is new and technical, its necessity can only be justified by some very meaningful examples of twisted vertex algebras—which indeed are the boson-fermion correspondences. Since the proofs of the statements in the narrative of the examples are lengthy enough to impede the overview, we give the proofs in the latter part of the paper, after we have introduced the requisite technical set of tools, namely the bicharacter construction. The bicharacter construction was first introduced in the context of vertex algebras in [6], the super bicharacters were introduced in [3]. The bicharacter construction is used in this paper in two ways. First, to give a general construction and description of variety of examples of twisted vertex algebras (see Theorem 4.39). We prove formulas for the analytic continuations, operator product expansions, vacuum expectation values using the bicharacters. (Such explicit formulas, for example the analytic continuations formulas, are very hard if not impossible to obtain otherwise even for specific examples.) Secondly, we use the bicharacter construction extensively to prove the properties of the new boson-fermion correspondence of type D-A (see Sections 5.4 and 5.8).

There is a major difference between the bicharacter description of a vertex algebra and the operator-based description typically used: In the operator-based description the examples are presented in terms of generating fields (vertex operators) and their OPEs (or commutation relations). With the bicharacter construction one starts instead with a (supercommutative super-cocommutative) Hopf algebra M and its free Leibnitz module (for the definition and examples of free Leibnitz module see sections 4.1 and 4.2). The commutation relations then result from the choice of a bicharacter r— different bicharacter r will dictate different commutation relations. Moreover, for each Hopf algebra M there are many choices of a bicharacter r, hence each such pair (M, r) will give rise to a different twisted vertex algebra, even if the underlying spaces of states are the same as Hopf algebras (see Theorem 4.39). It is in fact the field-state correspondence Y that changes with each choice of a bicharacter, and hence the fields in the vertex algebra (see Definition 4.20). This is the case for instance for the fermionic sides of both the B and D-A boson-fermion correspondences: they have identical spaces of states as Hopf algebras (which is not altogether surprising as they are both neutral fermions), but the collections of fields describing them differ substantially. Therefore in the bicharacter construction examples are grouped based on the underlying Hopf algebra M—one starts by keeping M the same, but changing the bicharacter r (Section 5). We want to stress that there is a variety of examples even after we fix the algebra M. This description based on the Hopf algebra M is further used to prove general formulas for the vacuum expectation values based on the Hopf algebra M (sections 5.2 and 5.6). The underlying Hopf structure explains why the vacuum expectation values for both the B and the D fermion are Pfaffians, although, of course, different ones. In the Section 5 the particular examples of the fermionic and bosonic examples are detailed and their properties are proved, including the statements from Sections 3.2 and 3.3.

2. Notation and background

In this section we list notations we will continuously use throughout the paper. We work over the field of complex numbers $\mathbb{C}$, and with the category of super vector spaces ($\mathbb{Z}_2$ graded vector spaces). The flip map τ is defined by

$$\tau(a\otimes b)=(-1)^{\tilde{a}\cdot\tilde{b}}(b\otimes a) \tag{2.1}$$

for any homogeneous elements a, b in the super vector space, where $\tilde{a}$, $\tilde{b}$ denote correspondingly the parity of a, b.

A superbialgebra H is a superalgebra, with compatible coalgebra structure (the coproduct and counit are algebra maps). Denote the coproduct and the counit by $\triangle$ and η. A Hopf superalgebra is a superbialgebra with an antipode S. For a superbialgebra H we will write $\triangle(a)=\sum\ a'\otimes a''$ for the coproduct of $a\in H$ (Sweedler's notation). That means we will usually omit the indexing in $\triangle(a)=\sum_k\ a'_k\otimes a''_k$, especially when it is clear from the context. Recall in a super Hopf algebra the product on $H\otimes H$ is defined by

$$(a\otimes b)(c\otimes d)=(-1)^{\tilde{b}\cdot\tilde{c}}(ac\otimes bd) \tag{2.2}$$

for any a, b, c, d homogeneous elements in H. A supercocommutative bialgebra is a superbialgebra with

$$\tau(\triangle(a))=\triangle(a).$$

A primitive element $a\in H$, where H is a Hopf super algebra, is such that $\triangle(a)=a\otimes 1+1\otimes a$, $\eta(a)=0$, and $S(a)=-a$. A grouplike element $g\in H$ is such that $\triangle(g)=g\otimes g,\ \eta(g)=1,\ S(g)=g^{-1}$.

Notation 2.1. *For any $a\in A$, where A is a commutative $\mathbb{C}$ algebra denote $a^{(n)}:=\frac{a^n}{n!}$.*

Definition 2.2. (The Hopf algebra $H_D=\mathbb{C}[D]$)
The Hopf algebra $H_D=\mathbb{C}[D]$ is the polynomial algebra with a primitive generator D. We have

$$\triangle D^{(n)}=\sum_{k+l=n}D^{(k)}\otimes D^{(l)}. \tag{2.3}$$

Definition 2.3. (The Hopf algebra $H^N_{T_\epsilon}$)
Let ϵ be a primitive root of unity of order N. The Hopf algebra $H^N_{T_\epsilon}$ is the Hopf algebra with a primitive generator D and a grouplike generator T_ϵ subject to the relations:

$$DT_\epsilon=\epsilon T_\epsilon D,\quad \textit{and } (T_\epsilon)^n=1 \tag{2.4}$$

$H^N_{T_\epsilon}$ has H_D as a Hopf subalgebra. Both H_D and $H^N_{T_\epsilon}$ are entirely even.

Notation 2.4. (The function spaces $\mathbf{F}_{\epsilon}^{N}(z,w)$, $\mathbf{F}_{\epsilon}^{N}(z,w)^{+,w}$) *Let ϵ be a primitive root of unity of order N. Denote by $\mathbf{F}_{\epsilon}^{N}(z,w)$ the space of rational functions in the formal variables z,w with only poles at $z=0, w=0$, $z=\epsilon^i w$, $i=1,\dots,N$:*

$$\mathbf{F}_{\epsilon}^{N}(z,w)=\mathbb{C}[z,w][z^{-1},w^{-1},(z-w)^{-1},(z-\epsilon w)^{-1},\cdots,(z-\epsilon^{N-1}w)^{-1}]. \quad (2.5)$$

Also, denote $\mathbf{F}_{\epsilon}^{N}(z,w)^{+,w}$ the space of rational functions in the formal variables z,w with only poles at $z=0$, $z=\epsilon^i w$, $i=1,\dots,N$:

$$\mathbf{F}_{\epsilon}^{N}(z,w)^{+,w}=\mathbb{C}[z,w][z^{-1},(z-w)^{-1},(z-\epsilon w)^{-1},\cdots,(z-\epsilon^{N-1}w)^{-1}]. \quad (2.6)$$

Note that we do not allow a pole at $w=0$ in $\mathbf{F}_{\epsilon}^{N}(z,w)^{+,w}$, i.e., if $f(z,w)\in\mathbf{F}_{\epsilon}^{N}(z,w)$, then $f(z,0)$ is well defined. Similarly let $\mathbf{F}_{\epsilon}^{N}(z_1,z_2,\dots,z_l)$ be the space of rational functions in variables $z_1,z_2,\dots,z_l$ with only poles at $z_m=0$, $m=1,\dots,l$, or at $z_i=\epsilon^k z_j$, and $i\neq j=1,\dots,l$, $k=1,\dots,N$. Lastly $\mathbf{F}_{\epsilon}^{N}(z_1,z_2,\dots,z_l)^{+,z_l}$ is the space of rational functions in variables $z_1,z_2,\dots,z_l$ with only poles at $z_m=0$, $m=1,\dots,l-1$, or at $z_i=\epsilon^k z_j$, $i\neq j=1,\dots,l$, $k=1,\dots,N$. If N is clear from the context, or the property doesn't depend on the particular value of N, we will just write $\mathbf{F}_{\epsilon}(z,w)$ or $\mathbf{F}_{\epsilon}(z,w)^{+,w}$.

Fact 2.5. *$\mathbf{F}_{\epsilon}^{N}(z,w)$ is an $H_{T_\epsilon}^N\otimes H_{T_\epsilon}^N$ (and consequently an $H_D\otimes H_D$) Hopf module by*

$$D_z f(z,w)=\partial_z f(z,w),\quad (T_\epsilon)_z f(z,w)=f(\epsilon z,w) \quad (2.7)$$
$$D_w f(z,w)=\partial_w f(z,w),\quad (T_\epsilon)_w f(z,w)=f(z,\epsilon w) \quad (2.8)$$

We will denote the action of elements $h\otimes 1\in H_{T_\epsilon}^N\otimes H_{T_\epsilon}^N$ on $\mathbf{F}_{\epsilon}(z,w)$ by $h_z\cdot$, correspondingly $h_w\cdot$ will denote the action of the elements $1\otimes h\in H_{T_\epsilon}^N\otimes H_{T_\epsilon}^N$.

Notation 2.6. *For a rational function $f(z,w)$ we denote by $i_{z,w}f(z,w)$ the expansion of $f(z,w)$ in the region $|z|\gg|w|$, and correspondingly for $i_{w,z}f(z,w)$. Similarly, we will denote by $i_{z_1,z_2,\dots,z_n}$ the expansion in the region $|z_1|\gg\cdots\gg|z_n|$.*

The following definitions are well known, they (or similar) can be found for instance in [16], [15], [21], [26] and others.

Definition 2.7. (Field) *A field $a(z)$ on a vector space V is a series of the form*

$$a(z)=\sum_{n\in\mathbf{Z}}a_{(n)}z^{-n-1},\quad a_{(n)}\in\mathrm{End}(V),\ \ \textit{such that}\ \ a_{(n)}v=0\ \ \forall\, v\in V,\ n\gg 0. \quad (2.9)$$

Remark 2.8. The coefficients $a_{(n)}$, $n\in\mathbb{Z}$, are often called modes. The indexing above is typically used in super vertex algebras, as in a super vertex algebra it can be made uniform: if a field $a(z)$ with this indexing is a vertex operator in a vertex algebra, then the modes in front of negative powers of z annihilate the vacuum vector (hence are called annihilation operators), and the modes in front of the positive powers of z are the creation operators. Hence denote

$$a(z)_- := \sum_{n\geq 0} a_n z^{-n-1}, \quad a(z)_+ := \sum_{n<0} a_n z^{-n-1}. \tag{2.10}$$

Definition 2.9. (Normal ordered product) *Let $a(z), b(z)$ be fields on a vector space V. Define normal ordered product of these fields by*

$$: a(z)b(w) := a(z)_+ b(w) + (-1)^{\tilde{a}\tilde{b}} b(w) a_-(z). \tag{2.11}$$

Remark 2.10. Let $a(z), b(z) \in End(V)[[z^{\pm 1}]]$ be fields on a vector space V. Then $: a(z)b(\lambda z) :$ and $: a(\lambda z)b(z) :$ are well defined elements of $End(V)[[z^{\pm 1}]]$, and are also fields on V for any $\lambda \in \mathbb{C}^*$.

For boson-fermion correspondence of type A and its super extension, the question "what mathematical structures does the correspondence equate?" has a precise answer: these two correspondences are isomorphisms of super vertex algebras. The definition of a super vertex algebra is well known, we refer the reader for example to [16], [15], [21], [26], as well as for notations, details and theorems. Super vertex algebras have two important properties which we would like to carry over to the case of twisted vertex algebras. These are the properties of analytic continuation and completeness with respect to operator product expansions (OPEs). In fact our definition of a twisted vertex algebra is based on enforcing these two properties. Recall we have for the OPE of two fields (see e.g. [21])

$$a(z)b(w) = \sum_{j=0}^{N-1} i_{z,w} \frac{c^j(w)}{(z-w)^{j+1}} + : a(z)b(w) :, \tag{2.12}$$

where $: a(z)b(w) :$ is the normal ordered product of the two fields. Moreover, we have $Res_{(z-w)} a(z)b(w)(z-w)^j = c^j(w) = (a_{(j)}b)(w)$, i.e., the coefficients of the OPEs are vertex operators in the **same** super vertex algebra. Since for the commutation relations only the singular part of the OPE matters, we abbreviate the OPE above as:

$$a(z)b(w) \sim \sum_{j=0}^{N-1} \frac{c^j(w)}{(z-w)^{j+1}}. \tag{2.13}$$

Also, an analytic continuation property for a super vertex algebra holds: for any $a_i \in V, i = 1, \ldots, k$, there exist a rational vector valued function

$$X_{z_1,z_2,\ldots,z_k} : V^{\otimes k} \to W[[z_1, z_2, \ldots, z_k]] \otimes \mathbf{F}^1_{\epsilon}(z_1, z_2, \ldots, z_k)^{+,z_k},$$

such that

$$Y(a_1, z_1)Y(a_2, z_2)\dots Y(a_k, z_k)1 = i_{z_1,z_2,\dots,z_k} X_{z_1,z_2,\dots,z_k}(a_1 \otimes a_2 \otimes \dots \otimes a_k).$$

For many examples, super vertex algebras are generated by a much smaller number of generating fields (see e.g. [21]), with imposing the condition that the resulting space of fields of the vertex algebra has to be closed under certain operations: for any vertex operator $a(z)$ the field $Da(z) = \partial_z a(z)$ has to also be a vertex operator in the vertex algebra. Also, the OPEs coefficients ($c^j(w)$ from (2.13)) and normal ordered products $:a(z)b(z):$ of any two vertex operators $a(z)$ and $b(w)$ have to be vertex operators in the vertex algebra. Note that the identity operator on V is always a trivial vertex operator in the vertex algebra corresponding to the vacuum vector $|0\rangle \in V$.

3. Twisted vertex algebras: definition, overview, examples

3.1. Twisted vertex algebras: definition and overview.

Definition 3.1. (Twisted vertex algebra of order N) *Twisted vertex algebra of order N is a collection of the following data (V, W, π_f, Y):*

- *the space of fields V: a vector super space V, which is an $H_{T_\epsilon}^N$ module, graded as an H_D-module;*
- *the space of states W: a vector super space, $W \subset V$;*
- *a linear surjective projection map $\pi_f : V \to W$, such that $\pi_f|_W = Id_W$*
- *a field-state correspondence Y: a linear map from V to the space of fields on W;*
- *a vacuum vector: a vector $1 = |0\rangle \in W \subset V$.*

This data should satisfy the following set of axioms:

- *vacuum axiom: $Y(1, z) = Id_W$;*
- *modified creation axiom: $Y(a, z)|0\rangle|_{z=0} = \pi_f(a)$, for any $a \in V$;*
- *transfer of action: $Y(ha, z) = h_z \cdot Y(a, z)$ for any $h \in H_{T_\epsilon}^N$;*
- *analytic continuation: For any $a, b, c \in V$ exists $X_{z,w,0}(a \otimes b \otimes c) \in W[[z, w]] \otimes \mathbf{F}_\epsilon(z, w)$ such that*
$$Y(a, z)Y(b, w)\pi_f(c) = i_{z,w} X_{z,w,0}(a \otimes b \otimes c) \tag{3.1}$$
- *symmetry: $X_{z,w,0}(a \otimes b \otimes c) = X_{w,z,0}(\tau(a \otimes b) \otimes c)$;*
- *Completeness with respect to Operator Product Expansions (OPE's): For each $i \in 0, 1, \dots, N-1$, $k \in \mathbb{Z}$, any $a, b, c \in V$, a, b–homogeneous w.r.to the grading by D, exist $l_k \in \mathbb{Z}$ such that*
$$Res_{z=\epsilon^i w} X_{z,w,0}(a \otimes b \otimes c)(z - \epsilon^i w)^k = \sum_s^{finite} w^{l_{k,i}^s} Y(v_{k,i}^s, w)\pi_f(c) \tag{3.2}$$
for some homogeneous elements $v_{k,i}^s \in V$, $l_{k,i}^s \in \mathbb{Z}$.

Remark 3.2. If V is an (ordinary) super vertex algebra, then the data $(V, V, \pi_f = Id_V, Y)$ is a twisted vertex algebra of order 1.

The definition of a twisted vertex algebra is very similar to the definition of a deformed chiral algebra given by E. Frenkel and Reshetikhin in [13].

Remark 3.3. (**Shift restriction**) The axiom/property requiring completeness with respect to the Operator Product Expansions (OPEs) is a weaker one than in the classical vertex algebra case. We can express this weaker axiom as follows (for more details see [2]): the OPE coefficient, the residue $Res_{z=\epsilon^i w} X_{z,w,0}(a \otimes b \otimes c)(z-\epsilon^i w)^k$, is a field $v(w)$ which may not be a vertex operator in the vertex algebra. But its doubly infinite sequence of modes is a finite sum of sequences of modes of vertex operators from the algebra. Each of these sequences of modes from the finite sum may require a different shift, i.e., may need to be multiplied by a different power $w^{-l_{k,i}}$. As was mentioned in the previous section, in a super vertex algebra we have a stronger property, requiring that the OPE coefficients automatically, without need of a shift, be vertex operators in the same vertex algebra. This stronger property cannot hold in the interesting examples, which forced the modification of the OPE completeness axiom (see remark 3.7 below). The "modified completeness with respect to OPE's" axiom is the weakest requirement one can impose, as in a physical theory on the one hand one needs to "close the algebra", but on the other hand it is the sequence of modes that is important, not so much its indexing. The axiom then requires that the sequence of modes is "in" the twisted vertex algebra, albeit after potential reindexing.

Definition 3.4. *(**Isomorphism of twisted vertex algebras**) Two twisted vertex algebras (V, W, π_f, Y) and $(\widetilde{V}, \widetilde{W}, \tilde{\pi}_f, \widetilde{Y})$ are said to be isomorphic via a linear bijective map $\Phi : V \to \widetilde{V}$ if $\Phi(|0\rangle_W) = |0\rangle_{\widetilde{W}}$ and the following holds: for any $v \in V$, $\tilde{v} \in \widetilde{V}$ and $w \in W$, $\tilde{w} \in \widetilde{W}$ we have*

$$\Phi(v) = \sum_{finite} c_k \tilde{v}_k, \quad c_k \in \mathbb{C}, \quad \tilde{v}_k \in \widetilde{V};$$

$$\Phi^{-1}(\tilde{v}) = \sum_{finite} d_m v_m, \quad d_m \in \mathbb{C}, \quad v_m \in V.$$

so that

$$\Phi(Y(v,z)w) = \sum_{finite} z^{l_k} c_k \widetilde{Y}(\tilde{v}_k, z)\tilde{\pi}_f \circ \Phi(w), \quad l_k \in \mathbb{Z}; \tag{3.3}$$

$$\Phi^{-1}(\widetilde{Y}(\tilde{v}, z)\tilde{w}) = \sum_{finite} z^{l_m} d_m Y(v_m, z)\pi_f \circ \Phi^{-1}(\tilde{w}), \quad l_m \in \mathbb{Z}. \tag{3.4}$$

Remark 3.5. The reason this definition is much more complicated then in the super vertex algebra case is the allowance for the shifts in the OPEs.

This can cause each of the summands in the linear sum $\Phi(v)$ to appear with a different shift in the sum of the corresponding vertex operators, e.g. (5.37) and (5.53), hence the allowance for the different powers of z in the above definition.

There are a variety of different vertex algebra like theories, each designed to describe different sets of examples of collections of fields. The best known is the theory of super vertex algebras (see e.g. [16], [21], [26], [5], [14]). The axioms of super vertex algebras are often given in terms of locality (see [21], [14]), as locality is a property that plays crucial importance for super vertex algebras ([28]). On the other hand, there are vertex algebra like objects which do not satisfy the usual locality property, but rather a generalization. Twisted vertex algebras are among them, but there are also generalized vertex algebras, Γ-vertex algebras, deformed chiral algebras, quantum vertex algebras. Unlike twisted vertex algebras, quantum vertex algebras do not satisfy the symmetry condition, and thus a locality-type property is very hard to write in that case. Thus for example the axioms for deformed chiral algebras ([13]) are given in terms of requiring existence of analytic continuations, and then the braided symmetry axiom is given in terms of the analytic continuations ([13]). Twisted vertex algebras occupy intermediate step between super vertex algebras and deformed chiral algebras. We choose here to define twisted vertex algebras with axioms closer to the deformed chiral algebra axioms, including the analytic continuation axiom. But twisted vertex algebras are closer to super vertex algebras in many ways. For one, they satisfy N-point locality (finitely many points of locality), see [2], unlike the deformed chiral algebras which have lattices of points of locality. Also, like for super vertex algebras, the axioms requiring existence of analytic continuation of product of two vertex operators plus the symmetry axiom do in fact enforce the property that analytic continuation of arbitrary product of fields exist; something that is not true for deformed chiral algebras (see [13]). This property, which we will not prove here (proof is given in [2]), is why we chose to give the axioms for twisted vertex algebra in the form above. Further, we will derive formulas for the analytic continuations using the bicharacter construction which we will use to derive vacuum expectation values identities.

Proposition 3.6. (Analytic continuation for arbitrary products of fields)
Let (V, W, π_f, Y) be a twisted vertex algebra. For any $a_i \in V, i = 1, \dots, k$, there exist a rational vector valued function

$$X_{z_1,z_2,\dots,z_k} : V^{\otimes k} \to W[[z_1, z_2, \dots, z_k]] \otimes \mathbf{F}^N_\epsilon(z_1, z_2, \dots, z_k)^{+,z_k},$$

such that

$$Y(a_1, z_1)Y(a_2, z_2)\dots Y(a_k, z_k)1 = i_{z_1,z_2,\dots,z_k} X_{z_1,z_2,\dots,z_k}(a_1 \otimes a_2 \otimes \dots \otimes a_k).$$

We do not discuss the axiomatics of twisted vertex algebras further in this paper due to length (for this see [2]). Instead we show that the concept of a twisted vertex algebra answers the question "what mathematical structures are the boson-fermion correspondences of type B, C and D-A?". We claim that these boson-fermion correspondences are isomorphisms of twisted vertex algebras.

Similarly to super vertex algebras, twisted vertex algebras are often generated by a smaller number of fields. We will not prove theorems on what constitutes a generating set of fields (see [2]), instead we will take an alternative approach and use the bicharacter construction which we will present in the next section. The bicharacter construction in some sense mimics the generation by a smaller set of generating fields, in that we start with a smaller set of data which then "generates" the entire set of data for the twisted vertex algebra. The bicharacter construction will thus replace the necessary theorems on generating sets of fields. We want to mention though, that if we have a set of generating fields, the full space of fields is in turn determined by requiring, as in a super vertex algebra, that it be closed under OPEs (see modification above); that for any vertex operator $a(z)$ the field $Da(z) = \partial_z a(z)$ has to be a vertex operator in the twisted vertex algebra. A new ingredient in a twisted vertex algebra is the requirement that the field $T_\epsilon a(z) = a(\epsilon z)$ is a vertex operator in the same twisted vertex algebra as well. Note that this immediately violates the stronger creation axiom for a classical vertex algebra, since:

$$(T_\epsilon a)(z)|0\rangle|_{z=0} = a(\epsilon z)|0\rangle|_{z=0} = a(z)|0\rangle|_{z=0} = \pi_f(a). \tag{3.5}$$

Hence any such field $T_\epsilon a(z)$ cannot belong to a classical vertex algebra. This is the reason we require the projection map π_f to be a part of our data for a twisted vertex algebra, as well as we modify the field-state correspondence with the **modified creation axiom**. The projection map π_f in the definition of a twisted vertex algebra (V, W, π_f, Y) could be made more general, for example one can omit the requirement that $W \subset V$ and make π_f a more general linear surjective projection map, but the current generality is sufficient for the examples we want to describe.

3.2. **Examples of twisted vertex algebras: the boson-fermion correspondence of type B.**

We now proceed with the two examples of a twisted vertex algebra of order 2 which give the two sides of the boson-fermion correspondence of type B. This correspondence was first introduced in [10], and was interpreted as an isomorphism of twisted vertex algebras in [1], but with no proofs. Since this is a case of twisted vertex algebras of order two, the root of unity is -1, and we will write T_{-1} instead of T_ϵ. The proofs of all the statements in this section are given in sections 5.3 and 5.7.

The fermionic side is generated by a single field $\phi^B(z)=\sum_{n\in\mathbf{Z}}\phi_n z^n$, with OPE given by:

$$\phi^B(z)\phi^B(w)\sim\frac{z-w}{z+w},\quad \text{in modes: } [\phi^B_m,\phi^B_n]_\dagger=2(-1)^m\delta_{m,-n}1. \tag{3.6}$$

The modes generate a Clifford algebra Cl_B, and the underlying space of states, denoted by F_B, of the twisted vertex algebra is a highest weight representation of Cl_B with the vacuum vector $|0\rangle$, such that $\phi^B_n|0\rangle=0$ for $n<0$. The space of fields, which is larger than the space of states, is generated via the field $\phi^B(z)$ together with its descendent $T_{-1}\phi^B(z)=\phi^B(-z)$. We call the resulting twisted vertex algebra the **free neutral fermion of type B**, and denote it also by F_B.

Remark 3.7. We can write the singular part the defining OPE, (3.6):

$$\phi^B(z)\phi^B(w)\sim\frac{-2w}{z+w},$$

which shows why we adopted the modification of the completeness with respect to OPEs. One of the residues here is $Res_{z=-w}X_{z,w,0}(a\otimes b\otimes c)=-2w\cdot\pi_f(c)=-2wId_W(\pi_f(c))$. Note that $-2wId_W$ is a field, but can not be a vertex operator in any vertex algebra as is. But a shift by w^{-1} will produce the field $-2Id_W$, which is the vertex operator corresponding to the state $-2|0\rangle$.

The boson-fermion correspondence of type B is determined once we write the image of the generating fields $\phi^B(z)$ (and thus of $T_{-1}\phi^B(z)=\phi(-z)$) under the correspondence. In order to do that, an essential ingredient is the **twisted** Heisenberg field $h(z)$ given by:

$$h(z)=\frac{1}{4}(:\phi^B(z)T_{-1}\phi^B(z):-1)=\frac{1}{4}(:\phi^B(z)\phi^B(-z):-1) \tag{3.7}$$

Lemma 3.8. *(*[10], [37], [1]*) The field $h(z)$ given above is a twisted Heisenberg field, it has only odd-indexed modes, $h(z)=\sum_{n\in\mathbb{Z}}h_{2n+1}z^{-2n-1}$, and has OPEs with itself given by:*

$$h(z)h(w)\sim\frac{zw(z^2+w^2)}{2(z^2-w^2)^2}, \tag{3.8}$$

Its modes, $h_n,\ n\in 2\mathbb{Z}+1$, generate a ***twisted*** *Heisenberg algebra $\mathcal{H}_{\mathbb{Z}+1/2}$ with relations $[h_m,h_n]=\frac{m}{2}\delta_{m+n,0}1,\ \ m,n$–odd integers.*

$\mathcal{H}_{\mathbb{Z}+1/2}$ has (up-to isomorphism) only one irreducible highest weight module $B_{1/2}\cong\mathbb{C}[x_1,x_3,\dots,x_{2n+1},\dots]$.

Lemma 3.9. *([10], [37], [1]) The space of states F_B can be decomposed as*

$$F_B = B_{1/2} \oplus B_{1/2} \cong \mathbb{C}[e^{\alpha}, e^{-\alpha}] \otimes \mathbb{C}[x_1, x_3, \dots, x_{2n+1}, \dots] = B_B, \tag{3.9}$$

where the isomorphism is as twisted Heisenberg modules for $\mathcal{H}_{\mathbb{Z}+1/2}$; e^{α} denotes the nontrivial highest weight vector. (We assume the extra relation $e^{2\alpha} \equiv 1$, i.e., $e^{\alpha} \equiv e^{-\alpha}$).

The right-hand-side, which we denote by B_B, is the underlying vector space of **states** of the bosonic side of the boson-fermion correspondence of type B.

Now we can write the image of the generating field $\phi^B(z) \mapsto e^{\alpha}(z)$, which will determine the correspondence of type B:

Lemma 3.10. *([1]) The bosonization of type B is determined by*

$$\phi^B(z) \mapsto e^{\alpha}(z) = \exp\Big(\sum_{k\geq 0} \frac{h_{-2k-1}}{k+1/2} z^{2k+1}\Big) \exp\Big(-\sum_{k\geq 0} \frac{h_{2k+1}}{k+1/2} z^{-2k-1}\Big) e^{\alpha}. \tag{3.10}$$

The fields $e^{\alpha}(z)$ and $e^{\alpha}(-z) = e^{-\alpha}(z)$ (observe the symmetry) generate the resulting **twisted** vertex algebra, which we denote also by B_B.

Note that one Heisenberg $\mathcal{H}_{\mathbb{Z}+1/2}$-module $B_{1/2}$ on its own can be realized as a **twisted module** for an ordinary super vertex algebra (see [16], [4] for details), but the point is that we need **two** of them glued together for the bosonic side of the correspondence. The two of them glued together as above no longer constitute a twisted module for an ordinary super vertex algebra.

Theorem 3.11. *([1]) The boson-fermion correspondence of type B is the isomorphism between the fermionic* ***twisted*** *vertex algebra F_B and the bosonic* ***twisted*** *vertex algebra B_B.*

Corollary 3.12. *The Schur Pfaffian identity follows from the equality between the vacuum expectation values:*

$$AC\langle 0|\phi^B(z_1)\dots\phi^B(z_{2n})|0\rangle = Pf\Big(\frac{z_i - z_j}{z_i + z_j}\Big)_{i,j=1}^{2n} =$$

$$= \prod_{i<j}^{2n} \frac{z_i - z_j}{z_i + z_j} = AC\langle 0|e^{\alpha}(z_1)\dots e^{\alpha}(z_{2n})|0\rangle$$

AC stands for Analytic Continuation.

3.3. Examples of twisted vertex algebras: the boson-fermion correspondence of type D-A.

Next are the two examples of a twisted vertex algebra of order 2 which give the two sides of the boson-fermion correspondence of type D-A. The boson-fermion correspondence of type D-A is new, and the bosonisation of type D is one of the main results of this paper. This correspondence was discussed during the work on both this paper and on [31], where the "multilocal fermionization" is discussed. The author thanks K. Rehren for the helpful discussions and a physics point of view on the subject. Here we will show the bosonization of type D (including the split of the neutral fermion space into a direct sum of bosonic spaces) and prove that the boson-fermion correspondence of type D-A is an isomorphism of twisted vertex algebras. The correspondence of type D-A can be generalized to arbitrary order $N \in \mathbb{N}$ (see below, and also [31]), but we start with the case of twisted vertex algebras of order two. Again we write T_{-1} instead of T_ϵ. The proofs of all the statements in this section are given in sections 5.4 and 5.8.

The fermionic side is generated by a single field $\phi^D(z) = \sum_{n\in \mathbf{Z+1/2}} \phi^D_n z^{-n-1/2}$ with OPE given by:

$$\phi^D(z)\phi^D(w) \sim \frac{1}{z-w}, \quad \text{in modes: } [\phi^D_m, \phi^D_n]_\dagger = \delta_{m,-n}1. \tag{3.11}$$

The modes generate a Clifford algebra Cl_D, with underlying space of states, denoted by F_D, the highest weight representation of Cl_D with the vacuum vector $|0\rangle$, such that $\phi_n|0\rangle = 0$ for $n < 0$. Here it is recognized that on its own the field $\phi^D(z)$ and its descendants $D^{(n)}\phi^D(z)$ generate an (ordinary) super-vertex algebra (see [21], [22]). It is important to note that on its own, this super-vertex algebra, called free neutral fermion vertex algebra, cannot be bosonized. But, if we take not only the field $\phi^D(z)$, but also its **twisted** descendant $T_{-1}\phi^D(z) = \phi^D(-z)$, they together with all their descendants will generate a **twisted** vertex algebra which will be bosonized. We call it the **free neutral fermion of type D-A**, and denote by abuse of language also by F_D. The twisted vertex algebra F_D is obviously bigger that the super-vertex algebra, and we can think of it as an orbifolded double cover of the super-vertex algebra.

The boson-fermion correspondence of type D-A is determined once we write the image of the generating fields $\phi^D(z)$ and $T_{-1}\phi^D(z) = \phi(-z)$ under the correspondence. In order to do that, an essential ingredient is once again the Heisenberg field $h(z)$ given by

$$h(z) = \frac{1}{2} :\phi^D(z)T_{-1}\phi^D(z):= \frac{1}{2} :\phi^D(z)\phi^D(-z): \tag{3.12}$$

Proposition 3.13. *The field $h(z)$ given above is a Heisenberg field, has only odd-indexed modes, $h(z) = \sum_{n\in\mathbb{Z}} h_n z^{-2n-1}$, (note the indexing), and has OPEs with itself given by:*

$$h(z)h(w) \sim \frac{zw}{(z^2-w^2)^2}. \tag{3.13}$$

*Its modes, $h_n,\ n\in\mathbb{Z}$, generate an **untwisted** Heisenberg algebra $\mathcal{H}_{\mathbb{Z}}$ with relations $[h_m,h_n]=m\delta_{m+n,0}1,\ \ m,n-$ integers.*

Unlike the twisted Heisenberg algebra, the untwisted Heisenberg algebra has infinitely many irreducible highest weight modules, labeled by the action of h_0.

Proposition 3.14. *The space of states F_D can be decomposed as*

$$W=F_D\cong\oplus_{i\in\mathbb{Z}}B_i\cong\mathbb{C}[e^{\alpha},e^{-\alpha}]\otimes\mathbb{C}[x_1,x_2,\dots,x_n,\dots]=B_D,\qquad(3.14)$$

$\mathbb{C}[e^{\alpha},e^{-\alpha}]$ denotes the Laurent polynomials with one variable e^{α}. The isomorphism above is as Heisenberg modules, where $e^{n\alpha}$ denotes the highest weight vector for the irreducible Heisenberg module B_n, with highest weight n.

We denote the vector space on the right-hand-side of this $\mathcal{H}_{\mathbb{Z}}$-module isomorphism by B_D.

Proposition 3.15. *The images of the generating fields $\phi^D(z)$ and $T_{-1}\phi^D(z)=\phi(-z)$, which determine the correspondence of type D-A, are as follows:*

$$\phi^D(z)\mapsto e_{\phi}^{-\alpha}(z)+e_{\phi}^{\alpha}(z),\quad (T\phi)^D(z)=\phi(-z)\mapsto e_{\phi}^{-\alpha}(z)-e_{\phi}^{\alpha}(z),\qquad(3.15)$$

where $e_{\phi}^{-\alpha}(z)$ and $e_{\phi}^{\alpha}(z)$ are defined by the the formulas

$$e_{\phi}^{-\alpha}(z)=\exp(-\sum_{n\geq1}\frac{h_{-n}}{n}z^{2n})\exp(\sum_{n\geq1}\frac{h_n}{n}z^{-2n})e^{-\alpha}z^{-2\partial_{\alpha}}=e_A^{-\alpha}(z^2),\qquad(3.16)$$

$$e_{\phi}^{\alpha}(z)=\exp(\sum_{n\geq1}\frac{h_{-n}}{n}z^{2n})\exp(-\sum_{n\geq1}\frac{h_n}{n}z^{-2n})e^{\alpha}z^{2\partial_{\alpha}+1}=ze_A^{\alpha}(z^2),\qquad(3.17)$$

Theorem 3.16. *The boson-fermion correspondence of type D-A is the isomorphism between the fermionic **twisted** vertex algebra F_D and the bosonic **twisted** vertex algebra B_D.*

Remark 3.17. The name "boson-fermion correspondence of type D-A" is given since the fermionic side is a double cover of the well know neutral fermion super vertex algebra that gives the basic representation of d_{∞}, see e.g. [24], [22]. On the other hand the operators $e_A^{\alpha}(z)$ and $e_A^{-\alpha}(z)$ in the right-hand side above are the vertex operators describing the boson-fermion correspondence of type A (see e.g. [21]).

Corollary 3.18. *The following Pfaffian identity follows from the equality between the vacuum expectation values:*

$$AC\langle 0|\phi^D(z_1)\dots\phi^D(z_{2n})|0\rangle = Pf\Big(\frac{1}{z_i - z_j}\Big)_{i,j=1}^{2n} =$$

$$= \frac{\sum_{i_1<i_2\cdots<i_n}^{2n} z_{i_1}z_{i_2}\cdots z_{i_n}\prod_{k<l}^{n}(z_{i_k}^2 - z_{i_l}^2)\prod_{p<q}^{n}(z_{j_p}^2 - z_{j_q}^2)}{\prod_{k,p}^{n}(z_{i_k}^2 - z_{j_p}^2)} =$$

$$= AC\langle 0|(e_\phi^{-\alpha}(z_1) + e_\phi^{\alpha}(z_1))\dots(e_\phi^{-\alpha}(z_{2n}) + e_\phi^{\alpha}(z_{2n}))|0\rangle$$

AC stands for Analytic Continuation, and $\{j_1, j_2, \dots j_n\} = \{1, 2, \dots 2n\} \setminus \{i_1, i_2, \dots i_n\}$.

The proofs will be given in section 5.8.

3.4. **Examples of twisted vertex algebras: the boson-fermion correspondence of type D-A of order N.**
The boson-fermion correspondence of type D-A from the previous section can be generalized to general order N as follows. We again consider the free field $\phi^D(z) = \sum_{n\in \mathbf{Z+1/2}}\phi_n^D z^{-n-1/2}$, with OPEs with itself given by (3.11). Let ϵ be an N-th order primitive root of unity and consider the following twisted vertex algebra descendants $T_\epsilon^i\phi^D(z) = \phi^D(\epsilon^i z)$, for any $0 \le i \le N-1$, with OPEs

$$T_\epsilon^i\phi^D(z)T_\epsilon^j\phi^D(w) \sim \frac{1}{\epsilon^i z - \epsilon^j w}.$$

Such OPEs are not allowed in a super vertex algebra, but are allowed in a twisted vertex algebra. These mean that on its own each of the fields $T_\epsilon^i\phi^D(z)$, for any $0 \le i \le N-1$ will generate a super vertex algebra, but the N of them "glued" together form a twisted vertex algebra. Again, this resembles the gluing together of the N sheets of the N-th root Riemann surface.

Lemma 3.19. *The field $h(z)$ given by*

$$h(z) = \frac{1}{N}\sum_{i=0}^{N-1}\epsilon^{i-1} : T_\epsilon^{i-1}\phi^D(z)T_\epsilon^i\phi^D(z) := \sum_{n\in\mathbb{Z}} h_n z^{-Nn-1} \tag{3.18}$$

is a Heisenberg field with OPE

$$h(z)h(w) \sim \frac{z^{N-1}w^{N-1}}{(z^N - w^N)^2}. \tag{3.19}$$

Thus the commutation relations $[h_m, h_n] = m\delta_{m+n,0}1$ for the Heisenberg algebra $\mathcal{H}_\mathbb{Z}$ hold.

Let

$$e_\phi^\alpha(w) = \frac{1}{N}(\sum_{i=0}^{N-1}\epsilon^{-i}T^i\phi^D(w)), \quad e_\phi^{\epsilon^k\alpha}(w) = \frac{1}{N}(\sum_{i=0}^{N-1}\epsilon^{(k-1)i}T^i\phi^D(w)).$$

Lemma 3.20. *The boson-fermion correspondence of order N is given by*

$$e_{\phi}^{\epsilon^k\alpha}(z)\mapsto \exp(\epsilon^{-k}\sum_{n\geq 1}\frac{h_{-n}}{n}z^{Nn})\exp(\epsilon^{k}\sum_{n\geq 1}\frac{h_{n}}{n}z^{-Nn})e_{\phi}^{\epsilon^k\alpha}z^{1-k+N\partial_\alpha}, \tag{3.20}$$

where $e_{\phi}^{\epsilon^k\alpha}$ identifies the highest weight vector of the Heisenberg submodule.

In these last two subsections we gave examples of twisted vertex algebras, which are **not** super-vertex algebras. These examples have a great importance on their own, as each pair gives the two sides of a boson-fermion correspondence, of type B and D-A correspondingly. The correspondence of type C, which was introduced in [9] and developed further in [32], is another example of a twisted vertex algebra isomorphism (see a brief discussion in section 5.10). Although there are many other examples of twisted vertex algebras, as we will see in theorem 4.39, the boson-fermion correspondences are important enough phenomena to justify this new definition of a twisted vertex algebra.

4. Bicharacter construction: a general way of constructing examples of twisted vertex algebras

4.1. **Super bicharacters and free Leibnitz modules.**

In this section we recall the ingredients of the super-bicharacter construction. The notations, definitions and results of the first subsection are not new, but we are using them extensively in what follows (the super case was introduced in [3], generalizing [6]).

Notation 4.1. *Henceforth we will assume that M is a Hopf supercommutative and supecocommutative superalgebra with antipode S. Here and below a, b, c and d are homogeneous elements of M.*

Definition 4.2. (Super-bicharacter) *Define a bicharacter on M to be a linear map r from $M\otimes M$ to $\mathbf{F}_\epsilon(z,w)$, such that*

$$r_{z_1,z_2}(1\otimes a)=\eta(a)=r_{z,w}(a\otimes 1), \tag{4.1}$$

$$r_{z,w}(ab\otimes c)=\sum (-1)^{\tilde{b}\tilde{c'}}r_{z,w}(a\otimes c')r_{z,w}(b\otimes c''), \tag{4.2}$$

$$r_{z,w}(a\otimes bc)=\sum (-1)^{\tilde{a''}\tilde{b}}r_{z,w}(a'\otimes b)r_{z,w}(a''\otimes c). \tag{4.3}$$

We say that a bicharacter r is even if $r_{z,w}(a\otimes b)=0$ whenever $\tilde{a}\neq\tilde{b}$.

From now on we will always work with *even* bicharacters. In most cases there are no nontrivial arbitrary bicharacters (see [3]). The identity bicharacter is given by $r(a\otimes b)=\eta(a)\otimes\eta(b)$.

Remark 4.3. The notion of super bicharacter is similar to the notion of a twist induced by Laplace pairing (or the more general concept of a Drinfeld twist) as described in [7].

Definition 4.4. (Symmetric bicharacter) *The transpose of a bicharacter is defined by*

$$r^{\tau}_{z,w}(a\otimes b)=r_{w,z}\circ\tau(a\otimes b). \tag{4.4}$$

A bicharacter r is called symmetric *if $r=r^{\tau}$.*

Definition 4.5. **($\mathbf{H^N_{T_\epsilon}}\otimes\mathbf{H^N_{T_\epsilon}}$*-covariant bicharacter)*** *Let M be a a Hopf supercommutative and supecocommutative superalgebra, r be a bicharacter on M. Suppose in addition M is an $H^N_{T_\epsilon}$-module algebra. We call the bicharacter $H^N_{T_\epsilon}\otimes H^N_{T_\epsilon}$-covariant if it additionally satisfies :*

$$r_{z,w}(h(a)\otimes g(b))=h_z g_w\cdot r_{z,w}(a\otimes b), \tag{4.5}$$

for all $a,b\in M,\ h,g\in H^N_{T_\epsilon}$.

We recall the following result from [6], generalized to the super case:

Lemma/Definition 4.6. ***(Free* H−*Leibnitz module)*** *Suppose M is a super commutative algebra and H is an entirely even cocommutative coalgebra. Then there is a universal supercommutative algebra $H(M)$ such that there is a map $h\otimes m\to hm:=h(m)$ from $H\otimes M$ to $H(M)$ such that $H(M)$ is a left module for H and*

$$h(mn)=\sum h^{'}(m)h^{''}(n),\quad h(1)=\eta(h), \tag{4.6}$$

for any $m,n\in M,\ h\in H$. We will call $H(M)$, defined as above, the "free H Leibnitz module of M" (or universal H-Leibnitz module of M).

Notes:

(1) $H(M)$ is the quotient of the tensor algebra $T(H\otimes M)$ modulo supercommutativity relations plus relations (4.6).
(2) An H-module which has the properties (4.6) is by definition an H-module algebra (see for example [25]), thus $H(M)$ is an H-module algebra. It is the universal H-module algebra containing M in the super-commutative category.
(3) If M is supercommutative and supercocommutative bialgebra (or Hopf algebra), then so is $H(M)$. The extension of comultiplication, the counit and the antipode from M to $H(M)$ is as follows: If $a\in M,\ h\in H$ we have $ha\in H(M)$ and we define

$$\triangle(ha)=\sum h'a'\otimes h''a'', \tag{4.7}$$

$$\eta(ha)=\eta(h)\eta(a), \tag{4.8}$$

$$S(ha)=S(h)(S(a)). \tag{4.9}$$

It is easy to check that the comultiplication, the counit and the antipode defined as above will turn $H(M)$ into a Hopf algebra. (Note that this is only true in the case of cocommutative Hopf algebra H, and some modifications are required in the case H is not entirely even; but we will only work with the case of H even cocommutative).

A source of examples of $H^N_{T_\epsilon}\otimes H^N_{T_\epsilon}$-covariant bicharacters is as follows:

Lemma 4.7. *(*[6]*)* **(Extension of bicharacters)** *Let $r : M\otimes M \to \mathbf{F}^N_\epsilon(z,w)$ be a bicharacter on M. Then we can extend extend this bicharacter to the free Leibnitz module $H^N_{T_\epsilon}(M)$ as follows: all elements in $H^N_{T_\epsilon}(M)$ are generated as an algebra from elements of the form $a = h\bar{a}$, $b = g\bar{b}$ for some $\bar{a},\bar{b}\in M$, $g,h\in H$. Thus define a bicharacter*

$$r : H^N_{T_\epsilon}(M)\otimes H^N_{T_\epsilon}(M) \to \mathbf{F}^N_\epsilon(z,w) \tag{4.10}$$

by

$$r_{z,w}(a\otimes b) = h_z g_w\cdot r_{z,w}(\bar{a}\otimes\bar{b}), \tag{4.11}$$

and extend it by linearity and using multiplicativity ((4.2) *and* (4.3)*) of the bicharacter to the whole of $H^N_{T_\epsilon}(M)$. The extended bicharacter r on $H^N_{T_\epsilon}(M)$ is $H^N_{T_\epsilon}\otimes H^N_{T_\epsilon}$-covariant.*

4.2. **Examples of free Leibnitz modules.**

Most of the modules underlying our vertex algebras in this paper are going to be free Leibnitz modules. The first two types of examples of free Leibnitz modules are entirely even, or **bosonic**.

Example 4.8. (The free Leibnitz modules $H_D(\mathbb{C}[h])$ and $H^N_{T_\epsilon}(\mathbb{C}[h])$)

The free H_D-Leibnitz module over the algebra $\mathbb{C}[h]$ (the polynomial algebra of a single variable, considered as a Hopf algebra with h a primitive element) is isomorphic to the polynomial algebra $\mathbb{C}[x_1,x_2,\dots,x_n,\dots]$. We can identify $x_1 = h$ and $D^{(n)}h = D^{(n)}x_1 = x_{n+1}$. From equation (2.3), we have that $x_n = D^{(n-1)}h$ are primitive: according to (4.7):

$$\triangle(D^{(l)}h) = (\sum_{p+q=l} D^{(p)}\otimes D^{(q)})(h\otimes 1 + 1\otimes h) = D^{(l)}h\otimes 1 + 1\otimes D^{(l)}h, \tag{4.12}$$

one similarly checks the counit and the anitpode. These variables commute and generate $H_D(\mathbb{C}[h])$. Thus $\mathbb{C}[x_1,x_2,\dots,x_n,\dots]$ is isomorphic to the free H_D module-algebra over $\mathbf{C}[h]$.

The free Leibnitz module $H^N_{T_\epsilon}(\mathbb{C}[h])$ is isomorphic as a Hopf algebra to the polynomial algebra with k groups of variables:
$\mathbb{C}[x^0_1,x^0_2,\dots,x^0_n,\dots,x^{N-1}_1,x^{N-1}_2,\dots,x^{N-1}_n,\dots,]$, by identifying

$$x^k_l = T^k D^{(l)}h, \quad k = 0,\dots,N-1, \quad n = 0,1,\dots,l,\dots. \tag{4.13}$$

Example 4.9. (The free Leibnitz modules over free abelian group algebras)
Let $L_1 = \mathbb{C}[\mathbb{Z}\alpha]$ be the group algebra of the rank-one free abelian group $\mathbb{Z}\alpha$. The group algebra is generated by $e^{m\alpha},\ m \in \mathbb{Z},$ with relations $e^{m\alpha}e^{n\alpha} = e^{(m+n)\alpha}$, $e^0 = 1$. Note that as an algebra $L_1 = \mathbb{C}[e^{\alpha}, e^{-\alpha}]$, with the relation $R:\ e^{\alpha}e^{-\alpha} = 1$. The Hopf algebra structure is determined by e^{α} and $e^{-\alpha}$ being grouplike.

Lemma 4.10. *The free H_D Leibnitz module $H_D(L_1)$ is isomorphic as an algebra to $L_1 \otimes \mathbb{C}[h]$.*

Proof. Since $H_D(L_1)$ is a free Leibnitz module, we can define an element $h = (De^{\alpha})e^{-\alpha}$. It follows that h is primitive, we have

$$\Delta(h) = \Delta(De^{\alpha})\Delta(e^{-\alpha}) = (De^{\alpha}\otimes e^{\alpha} + (e^{\alpha}\otimes De^{\alpha}))(e^{-\alpha}\otimes e^{-\alpha}) = h\otimes 1 + 1\otimes h.$$

Similarly one checks that $\epsilon(h) = 0$. Thus, since $H_D(L_1)$ is a free Leibnitz module, it has the subalgebra $H_D(\mathbf{C}[h]) = \mathbf{C}[x_1, x_2, \dots, x_n, \dots]$. Thus, $L_1 \otimes H_D(\mathbf{C}[h])$ is (isomorphic to) a subalgebra in $H_D(L_1)$. Conversely, for any $m \in \mathbf{Z}$, since $De^{\alpha} = h\cdot e^{\alpha}$ we can write

$$De^{m\alpha} = D(e^{\alpha})^m = m(De^{\alpha})(e^{\alpha})^{m-1} = m(he^{\alpha})(e^{\alpha})^{m-1} = mh\cdot e^{m\alpha}$$

and so for any $m, n \in \mathbf{Z}$ the element $D^n(e^{m\alpha})$ is in $M \otimes H_D(\mathbf{C}[h])$, and so $H_D(L_1)$ is (isomorphic to) a subalgebra in $M \otimes H_D(\mathbf{C}[h])$. Thus $H_D(L_1)$ is isomorphic to $L_1 \otimes \mathbb{C}[h] \simeq L_1 \otimes \mathbf{C}[x_1, x_2, \dots, x_n, \dots],\ n \in \mathbf{N}$. □

Note that the primitive element h is of particular importance for the boson-fermion correspondences, as it generates a Heisenberg subalgebra for a variety of choices for a bicharacter.

The free Leibnitz module $H^N_{T_\epsilon}(L_1)$ is isomorphic to $L_N \otimes H^N_{T_\epsilon}(\mathbb{C}[h])$, where L_N is the group algebra $L_N = \mathbb{C}[\mathbb{Z}\alpha_1, \mathbb{Z}\alpha_2, \dots \mathbb{Z}\alpha_N]$ of the free abelian group of rank N (one can identify $T^k e^{\alpha}$, which is grouplike, with e^{α_k}).

One proceeds similarly with the free Leibnitz modules over the free abelian group of any rank.

The other examples we will use throughout the paper are **fermionic**, or super algebras.

Example 4.11. (The free Leibnitz modules $H_D(\mathbb{C}\{\phi\})$ and $H^N_{T_\epsilon}(\mathbb{C}\{\phi\})$)
Let $\mathbb{C}\{\phi\}$ be the Grassmann algebra generated by one odd primitive element ϕ, $\phi\cdot\phi = 0$. Then the free Leibnitz module $H_D(\mathbb{C}\{\phi\})$ is the Grassmann algebra with odd anticommuting generators $\phi^n = D^{(n)}\phi$, $\phi^n\phi^m + \phi^m\phi^n = 0$, which one checks to be primitive.

Similarly, the free Leibnitz module $H^N_{T_\epsilon}(\mathbb{C}\{\phi\})$ is the Grassmann algebra with odd anticommuting generators

$$\phi^{n,k} = D^{(n)}T^k_\epsilon\phi, \quad k = 0, \dots, N-1, \quad n = 0, 1, \dots, l, \dots. \tag{4.14}$$

Note that the ordering of the operators $D^{(n)}$ and T_ϵ^k matters; one should be careful to be consistent, as it may result in rescaling of the basis: $\phi^{n,k} = D^{(n)}T_\epsilon^k\phi = \epsilon^{kn}T_\epsilon^k D^{(n)}\phi$.

Of particular interest for the boson-fermion correspondences is going the case of $N = 2$: the free Leibnitz module $H^2_{T_\epsilon}(\mathbb{C}\{\phi\})$ is algebra-isomorphic to $H_D(\mathbb{C}\{\phi, T\phi\})$, where write $T = T_\epsilon = T_{-1}$ ($\epsilon = -1$ in this case). Both the boson-fermion correspondences (the type B, and the type D-A) have $H_D(\mathbb{C}\{\phi, T\phi\})$ as underlying space of fields on its fermionic side. Of particular interest is the element

$$h_\phi = \phi T\phi, \quad \text{with} \quad Th_\phi = -h_\phi. \tag{4.15}$$

This element is even, and although it is not primitive, we will see in the later sections that the element h_ϕ generates a Heisenberg subalgebra for particular choices of bicharacter.

Similar to the last example of free Leibnitz modules are the free Leibnitz modules $H_D(\mathbb{C}\{\phi, \psi\})$ and $H^N_{T_\epsilon}(\mathbb{C}\{\phi, \psi\})$, which appear on the fermionic side of the boson-fermion correspondence of type A.

Recall we assume that M is a (super)commutative, (super)cocommutative Hopf algebra. To unclutter the language, we will just write commutative, cocommutative, omitting the term "super" as long as the parity is clear from the context.

4.3. **Exponential map and its properties; Nonsingular twisted vertex algebras.**

Definition 4.12. (Nonsingular vertex algebra) *We call a twisted vertex algebra nonsingular if the analytic continuations $X_{z,w,0}(a \otimes b \otimes c)$ have no poles for any $a, b, c \in V$.*

For super vertex algebras this definition coincides with the notion of a holomorphic super vertex algebra introduced in the previous literature (see for example [21], [26]). A holomorphic super-vertex algebra is in fact just a commutative associative unital differential algebra: if V is a holomorphic super-vertex algebra, for any $a, b \in V$ we have (see for example [21], [26]):

$$Y(a,z)b = (e^{zD}a)b, \quad \text{where} \quad e^{zD} = \sum_{n\geq 0} z^n D^{(n)}$$

Thus the fields in a holomorphic super-vertex algebra are uniquely determined by the unique "exponential map" e^{zD}. In the case of twisted vertex algebras the situation is not as simple, as the "exponential map" is not unique. There are a variety of examples of "exponential maps" for twisted vertex algebras that would satisfy the properties of a nonsingular twisted vertex algebra. Note that for twisted vertex algebras, we also have the concept of a projection map, thus we really consider **pairs** of exponential and projection maps that satisfy the properties of a nonsingular twisted vertex algebra. Hence we will proceed to define a standard pair of projection and exponential maps on given free $H^N_{T_\epsilon}$ Leibnitz modules. In what follows let M be a commutative cocommutative Hopf algebra, V be the free Leibnitz module $V = H^N_{T_\epsilon}(M)$. Note that the free Leibnitz module $W = H_D(M)$ is a sub-Hopf algebra of V, and thus we can use the exponential map e^{zD} on W. Moreover, each element in V can be written uniquely as a linear combination of elements of the form $a = \prod_{i=0}^{N-1} a_i$, where $a_i = T^i_\epsilon \bar{a}_i$, for some $\bar{a}_i \in W$.

Definition 4.13. (T-projection Map $\pi_{\mathbf{T}}$) *Let M be a commutative cocommutative Hopf algebra, let V be the free Leibnitz module $V = H^N_{T_\epsilon}(M)$, $W = H_D(M)$, and let $a \in V$ is such that $a = \prod_{i=0}^{N-1} a_i$, where $a_i = T^i_\epsilon \bar{a}_i$ for some $\bar{a}_i \in W$. Define the projection map $\pi_T : V \to W$ to be the algebra homomorphism map defined by:*

$$\pi_T(a_i) = \bar{a}_i, , \quad i = 1, \dots, N-1, \quad \pi_T(a) = \prod_{i=0}^{N-1} \bar{a}_i; \tag{4.16}$$

Since V is the span of such elements a as above, extend π_T to V by linearity.

Definition 4.14. (Exponential Map $\mathcal{E}_{\mathbf{z}}$) *Let as above $a_i = T^i_\epsilon \bar{a}_i$, $\bar{a}_i \in W$, $a = \prod_{i=0}^{N-1} a_i$. Define the map $\mathcal{E}_z : V \to W[[z]]$ to be the algebra homomorphism map such that*

$$\mathcal{E}_z(\bar{a}_i) = e^{zD}\bar{a}_i, \quad \text{for any } \ \bar{a}_i \in W \tag{4.17}$$

$$\mathcal{E}_z(a_i) = e^{\epsilon^i zD}\bar{a}_i, \quad i = 0, \dots, N-1; \tag{4.18}$$

$$\mathcal{E}_z(\prod_{i=0}^{N-1} a_i) = \prod_{i=0}^{N-1} e^{\epsilon^i zD}\bar{a}_i; \tag{4.19}$$

and extend $\mathcal{E}_z$ by linearity to V.

Example 4.15. This example points out that one has to be careful with the specific order between $D^{(n)}$ and T^i_ϵ which we implicitly used in the definition of the exponential map $\mathcal{E}_z$. Let as above $a_i = T^i_\epsilon \bar{a}_i$, $\bar{a}_i \in W$, and let $n \in \mathbb{N}$. To calculate $\mathcal{E}_z(D^{(n)}a_i)$, we need

$$D^{(n)}a_i = D^{(n)}T^i_\epsilon \bar{a}_i = \epsilon^{ni} T^i_\epsilon D^{(n)}\bar{a}_i; \tag{4.20}$$

where now $D^{(n)}\bar{a}_i \in W$, and thus

$$\mathcal{E}_z(D^{(n)}a_i) = \epsilon^{ni} e^{\epsilon^i zD}(D^{(n)}\bar{a}_i) = \epsilon^{ni} D^{(n)}(e^{\epsilon^i zD}\bar{a}_i) = \epsilon^{ni} D^{(n)}\mathcal{E}_z(a_i), \tag{4.21}$$

This equality plays part in the "transfer of action" property of $\mathcal{E}_z$, and is also used in proving the Modified Expansion property of the exponential map.

Lemma 4.16. (Properties of the Exponential Map $\mathcal{E}_{\mathbf{z}}$) *Let M be a commutative cocommutative Hopf algebra, let V be the free Leibnitz module $V = H^N_{T_\epsilon}(M)$, r is a $H^N_{T_\epsilon}\otimes H^N_{T_\epsilon}$-covariant bicharacter on V. Let $\pi_T : V \to W$ and $\mathcal{E}_z : V \to W[[z]]$ be the pair projection-exponential map defined above. This pair of maps satisfies the following properties:*

- *Vacuum property: $\mathcal{E}_z(1) = 1$, where 1 is the unit in V;*
- *Modified creation property: $\mathcal{E}_z(a)|_{z=0} = \pi_T(a)$, for any $a \in V$;*
- *Transfer of action: $\mathcal{E}_z(ha) = h_z\mathcal{E}_z(a)$, for any $h \in H^N_{T_\epsilon}$, $a \in V$;*
- *Multiplicativity: $\mathcal{E}_z(ab) = \mathcal{E}_z(a)\mathcal{E}_z(b)$, for any $a, b \in V$;*
- *Grouplike: $\triangle\mathcal{E}_z(a) = \mathcal{E}_z(a') \otimes \mathcal{E}_z(a'')$;*
- *Compatibility with bicharacters: $i_{z,w}r_{z,w}(a\otimes b) = r_{z,0}(a\otimes \mathcal{E}_w(b))$, for any $a, b \in V$.*
- *Modified expansion: $\mathcal{E}_z(a) = \sum_{n\geq 0}(z-\epsilon^i w)^n\mathcal{E}_w(T^iD^{(n)}a)$,*

Proof. The proofs for most of the properties are straightforward, and use the similar properties of the ordinary exponential map e^{zD} and the definition of the map $\mathcal{E}_z$ via the projection map. □

Remark 4.17. Note that as we saw in the example 4.15 above $D^{(n)}$ and $\mathcal{E}_w$ do not commute. Thus even though $\sum_{n\geq 0}(z-\epsilon^i w)^n\mathcal{E}_w(T^iD^{(n)}a) = \sum_{n\geq 0}\epsilon^{-in}(z-\epsilon^i w)^n\mathcal{E}_w(D^{(n)}Ta)$, and $\sum_{n\geq 0}\epsilon^{-in}(z-\epsilon^i w)^nD^{(n)} = e^{\epsilon^{-i}(z-\epsilon^{-i}w)D}$, the modified expansion property can **not** be rewritten using $e^{\epsilon^{-i}(z-\epsilon^{-i}w)D}$. That is the reason for the lack of a "modified associativity" property.

Lemma 4.18. (Nonsingular twisted vertex algebra) *Let M be a commutative cocommutative Hopf algebra, let V be the free Leibnitz module $V = H^N_{T_\epsilon}(M)$, let $\pi_T : V \to W$ be the projection map from definition 4.13. The map $\mathcal{E}_z : V \to W[[z]]$ defines a structure of a nonsingular twisted vertex algebra $(V, W, \pi_f = \pi_T, Y)$ by:*

$$Y(a,z)\pi_T(b) = \mathcal{E}_z(a)\cdot\pi_T(b) \quad \textit{for any} \quad a, b \in V. \tag{4.22}$$

4.4. Vertex operators, analytic continuations, OPEs and normal ordered products from a bicharacter.

In this subsection we combine together the bicharacter construction to produce fields and vertex algebras from a bicharacter. A space of fields in the vertex algebra is given by the free Leibnitz module $V = H^N_{T_\epsilon}(M))$, a space of states by the free Leibnitz module $W = H_D(M) \subset V$. This is part of the data needed for a twisted vertex algebra. Now we will define the fields and the field-state correspondence via a bicharacter.

Definition 4.19. (Two-variable fields from a bicharacter) *Let M be a commutative cocommutative Hopf algebra, let V be the free Leibnitz module $V = H^N_{T_\epsilon}(M)$, r a $H^N_{T_\epsilon} \otimes H^N_{T_\epsilon}$-covariant bicharacter on V with values in $\mathbf{F}_\epsilon(z,w)^{+,w}$, $W = H_D(M)$ be the free H_D-Leibnitz sub-module-algebra of V. Let $\mathcal{E}_z$ be the exponential map $\mathcal{E}_z : V \to W[[z]]$ defined in 4.14. Define a singular multiplication map*

$$X_{z,w}\colon V^{\otimes 2} \to W[[z,w]] \otimes \mathbf{F}^N_\epsilon(z,w), \tag{4.23}$$

by

$$X_{z,w}(a \otimes b) = \sum (-1)^{\tilde{a''}\tilde{b'}} (\mathcal{E}_z a')(\mathcal{E}_w b') r_{z,w}(a'' \otimes b''), \tag{4.24}$$

where a, b are homogeneous elements of the super space V. The map $X_{z,w}$ is extended by linearity to the whole of V.

Definition 4.20 (Vertex operators $Y(a,z)$ and field-state correspondence). *Let V, W, $\mathcal{E}_z$ be as above, $\pi_T : V \to W$ be the projection map defined in 4.13. Define the vertex operator $Y(a,z)$ associated to $a \in V$ by*

$$Y(a,z)\pi_T(b) = X_{z,0}(a \otimes b) = \sum (-1)^{\tilde{a''}\tilde{b'}} (\mathcal{E}_z a')\pi_T(b') r_{z,0}(a'' \otimes b''), \tag{4.25}$$

for any $b \in V$. $Y(a,z)$ is a field on W and the map $Y : a \in V \to Y(a,z)$ is a field-state correspondence for the twisted vertex algebra with space of fields V and space of states W.

Here we are implicitly using the modified creation property of the exponential map (lemma 4.16).

Remark 4.21. This definition consistent, i.e., for each $\bar{b} \in W$ the vertex operator acting on $\bar{b}$ is independent from the choice of the $b \in V$ used in the definition, due to the following: If $\bar{b} = \pi_T(b_1) = \pi_T(b_2)$, then from the $H^N_{T_\epsilon} \otimes H^N_{T_\epsilon}$-covariance of the bicharacter r it follows that $r_{z,0}(a \otimes b_1) = r_{z,0}(a \otimes b_2) = r_{z,0}(a \otimes \bar{b})$. Also, since the map $\pi_T(b)$ is a surjection, this definition is sufficient for any $\bar{b} \in W$.

Remark 4.22. Note that **any** $H^N_{T_\epsilon} \otimes H^N_{T_\epsilon}$-covariant bicharacter r will produce a (different) field-state correspondence, thus even with the same space of fields V and space of states W we can get a variety of examples of field-state correspondences by choosing different bicharacters on V.

Lemma 4.23. (n = 2 *Analytic continuation*) *Let V, W, $\mathcal{E}_z$, $\pi_T : V \to W$ be as above. We have for any $a, b \in V$*

$$Y(a,z)1 = \mathcal{E}_z a, \tag{4.26}$$

$$i_{z,w} X_{z,w}(a \otimes b) = Y(a,z)\mathcal{E}_w b = Y(a,z)Y(b,w)1. \tag{4.27}$$

Proof. (4.26) follows from the vacuum property of the exponential map (see lemma 4.16). Note that since $W = H_D(M)$ is a Hopf subalgebra of V, the unit 1 is in W. Thus we have

$$Y(a,z)1 = \sum(-1)^0(\mathcal{E}_z a')1 r_{z,0}(a''\otimes 1) = \sum(\mathcal{E}_z a')\eta(a'') = \mathcal{E}_z a$$

The equation (4.27) follows from the "compatibility with bicharacters" and the "grouplike" properties of the exponential map (see lemma 4.16):

$$\begin{aligned} i_{z,w}X_{z,w}(a\otimes b) &= \sum(-1)^{\tilde{a''}\tilde{b'}}(\mathcal{E}_z a')(\mathcal{E}_w b')i_{z,w}r_{z,w}(a''\otimes b'') = \\ &= \sum(-1)^{\tilde{a''}\tilde{b'}}(\mathcal{E}_z a')(\mathcal{E}_w b')r_{z,0}(a''\otimes \mathcal{E}_w b'') = \\ &= \sum(-1)^{\tilde{a''}\tilde{b'}}(\mathcal{E}_z a')(\mathcal{E}_w b)' r_{z,0}(a''\otimes (\mathcal{E}_w b)'') = Y(a,z)\mathcal{E}_w b, \end{aligned}$$

since $\pi_T(\mathcal{E}_w b) = \mathcal{E}_w b$ for any $b\in V$. □

Thus we have established a field-state correspondence Y, and in fact have also proved that it satisfies the analytic continuation property of twisted vertex algebras for $n = 2$. It is immediate that this field-state correspondence satisfies also the following required properties for a twisted vertex algebra:

Lemma 4.24. *Let V, W, $\mathcal{E}_z$, $\pi_f = \pi_T : V \to W$, $Y : a\in V \to Y(a,z)$ be as above. This data satisfies the following properties:*

- *vacuum axiom:* $Y(1,z) = Id_W$;
- *modified creation axiom:* $Y(a,z)|0\rangle|_{z=0} = \pi_f(a)$, *for any* $a\in V$;
- *transfer of action:* $Y(ha,z) = h_z\cdot Y(a,z)$ *for any* $h\in H^N_{T_\epsilon}$.

The vacuum property and the modified creation property follow immediately from the corresponding properties of the exponential map. For the transfer of action property:

$$Y(ha,z)\pi_T(b) = X_{z,0}(ha\otimes b) = \sum(-1)^{\tilde{a''}\tilde{b'}}(\mathcal{E}_z(ha)')\pi_T(b')r_{z,0}((ha)''\otimes b''),$$

which from the property of free Leibnitz modules equals

$$\sum(-1)^{\tilde{a''}\tilde{b'}}(\mathcal{E}_z(h'a')\pi_T(b')r_{z,0}(h''a''\otimes b'').$$

From the transfer of action property of the exponential map, and the covariance of the bicharacter we have

$$\begin{aligned} &= \sum(-1)^{\tilde{a''}\tilde{b'}}\big((h')_z\cdot \mathcal{E}_z(a')\big)\pi_T(b')\big((h'')_z\cdot r_{z,0}(a''\otimes b'')\big) = \\ &= h_z\cdot\big(\sum(-1)^{\tilde{a''}\tilde{b'}}(\mathcal{E}_z(a)')\pi_T(b')r_{z,0}((a)''\otimes b'')\big) = h_z\cdot\big(Y(a,z)\pi_T(b)\big). \end{aligned}$$ □

One of the main advantages of the bicharacter construction is that there are explicit formulas for all the analytic continuation maps $X_{z_1,\dots,z_n}$ in terms of the bicharacter, similar to the formula (4.24). We will start with the formula for X_{z_1,z_2,z_3} as it is needed also for the Operator Product Expansions.

Definition 4.25. (Three-variable fields from a bicharacter) *Let V, W, $\mathcal{E}_z$, $\pi_T : V \to W$ be as above. Let a, b, c be arbitrary homogeneous elements of the super space V. Define the three variable field*

$$X_{z_1,z_2,z_3} \colon V^{\otimes 3} \to W[[z_1, z_2, z_3]] \otimes \mathbf{F}^N_{\epsilon}(z_1, z_2, z_3), \tag{4.28}$$

by

$$X_{z_1,z_2,z_3}(a \otimes b \otimes c) =$$
$$= \sum (-1)^{f(\tilde{a},\tilde{b},\tilde{c})} \mathcal{E}_{z_1} a^{(1)} \mathcal{E}_{z_2} b^{(1)} \mathcal{E}_{z_3} c^{(1)} r_{z_1,z_2}(a^{(2)} \otimes b^{(2)}) r_{z_1,z_3}(a^{(3)} \otimes c^{(2)}) r_{z_2,z_3}(b^{(3)} \otimes c^{(3)}),$$

where $f(\tilde{a},\tilde{b},\tilde{c}) = \widetilde{b^{(3)}}(\widetilde{c^{(1)}} + \widetilde{c^{(2)}}) + (\widetilde{a^{(2)}} + \widetilde{a^{(3)}})(\widetilde{b^{(1)}} + \widetilde{c^{(1)}}) + \widetilde{a^{(3)}}\widetilde{b^{(2)}} + \widetilde{b^{(2)}}\widetilde{c^{(1)}}$. *Here as usual we denote $\Delta^2(a) = a^{(1)} \otimes a^{(2)} \otimes a^{(3)}$ for any $a \in V$. The map X_{z_1,z_2,z_3} is extended to the whole of V by linearity.*

Lemma 4.26 ($\mathbf{n = 3}$ Analytic continuation). *Let V, W, $\mathcal{E}_z$, $\pi_T : V \to W$ be as above. We have for any $a, b, c \in V$*

$$i_{z_1,z_2,z_3} X_{z_1,z_2,z_3}(a \otimes b \otimes c) = Y(a, z_1)Y(b, z_2)\mathcal{E}_{z_3} c = Y(a, z_1)Y(b, z_2)Y(c, z_3)1.$$

Proof. Again from the "compatibility with bicharacters" and the "grouplike" property of the exponential map, and Definition 4.20 we have

$$\begin{aligned}
& i_{z_1,z_2,z_3} X_{z_1,z_2,z_3}(a \otimes b \otimes c) = \\
& = \sum (-1)^{f(\tilde{a},\tilde{b},\tilde{c})} \mathcal{E}_{z_1} a^{(1)} \mathcal{E}_{z_2} b^{(1)} \mathcal{E}_{z_3} c^{(1)} \cdot \\
& \qquad \cdot i_{z_1,z_2} r_{z_1,z_2}(a^{(2)} \otimes b^{(2)}) i_{z_1,z_3} r_{z_1,z_3}(a^{(3)} \otimes c^{(2)}) i_{z_2,z_3} r_{z_2,z_3}(b^{(3)} \otimes c^{(3)}) = \\
& = \sum (-1)^{f(\tilde{a},\tilde{b},\tilde{c})} \mathcal{E}_{z_1} a^{(1)} \mathcal{E}_{z_2} b^{(1)} \mathcal{E}_{z_3} c^{(1)} \cdot \\
& \qquad \cdot r_{z_1,0}(a^{(2)} \otimes \mathcal{E}_{z_2} b^{(2)}) r_{z_1,0}(a^{(3)} \otimes \mathcal{E}_{z_3} c^{(2)}) r_{z_2,0}(b^{(3)} \otimes \mathcal{E}_{z_3} c^{(3)} = \\
& = \sum (-1)^{(\widetilde{b''})''(\tilde{c'} + (\widetilde{c''})') + ((\widetilde{a''})' + (\widetilde{a''})'')(\tilde{b'} + \tilde{c'}) + (\widetilde{b''})'\tilde{c'} + (\widetilde{a''})''(\widetilde{b''})'} \mathcal{E}_{z_1} a' \mathcal{E}_{z_2} b' \mathcal{E}_{z_3} c' \cdot \\
& \qquad \cdot r_{z_1,0}((a'')' \otimes (\mathcal{E}_{z_2}(b'')')) r_{z_1,0}((a'')'' \otimes \mathcal{E}_{z_3}(c'')')) r_{z_2,0}((b'')'' \otimes \mathcal{E}_{z_3}(c'')'' = \\
& = \sum (-1)^{(\widetilde{b''})''(\tilde{c'} + (\widetilde{c''})') + \widetilde{a''}\tilde{b'} + \widetilde{a''}\tilde{c'} + (\widetilde{b''})'\tilde{c'}} \mathcal{E}_{z_1} a' \mathcal{E}_{z_2} b' \mathcal{E}_{z_3} c' \cdot \\
& \qquad \cdot r_{z_1,0}(a'' \otimes (\mathcal{E}_{z_2}((b'')')(\mathcal{E}_{z_3} c'')')) r_{z_2,0}((b'')'' \otimes (\mathcal{E}_{z_3} c'')'') = \\
& = Y(a, z_1)Y(b, z_2)\mathcal{E}_{z_3} c.
\end{aligned}$$

□

Similar formulas can be derived for any $X_{z_1,\ldots,z_n}$, $n \in \mathbf{N}$. Before we give those, as a corollary of the formula above we will derive a formula for Operator Product Expansions (OPEs). Let r be a $H^N_{T_\epsilon} \otimes H^N_{T_\epsilon}$-covariant bicharacter on V, with values in $\mathbf{F}^N_{\epsilon}(z, w)^{+,w}$. For any $a, b \in V$ the bicharacter $r_{z,w}(a \otimes b)$ is just a function of z and w in $\mathbf{F}^N_{\epsilon}(z, w)$ and can be expanded as a Laurent series around $z = \epsilon^i w$ for any $i = 0, 1, \ldots, N-1$:

$r_{z,w}(a\otimes b) = \sum_{l=0}^{M_{a,b}-1} \frac{f^{i,l}_{a,b}}{(z-\epsilon^i w)^{l+1}} + reg$. We denote by $M_{a,b}$ the order of the pole at $z=\epsilon^i w$ and note that $f^{i,l}_{a,b} = f^{i,l}_{a,b}(w)$ is a function only of w.

Recall we usually omit writing the indexing in $\triangle(a) = \sum_p a'_p\otimes a''_p$, and write it just as $\triangle(a) = \sum a'\otimes a''$ to unclutter notation, but this summation is always implicitly present.

Theorem 4.27. (Bicharacter formula for the Residues) *Let V, W, $\mathcal{E}_z$ and π_T be as above. Let r be a $H^N_{T_\epsilon}\otimes H^N_{T_\epsilon}$-covariant bicharacter on V, with values in $\mathbf{F}^N_\epsilon(z,w)$, denote $M_{pq} = M_{a''_p,b''_q}$. For any $a,\ b\in V$ and any $0\leq k\leq M_{pq}-1$ we have*

$$\mathrm{Res}_{z=\epsilon^i w} X_{z,w,0}(a\otimes b\otimes c)(z-\epsilon^i w)^k dz =$$

$$= \sum_{p,q}\sum_{l=k}^{M_{pq}-1} (-1)^{\tilde{a''}\tilde{b'}} f^{i,l}_{a'',b''} Y\big((T^i_\epsilon D^{(l-k)}a').b', w\big)\pi_T(c).$$

Proof. By using coassociativity and cocommutativity we have from 4.25

$$X_{z,w,0}(a\otimes b\otimes c) = \sum_{p,q,r}(-1)^{(\tilde{b''})''(\tilde{c'}+(\tilde{c''})')+((\tilde{a''})'+(\tilde{a''})'')(\tilde{b'}+\tilde{c'})+(\tilde{a''})''(\tilde{b''})'+(\tilde{b''})'\tilde{c'}}\cdot$$

$$\cdot(\mathcal{E}_z a')(\mathcal{E}_w b')\pi_T(c')r_{z,w}((a'')'\otimes(b'')'))r_{z,0}((a'')''\otimes c'')')r_{w,0}((b'')''\otimes(c'')'') =$$

$$= \sum_{p,q,r}(-1)^{(\tilde{b''})'(\tilde{c'}+(\tilde{c''})')+((\tilde{a''})''+(\tilde{a''})')(\tilde{b'}+\tilde{c'})+(\tilde{a''})'(\tilde{b''})''+(\tilde{b''})''\tilde{c'}+(\tilde{a''})'(\tilde{a''})''+(\tilde{b''})'(\tilde{b''})''}\cdot$$

$$\cdot(\mathcal{E}_z a')(\mathcal{E}_w b')\pi_T(c')r_{z,w}((a'')''\otimes(b'')''))r_{z,0}((a'')'\otimes c'')')r_{w,0}((b'')'\otimes(c'')'')$$

$$= \sum_{p,q,r}(-1)^{(\tilde{b'})''(\tilde{c'}+(\tilde{c''})')+(\tilde{a''}+(\tilde{a'})'')((\tilde{b'})'+\tilde{c'})+(\tilde{a'})''\tilde{b''}+\tilde{b''}\tilde{c'}+(\tilde{a'})''\tilde{a''}+(\tilde{b'})''\tilde{b''}}\mathcal{E}_z(a')'\mathcal{E}_w(b')'\cdot$$

$$\cdot\pi_T(c')\cdot r_{z,w}(a''\otimes b''))r_{z,0}((a')''\otimes(c'')')r_{w,0}((b')''\otimes(c'')'').$$

Note that $r_{z,0}((a')''\otimes(c'')')$ is nonsingular at $z=\epsilon^i w$, and therefore can be expanded in a power series in $(z-\epsilon^i w)$:

$$r_{z,0}((a')''\otimes(c'')') = \sum_{j\geq 0}\big((\partial_z)^{(j)}r_{z,0}((a')''\otimes c'')')\big)|_{z=\epsilon^i w}(z-\epsilon^i w)^j$$

$$= \sum_{j\geq 0} r_{\epsilon^i w,0}(D^{(j)}(a')''\otimes c'')'))(z-\epsilon^i w)^j = \sum_{j\geq 0} r_{w,0}(T^i_\epsilon D^{(j)}(a')''\otimes c'')'))(z-\epsilon^i w)^j$$

$$= \sum_{j\geq 0}\epsilon^{-ij} r_{w,0}(D^{(j)}T^i_\epsilon(a')''\otimes c'')'))(z-\epsilon^i w)^j.$$

We used the fact that the bicharacter is $H^N_{T_\epsilon}$ covariant. Next we use the modified expansion property of the exponential map, see lemma 4.16, and the fact that $f^l_{a'',b''}=0$ unless $\tilde{a''}=\tilde{b''}$, as the bicharacters are even.

$$\mathrm{Res}_{z=\epsilon^i w}\, X_{z,w,0}(a\otimes b\otimes c)(z-\epsilon^i w)^k$$

$$=\sum_{p,q,r}(-1)^{f(\tilde{a},\tilde{b},\tilde{c})+(\tilde{a'})''\tilde{a''}+(\tilde{b'})''\tilde{b''}} r_{w,0}((b')''\otimes(c'')'')\cdot$$

$$\cdot\, Res_{z=\epsilon^i w}\big((\mathcal{E}_z(a')')(\mathcal{E}_w(b')')\pi_T(c')(\sum_j \epsilon^{-ij} r_{w,0}(D^{(j)}T^i_\epsilon(a')''\otimes(c'')'))(z-\epsilon^i w)^{j+k})\cdot$$

$$\cdot\, r_{z,w}(a''\otimes b''))\big)$$

$$=\sum_{p,q,r}(-1)^{f(\tilde{a},\tilde{b},\tilde{c})+(\tilde{a'})''\tilde{a''}+(\tilde{b'})''\tilde{b''}} r_{w,0}((b')''\otimes(c'')'')\cdot$$

$$\cdot\, Res_{z=\epsilon^i w}\big(\sum_{n,j\geq 0}\epsilon^{-i(n+j)}(z-\epsilon^i w)^{n+j+k}\mathcal{E}_w(D^{(n)}Ta')')(\mathcal{E}_w(b')')\pi_T(c')\cdot$$

$$\cdot\, r_{w,0}(D^{(j)}(a')''\otimes(c'')')r_{z,w}(a''\otimes b''))\big)=$$

$$=\sum_{p,q,r}(-1)^{(\tilde{b'})''(\tilde{c'}+(\tilde{c''})')+(\tilde{a''}+(\tilde{a'})'')((\tilde{b'})'+\tilde{c'})+(\tilde{a'})''\tilde{b''}+\tilde{b''}\tilde{c'}+(\tilde{a'})''\tilde{a''}+(\tilde{b'})''\tilde{b''}}\cdot$$

$$\cdot\sum_{l=k}^{M_{pq}-1}\mathcal{E}_w((D^{(l-k)}Ta')')(\mathcal{E}_w(b')')\pi_T(c')r_{w,0}(D^{(l-k)}(Ta')''\otimes(c'')'))\cdot$$

$$\cdot\, r_{w,0}((b')''\otimes c'')''))))\big)f^k_{a'',b''}=$$

$$=\sum_{p,q,r}(-1)^{(\tilde{b'})''\tilde{c'}+\tilde{a''}((\tilde{b'})'+\tilde{c'})+(\tilde{a'})''\tilde{c'}+(\tilde{a'})''\tilde{b''}+\tilde{b''}\tilde{c'}+(\tilde{a'})''\tilde{a''}+(\tilde{b'})''\tilde{b''}}\cdot$$

$$.\Big(\sum_{l=k}^{M_{pq}-1}\big(\mathcal{E}_w((D^{(l-k)}Ta')(b'))')\pi_T(c')r_{w,0}((D^{(l-k)}Ta'b')''\otimes c'')f^l_{a'',b''}\Big)=$$

$$=\sum_{p,q,r}\sum_{l=k}^{M_{pq}-1}(-1)^{((\tilde{b'})''+(\tilde{a'})\tilde{c'}+\tilde{a''}\tilde{b'}}\mathcal{E}_w\big(((D^{(l-k)}a')(b'))'\pi_T(c')\big)\cdot$$

$$\cdot\, r_{w,0}((D^{(l-k)}Ta'b')''\otimes c'')f^k_{a'',b''}=$$

$$=\sum_{p,q}\sum_{l=k}^{M_{pq}-1}(-1)^{\tilde{a''}\tilde{b'}}f^l_{a'',b''}Y((D^{(l-k)}Ta').b',w)\pi_T(c).$$

□

Corollary 4.28. *Let V, W, r be as above, let again $M_{pq} = M_{a''_p, b''_q}$. For any $a,\ b\in V$ we have*

$$Y(a,z)Y(b,w) = i_{z,w}\sum_{p,q}\sum_{k=0}^{M_{pq}-1}\frac{\sum_{l=M_{p,q}-1-k}^{M_{p,q}-1}(-1)^{\tilde{a''}\tilde{b'}}f^{i,l}_{a'',b''}Y\big((T^i_\epsilon D^{(l-k)}a').b',w\big)}{(z-\epsilon^i w)^{k+1}} + Reg^{\epsilon^i}_{(z,w)}(a\otimes b).$$

The term $Reg^{\epsilon^i}_{(z,w)}(a\otimes b)$ denotes the regular part in the Laurent expansion above, it depends on $a, b\in V$, z and w, and $i\in\{0,1,\dots N-1\}$.

Note that $Reg^{\epsilon^i}_{(z,w)}(a\otimes b)$ is non-singular for $z=\epsilon^i w$, but it can still be singular at $z=\epsilon^j w$ for $j\neq i$, and thus $Reg^{\epsilon^i}_{(z,w)}(a\otimes b)$ can potentially differ from the normal ordered product $:a(z)b(w):$. On the other hand, as the normal ordered product (2.11) is the regular part of an OPE of two fields, it is clear that if we have a single pole at $z=\epsilon^i w$, then $Reg^{\epsilon^i}_{(z,w)}(a\otimes b) = :a(z)b(w):$. (For proof of that the normal ordered product is the regular part of the OPE of two fields in this more general context, see [2]). The above formula simplifies in the case of simple pole:

Corollary 4.29. (Bicharacter formula for OPEs for simple poles) *Let V, W, r be as above, and let $a,\ b\in V$ are such that the bicharacters $r_{z,w}(a''\otimes b'')$ have at most simple poles at each a'', b''. Then*

$$Y(a,z)Y(b,w) = i_{z,w}\sum_{p,q}\sum_i(-1)^{\tilde{a''}\tilde{b'}}f^{i,0}_{a'',b''}\frac{Y((T^i a').b',w)}{z-\epsilon^i w} + :a(z)b(w): . \tag{4.29}$$

We derived formulas for the analytic continuations of products of two fields, as well as formulas for the OPEs and normal order products via the bicharacter. These formulas always hold for any $H^N_{T_\epsilon}\otimes H^N_{T_\epsilon}$-covariant bicharacter on V, but if we want the vertex operators given by these formulas to satisfy all the axioms for a twisted vertex algebra the restriction that remains is the shift restriction, see remark 3.3. Hence we need to impose the following restriction on the bicharacter:

Definition 4.30. (Shift restricted bicharacter) *Let r be a $H^N_{T_\epsilon}\otimes H^N_{T_\epsilon}$-covariant bicharacter on V, with values in $\mathbf{F}^N_\epsilon(z,w)$. Let as above $f^{i,l}_{a,b}$ stands for the coefficient in the expansion*

$$r_{z,w}(a\otimes b) = \sum_{l=0}^{M-1}\frac{f^{i,l}_{a,b}(w)}{(z-\epsilon^i w)^{l+1}} + \sum_{l=0}^{\infty} f^{i,-l-1}_{a,b}(w)\cdot(z-\epsilon^i w)^l$$

as a Laurent series in $(z-\epsilon^i w)$. We call the bicharacter r ***shift-restricted*** *if for any $a, b \in V$ $\ f_{a,b}^{i,l} = c_{a,b}^{i,l} \cdot w^{s_{a,b}^{i,l}}$, where $c_{a,b}^{i,l}$ is a constant, $c_{a,b}^{i,l} \in \mathbb{C}$, and $s_{a,b}^{i,l} \in \mathbb{Z}$.*

Examples of shift-restricted bicharacters are given by functions in $\mathbf{F}_\epsilon^N(z,w)$ which have separately homogeneous numerators and denominators.

Lemma 4.31. (Bicharacter formula for the normal order product for simple poles) *Let V, W, r be as above, and let $a,\ b \in V$ are such that the bicharacters $r_{z,w}(a''_k \otimes b''_l)$ have at most simple poles at each a''_k, b''_l. Then the normal order product $:Y(a,z)Y(b,z):$ of the fields $Y(a,z)$ and $Y(b,w)$ is given by*

$$:Y(a,z)Y(b,z) := \sum_i \sum_{k,l} c^{i,-1}_{a''_k, b''_l} (-1)^{\tilde{a}''_k \tilde{b}'_l} Y(a'_k . b'_l, z) \tag{4.30}$$

We want to finish this section by giving the general formulas for multi-variable fields and analytic continuation of products of fields via bicharacter. Recall the extended Sweedler notation for an element a in a commutative and cocommutative Hopf algebra, $n \in \mathbb{N},\ n \geq 2$:

$$\triangle^{n-1}(a) = \sum_s \ a_s^{(1)} \otimes a_s^{(2)} \otimes \ldots \otimes a_s^{(n)}, \tag{4.31}$$

which again we will often write omitting the index s to shorten the notation.

Notation 4.32. (Coproduct matrices)
Let $a_1, a_2, \ldots, a_n$ be n elements of a commutative and cocommutative Hopf algebra. We can arrange the terms of the l-coproducts of these elements as sets of n by $(l+1)$ matrices $M^{\vec{k}}_{\Delta^l}(a_1, a_2, \ldots, a_n) = ((a_i^{(j)})_{\vec{k}})_{j=1}^{l+1}$, where $\vec{k} = (k_1, k_2, \ldots, k_n)$ is the coproduct index.

Note that this is not one matrix, but a set of matrices, indexed by $\vec{k}$, with cardinality dependent on how many different elements of the coproducts of $a_1, a_2, \ldots, a_n$ there are. Since we are going to encounter a lot of sign contributions, we introduce the following notation:

Notation 4.33. (Sign notation) *Let $M = (m_{ij})_{i,j=1}^n$ be an n by n matrix with elements $m_{ij} \in V$. Let $\boldsymbol{sign}(M) = \boldsymbol{sign}((m_{ij})_{i,j=1}^n)$ denote the following sign value:*

$$\boldsymbol{sign}(M) = (-1)^{\sum_{i=2}^n \sum_{j=1}^{i-1} \sum_{k=2}^n \tilde{m}_{i1} \tilde{m}_{jk}} \cdot (-1)^{\sum_{i=2}^n \sum_{j=2}^n (\sum_{k<i} \sum_{l \geq i+j-k} \tilde{m}_{ij} \tilde{m}_{kl})}.$$

If $l = n-1$ in the coproduct matrices above, $M^{\vec{k}}_{\Delta^{n-1}}(a_1, a_2, \ldots, a_n)$ are square matrices, and we can calculate $\mathbf{sign}(M^{\vec{k}}_{\Delta^{n-1}}(a_1, a_2, \ldots, a_n))$ for each one.

Example 4.34. Let $M = \mathbb{C}\{\phi\}$, as in example 4.11, ϕ is odd. We have $\Delta(\phi) = \phi\otimes 1 + 1\otimes\phi$. Thus there are two matrices $M^{\vec{k}}_{\Delta}(\phi, 1)$, $\vec{k} \in \{(1,1),\ (2,1)\}$: $M^{(1,1)}_{\Delta}(\phi,1) = \begin{pmatrix} \phi & 1 \\ 1 & 1 \end{pmatrix}$, $\quad M^{(2,1)}_{\Delta}(\phi,1) = \begin{pmatrix} 1 & \phi \\ 1 & 1 \end{pmatrix}$. Both of them have $\mathbf{sign} = 1$.

There are four matrices $M^{\vec{k}}_{\Delta}(\phi,\phi)$, i.e., $\vec{k} \in \{(1,1),\ (1,2),\ (2,1),\ (2,2)\}$:
$M^{(1,1)}_{\Delta}(\phi,\phi) = \begin{pmatrix} \phi & 1 \\ \phi & 1 \end{pmatrix}$, $\quad M^{(1,2)}_{\Delta}(\phi,\phi) = \begin{pmatrix} \phi & 1 \\ 1 & \phi \end{pmatrix}$,
$M^{(2,1)}_{\Delta}(\phi,\phi) = \begin{pmatrix} 1 & \phi \\ \phi & 1 \end{pmatrix}$, $\quad M^{(2,2)}_{\Delta}(\phi,\phi) = \begin{pmatrix} 1 & \phi \\ 1 & \phi \end{pmatrix}$.
We have $\mathbf{sign}(M^{(1,1)}_{\Delta}) = \mathbf{sign}(M^{(1,2)}_{\Delta}) = \mathbf{sign}(M^{(2,2)}_{\Delta}) = 1$, $\mathbf{sign}(M^{(2,1)}_{\Delta}) = -1$.

We need the following definition:

Definition 4.35. (n-characters) *Let M be a commutative and cocommutative Hopf algebra, and let $r : M \otimes M \to \mathbf{F}^N_{\epsilon}(z,w)$ be a super bicharacter on M. Let $a_1, a_2, \dots, a_n$ be n elements of M. Define an n-character $r_n : M^{\otimes n} \to \mathbf{F}^N_{\epsilon}(z_1, z_2, \dots, z_n)$ by*

$$
\begin{aligned}
&r_{z_1,z_2,\dots,z_n}(a_1 \otimes a_2 \otimes \ldots \otimes a_n) = \\
&= \sum_{coproducts} r_{z_1,z_2}(a_1^{(1)} \otimes a_2^{(1)}) r_{z_1,z_3}(a_1^{(2)} \otimes a_3^{(1)}) \ldots r_{z_1,z_n}(a_1^{(n-1)} \otimes a_n^{(1)}) \cdot \\
&\cdot r_{z_2,z_3}(a_2^{(2)} \otimes a_3^{(2)}) \ldots r_{z_2,z_n}(a_2^{(n-1)} \otimes a_n^{(2)}) \ldots r_{z_{n-1},z_n}(a_{n-1}^{(n-1)} \otimes a_n^{(n-1)}).
\end{aligned}
$$

In particular, a tri-character $r_3 : M \otimes M \otimes M \to \mathbf{F}^N_{\epsilon}(z_1, z_2, z_3)$ is given by

$$
\begin{aligned}
&r_{z_1,z_2,z_3}(a_1 \otimes a_2 \otimes a_3) = \sum_{coproducts} r_{z_1,z_2}(a_1' \otimes a_2') r_{z_1,z_3}(a_1'' \otimes a_3') r_{z_2,z_3}(a_2'' \otimes a_3'') = \\
&= \sum_{k_1,k_2,k_3} r_{z_1,z_2}((a_1')_{k_1} \otimes (a_2')_{k_2}) r_{z_1,z_3}((a_1'')_{k_1} \otimes (a_3')_{k_3}) r_{z_2,z_3}((a_2'')_{k_2} \otimes (a_3'')_{k_3}).
\end{aligned}
$$

Definition 4.36. (Multivariable fields from a bicharacter) *Let V, W, $\mathcal{E}_z$, $\pi_T : V \to W$ be as above. Let $a_1, a_2, \dots, a_n$ be any n homogeneous elements of V. Define the n-variable field*

$$
X_{z_1,z_2,\dots,z_n} : V^{\otimes n} \to W[[z_1, z_2, \dots, z_n]] \otimes \mathbf{F}^N_{\epsilon}(z_1, z_2, \dots, z_n), \tag{4.32}
$$

by

$$
\begin{aligned}
&X_{z_1,z_2,\dots,z_n}(a_1 \otimes a_2 \otimes \ldots \otimes a_n) = \\
&= \sum_{\vec{k}} \boldsymbol{sign}(M^{\vec{k}}_{\Delta^{n-1}}(a_1, a_2, \dots, a_n)) \mathcal{E}_{z_1} a_1' \mathcal{E}_{z_2} a_2' \cdots \mathcal{E}_{z_n} a_n' \cdot \\
&\qquad \cdot r_{z_1,z_2,\dots,z_n}(a_1'' \otimes a_2'' \otimes \ldots \otimes a_n'')
\end{aligned}
$$

Lemma 4.37. *(* ***Analytic continuation****) Let V, W, $\mathcal{E}_z$, $\pi_T : V \to W$ be as above. We have*

$$i_{z_1,z_2,\dots,z_n} X_{z_1,z_2,\dots,z_n}(a_1 \otimes a_2 \otimes \dots \otimes a_n) = Y(a_1, z_1)Y(a_2, z_2)\dots \mathcal{E}_{z_n} c = \\ = Y(a_1, z_1)Y(a_2, z_2)\dots Y(a_n, z_n)1$$

for any $a_1, a_2, \dots, a_n \in V$.

Proof. Very similar to the proof of lemma 4.26. □

Remark 4.38. (**Keeping track of sign contributions with variables**) When dealing with the multivariable fields the use of the **sign** notation can be restated as follows: In the definition 4.36 one can think of the variables $z_1, z_2, \dots, z_n$ as "attached" to the arguments $a_1, a_2, \dots, a_n$ and correspondingly their coproducts. One then multiplies by a minus sign whenever an **odd** element with attached variable z_j appears before ("transposes") another **odd** element with attached variable z_i, where $i < j$. There is no sign contribution unless both elements are odd.

We summarize the entire construction in the main bicharacter theorem:

Theorem 4.39. *Let M be a commutative cocommutative Hopf algebra, let V be the free Leibnitz module $V = H^N_{T_\epsilon}(M)$, r be a shift-restricted $H^N_{T_\epsilon} \otimes H^N_{T_\epsilon}$-covariant symmetric bicharacter on V with values in in $\mathbf{F}^N_\epsilon(z, w)^{+,w}$, $W = H_D(M)$ be the free H_D-Leibnitz sub-module-algebra of V. Let $\pi_T : V \to W$ be the projection map as in definition 4.13 and Y be the field-state correspondence defined by* (4.25)*, via* (4.24)*. The set of data (V, W, π_T, Y) constructed as above satisfies the definition of a twisted vertex algebra for* ***any*** *shift-restricted supercommutative $H^N_{T_\epsilon} \otimes H^N_{T_\epsilon}$-covariant bicharacter on V.*

5. Examples of twisted vertex algebras based on a bicharacter

In most of the examples in the literature vertex operators are presented in terms of generating fields and commutation relations. With the bicharacter construction as we saw in the previous section (Theorem 4.39) one starts instead with the commutative cocommutative Hopf algebra M and its free Leibnitz module $H^N_{T_\epsilon}(M)$; the OPEs and thus the commutation relations are then dictated by the choice of the bicharacter r. Moreover, for each commutative cocommutative Hopf algebra M there are many choices of a symmetric bicharacter r, and so each such pair (M, r) will give rise to a different twisted vertex algebra (V, W, π_T, Y), even if the spaces V and W are the same, as algebras— since the field-state correspondence Y changes with the choice of a bicharacter. This is the case in particular for the fermionic sides of the B and the D-A correspondences: for them the space of fields V and the space of states W coincide as free Leibnitz modules, but the generating fields for each are different, and so one gets highest weight modules of different Clifford algebras. Hence for the bicharacter construction examples are grouped based on the Hopf algebra M, i.e. one starts by keeping M the same, but changing the bicharacter r on M. We want to stress the fact that there is a variety of examples even after we fix the algebra M. In the following we will list the examples grouped by the underlying Hopf algebra M.

5.1. Twisted vertex algebras based on $\mathbb{C}\{\phi\}$ and a choice of a bicharacter.

Fix $M = \mathbb{C}\{\phi\}$ from example 4.11. To define a $H^N_{T_\epsilon} \otimes H^N_{T_\epsilon}$-covariant bicharacter on $H^N_{T_\epsilon}(M)$, one is only allowed to chose $r_{z,w}(\phi \otimes \phi)$, as all the other values of the bicharacter on $H^N_{T_\epsilon}(\mathbb{C}\{\phi\})$ would be in turn determined by the covariance and the bicharacter properties (see section 4.1). Thus a twisted vertex algebra V based on $M = \mathbb{C}\{\phi\}$ will be determined entirely by a supersymmetric bicharacter value $r_{z,w}(\phi \otimes \phi)$.

Let us specialize further, and consider the case of $N = 2$–twisted vertex algebra ($\epsilon = -1$). From example 4.11, the free Leibnitz module $H^2_{T_{-1}}(\mathbb{C}\{\phi\})$ is isomorphic to $H_D(\mathbb{C}\{\phi, T\phi\})$. Before proceeding to specific examples (dependent on the value of $r_{z,w}(\phi \otimes \phi)$), we want to present a formula for the vacuum expectation values valid for any choice of $r_{z,w}(\phi \otimes \phi)$.

5.2. Twisted vertex algebras based on $\mathbb{C}\{\phi\}$: Pfaffian vacuum expectation values.

Let $\langle \ | \ \rangle : W \otimes W \to \mathbb{C}$ be a symmetric bilinear form on the space of states $W = H_D(\mathbb{C}\{\phi\})$. There is a very important concept of an invariant bilinear form on a vertex algebra, for details see for example [27] and [36]. It is not our goal here to define a general invariant bilinear form for a twisted vertex algebra, but for our bicharacter construction we will require that any bilinear form is such that the vacuum vector $1 = |0\rangle$ is orthogonal to all other generators of the algebra $W = H_D(\mathbb{C}\{\phi\})$ and has norm 1, i.e.,

$$\langle 1 \mid 1 \rangle = \langle \langle 0| \mid |0\rangle \rangle = 1. \tag{5.1}$$

By abuse of notation we will just write $\langle 0 \mid 0 \rangle$ instead of $\langle \langle 0| \mid |0\rangle \rangle$. We can extend this form to $W((z_1, z_2, \dots)) \otimes W((z_1, z_2, \dots)) \to \mathbb{C}((z_1, z_2, \dots))$ by bilinearity. The values $\langle 0 \mid Y(a, z_1)Y(a, z_2) \dots Y(a, z_n)|0\rangle$ of the bilinear form are usually called vacuum expectation values.

Proposition 5.1. *Let V be a twisted vertex algebra based on $M = \mathbb{C}\{\phi\}$ and a supersymmetric bicharacter r (in particular, $V = H_D(\mathbb{C}\{\phi, T\phi\})$ and $W = H_D(\mathbb{C}\{\phi\})$). Denote by $\phi(z)$ the field $Y(\phi, z)$ produced by definition 4.25, via* (4.24)*. The following formula for the vacuum expectation values holds:*

$$\langle 0 \mid \phi(z_1)\phi(z_2)\ldots\phi(z_{2n})|0\rangle = i_z Pf\big(r_{z_i,z_j}(\phi\otimes\phi)\big)_{i,j=1}^{2n}. \qquad (5.2)$$

Here as usual Pf denotes the Pfaffian of an antisymmetric matrix and i_z stands for the expansion $i_{z_1,z_1,\ldots,z_{2n}}$.

Note that the matrix on the right-hand side is antisymmetric since the bicharacter r is symmetric and ϕ is odd, i.e., $r_{z_i,z_j}(\phi\otimes\phi) = -r_{z_j,z_i}(\phi\otimes\phi)$.

Proof. To calculate the vacuum expectation values we calculate instead the vacuum expectation values of the analytic continuation $X_{z_1,z_2,\ldots,z_{2n}}(\phi\otimes\phi\ldots\phi)$. We will use Lemma 4.37 which gives us a formula for the analytic continuation in terms of the bicharacter. Since ϕ is a primitive element, we have

$$\triangle^{2n}(\phi) = \phi\otimes 1\otimes\ldots\otimes 1 + 1\otimes\phi\otimes\ldots\otimes 1 + 1\otimes 1\otimes\phi\otimes\ldots\otimes 1 + \ldots 1\otimes 1\otimes\ldots\otimes\phi \quad (5.3)$$

We need three observations:

(1) Since for the bilinear form the vacuum vector $1 = |0\rangle$ spans an orthogonal subspace on its own (and in particular is orthogonal to ϕ and its descendants), the only contributions to the vacuum expectation values will come from the terms in the multivariable field where the coproducts have 1 as a first term; the other terms will not contribute. That forces us to work with the $(2n)$-character $r_{z_1,z_2,\ldots,z_{2n}}(\phi\otimes\phi\otimes\ldots\otimes\phi)$.
(2) To continue the previous observation, no sign contribution will come from the first (-1) factor in the $\mathbf{sign}(M^{\vec{k}}_{\Delta^{2n-1}}(\phi,\phi,\ldots,\phi))$ as the only contributing matrices are those with the first columns consisting entirely of 1s (as 1 is even).
(3) Since ϕ is a primitive element we have $r_{z,w}(\phi\otimes 1) = r_{z,w}(1\otimes\phi) = 0$ for any bicharacter. Thus the only contributions in the $(2n)$-character $r_{z_1,z_2,\ldots,z_{2n}}(\phi\otimes\phi\otimes\ldots\otimes\phi)$ will come from the following situation: Consider a matrix $M^{\vec{k}}_{\Delta^{2n-1}}(\phi,\phi,\ldots,\phi)$ with first column entirely consisting of 1s. A nonzero summand in the $(2n)$-character $r_{z_1,z_2,\ldots,z_{2n}}(\phi\otimes\phi\otimes\ldots\otimes\phi)$ will be a product of nonzero bicharacter factors, and that happens when we have a sequence of either $(1,1)$ pairs (trivial, as $r_{z,w}(1\otimes 1) = 1$), or (ϕ,ϕ) pairs (nontrivial). If there is a mixed pair $(1,\phi)$ or $(\phi,1)$ as a factor in a summand, that summand will be 0. So a nonzero summand will have exactly n such nontrivial contributing pairs (ϕ,ϕ) , and each pair forms one bicharacter $r_{z_i,z_j}(\phi\otimes\phi)$. The sign contribution will come only from the nontrivial pairs.

Thus, we have

$$X_{z_1,z_2,\dots,z_{2n}}(\phi\otimes\phi\otimes\ldots\otimes\phi)=$$
$$=\sum_{\text{coproducts}}\mathbf{sign}(M^{\vec{k}}_{\Delta^{2n-1}}(\phi,\phi,\dots,\phi))\mathcal{E}_{z_1}\phi'\mathcal{E}_{z_2}\phi'\cdots\mathcal{E}_{z_{2n}}\phi'\cdot$$
$$\cdot\, r_{z_1,z_2,\dots,z_{2n}}(\phi''\otimes\phi''\otimes\ldots\otimes\phi'')=$$
$$=\sum_{\substack{\text{contr.}\\ \text{coproducts}}}\mathbf{sign}(M^{\vec{k}}_{\Delta^{2n-1}}(\phi,\phi,\dots,\phi))1\cdot r_{z_1,z_2,\dots,z_{2n}}(\phi\otimes\phi\otimes\ldots\otimes\phi)+\cdots=$$
$$=\sum_P\epsilon(P)1\cdot r_{z_{i_1},z_{i_2}}(\phi\otimes\phi)r_{z_{i_3},z_{i_4}}(\phi\otimes\phi)\cdots r_{z_{i_{2n-1}},z_{i_{2n}}}(\phi\otimes\phi)+\text{other terms.}$$

The sum is over all permutations such that $i_1<i_2,\ i_3<i_4,\dots,\ i_{2n-1}<i_{2n}$, $i_1<i_3<\cdots<i_{2n-1}$. The sign contribution from any **contributing** matrix $M^{\vec{k}}_{\Delta^{2n-1}}(\phi,\phi,\dots,\phi))$ is precisely the sign of the corresponding permutation, since ϕ is odd (see remark 4.38 and observation 3 above). That produces precisely the Pfaffian $Pf\big(r_{z_i,z_j}(\phi\otimes\phi)\big)_{i,j=1}^{2n}$. □

Next we will consider examples of twisted vertex algebras arising from specific bicharacter values for $r_{z,w}(\phi\otimes\phi)$ by use of Theorem 4.39.

5.3. Twisted vertex algebras based on $\mathbb{C}\{\phi\}$: the neutral free fermion of type B.

We continue working with space of fields $V=H^2_T(\mathbb{C}\{\phi\})\equiv H_D(\mathbb{C}\{\phi,T\phi\})$, and space of states $W=H_D(\mathbb{C}\{\phi\})$. The projection map (recall definition 4.13) in this case is just the algebra homomorphism defined by $\pi_T(T\phi)=\phi$.

Let the covariant bicharacter $r^B : H_D(\mathbb{C}\{\phi,T\phi\})\otimes H_D(\mathbb{C}\{\phi,T\phi\})\to\mathbf{F}_{-1}(z,w)^{+,w}$ be defined by

$$r^B_{z,w}(\phi\otimes\phi)=\frac{z-w}{z+w}\tag{5.4}$$

Note that the bicharacter r^B is symmetric, as it is symmetric on the generator ϕ, it is also shift-restricted, and has a simple single pole. From theorem 4.39 we know that we will get an example of a twisted vertex algebra. We claim that the N=2 twisted vertex algebra corresponding to the pair $(\mathbb{C}\{\phi\},r^B)$ is the free fermion of type B. To prove that, we need to show that the field $\phi^B(z)$ (corresponding to the element ϕ via the field-state correspondence defined by (4.25)) satisfies the commutation relations for the free fermion field of type B that was introduced in section 3.2. We can use corollary 4.29 to calculate the OPE of $\phi^B(z)\phi^B(w)$. The only singular bicharacter from any of the coproducts ϕ'' and ϕ'' is $r^B_{z,w}(\phi\otimes\phi)$. Thus

$$\phi^B(z)\phi^B(w)\sim i_{z,w}\sum(-1)^{\tilde{\phi''}\tilde{\phi'}}f^{1,0}_{\phi'',\phi''}\frac{Y((T\phi').\phi',w)}{(z+w)}\sim$$
$$\sim i_{z,w}(-1)^{\tilde{\phi}\tilde{1}}f^{1,0}_{\phi,\phi}\frac{Y((T1).1,w)}{(z+w)}\sim-\frac{2w\cdot 1}{z+w}$$

This OPE coincides with (3.6) and if we index $\phi^B(z)=\sum_{n\in\mathbf{Z}}\phi_n z^n$, this OPE is well known to correspond to the required anticommutation relations $[\phi^B_m,\phi^B_n]_\dagger=2(-1)^m\delta_{m,-n}1$ of the Clifford algebra Cl_B.

Lemma 5.2. *The normal ordered product field $h(z)$ from (3.7) defined by $\frac{1}{4}(:\phi^B(z)\phi^B(-z):-1)$ corresponds to*

$$h(z)=\frac{1}{4}Y(\phi\cdot T\phi,z),\quad h=\frac{1}{4}\phi\cdot T\phi. \tag{5.5}$$

Proof. From lemma 4.31

$$:\phi^B(z)\phi^B(-z):=:\phi^B(z)T\phi^B(z):=\sum(-1)^{\tilde{\phi''}\tilde{\phi'}}f^{1,-1}_{\phi'',\phi''}Y(\phi'\cdot(T\phi)',z)=$$
$$=(-1)^{\tilde{\phi}\tilde{1}}f^{1,-1}_{\phi,\phi}Y(1\cdot 1,z)+(-1)^{\tilde{\phi}\tilde{1}}f^{1,-1}_{1,1}Y(\phi\cdot T\phi,z)=1_W+Y(\phi\cdot T\phi,z)$$

□

To calculate the OPEs (3.8) one uses theorem 4.27 to get

$$h_\phi(z)h_\phi(w)\sim+Y(1,w)r_{z,w}(h_\phi\otimes h_\phi)\sim\frac{8zw(z^2+w^2)}{(z^2-w^2)^2}. \tag{5.6}$$

Since the fermionic side of of the boson-fermion correspondence of type B is known, see [10] and [37], we omit most of the calculations.

We have $Th=\frac{1}{4}T\phi\cdot\phi=-h$, and since $Y(Th,z)=Y(h,-z)$ from the transfer of action axiom of twisted vertex algebras, we have $Y(h,-z)=-Y(h,z)$ in the twisted vertex algebra. Hence we have only **odd** powers of z in $Y(h,z)$, and the indexing in $h(z)=\sum_{n\in\mathbb{Z}}h_{2n+1}z^{-2n-1}$ is implied. The commutation relations $[h_m,h_n]=\frac{m}{2}\delta_{m+n,0}1$ for the Heisenberg algebra $\mathcal{H}_{\mathbb{Z}+1/2}$ then follow from the OPE in a standard calculation.

Next one needs to show the decomposition of the space of states $F_B=W=H_D(\mathbb{C}\{\phi\})$ into Heisenberg modules, Lemma 3.9 This decomposition was done first in [10] and [37], and the calculations using the bicharacter formulas (4.25) and (4.24) are also available.

To calculate the image of the generating field $\phi^B(z)\mapsto e^\alpha(z)$ we first calculate the OPE $h(z)\phi^B(w)$. From corollary 4.29, as all the poles are simple here we get

$$h(z)\phi^B(w)\sim\frac{1}{4}\Big(\frac{2w}{z-w}+\frac{2w}{z+w}\Big)Y(\phi,w)\sim\frac{zw}{z^2-w^2}\phi^B(w).$$

Once we know the OPE above, and we have the exact description of the split of W into irreducible Heisenberg submodules, we can use the standard calculational lemmas (see for example [23] and [34]) and immediately get that the exponential boson formula (3.10) holds for the field $\phi^B(w)$.

5.4. **Twisted vertex algebras based on $\mathbb{C}\{\phi\}$: the neutral free fermion of type D-A.**

We are again working with space of fields $V = H^2_T(\mathbb{C}\{\phi\}) \equiv H_D(\mathbb{C}\{\phi, T\phi\})$, and space of states $W = H_D(\mathbb{C}\{\phi\})$. The projection map (recall definition 4.13) is again, as in the previous section, the algebra homomorphism defined by $\pi_T(T\phi) = \phi$. We choose a different bicharacter r as follows.

Let the bicharacter $r^D : H_D(\mathbb{C}\{\phi, T\phi\}) \otimes H_D(\mathbb{C}\{\phi, T\phi\}) \to \mathbf{F}_{-1}(z, w)^{+,w}$ be defined by

$$r^D_{z,w}(\phi \otimes \phi) = \frac{1}{z-w} \tag{5.7}$$

The bicharacter r^D is symmetric, it is shift-restricted, and has a simple single pole. From theorem 4.39 the pair $(\mathbb{C}\{\phi\}, r^D)$ will produce an example of a twisted vertex algebra. Since this is the new example of the boson-fermion correspondence of type D-A, we will go carefully over the details.

We use corollary 4.29 to calculate the OPE of $\phi^D(z)\phi^D(w)$. Again, the only nontrivial bicharacter from any of the coproducts ϕ'' and ϕ'' is $r^D_{z,w}(\phi \otimes \phi)$ (ϕ primitive implies $r^D_{z,w}(\phi \otimes 1) = r^D_{z,w}(1 \otimes \phi) = 0$ which is true for any bicharacter and a primitive element). Thus from corollary 4.29

$$\phi^D(z)\phi^D(w) \sim i_{z,w} \sum (-1)^{\tilde{\phi''}\tilde{\phi'}} f^{1,0}_{\phi'',\phi''} \frac{Y((T\phi').\phi', w)}{(z-w)} \sim \frac{1}{z-w}$$

This OPE coincides with (3.11) and corresponds to the anticommutation relations:

$$[\phi^D(z), \phi^D(w)]_\dagger = i_{z,w}\frac{1}{z-w} + i_{w,z}\frac{1}{w-z} = (i_{z,w} - i_{w,z})\frac{1}{z-w}.$$

Using the notation $\delta(z-w) = (i_{z,w} - i_{w,z})\frac{1}{z-w} = \sum_{j\in\mathbb{Z}} z^{-j-1}w^j$ we can write

$$[\phi^D(z), \phi^D(w)]_\dagger = \delta(z-w),$$

which if we index the field $\phi^D(z) = \sum_{n\in\mathbf{Z+1/2}} \phi^D_n z^{-n-1/2}$ will give us anti-commutation relations:

$$[\phi^D_m, \phi^D_n]_\dagger = \delta_{m,-n}1.$$

This field $\phi^D(z)$ is well known as the neutral free fermion (of type D). Since in the OPE of $\phi^D(z)$ the only pole is at $z = w$, it is immediate that $\phi^D(z)$ on its own will generate a super vertex algebra (see e.g. [21], [24], [35]).

$\phi^D(z)$ cannot be bosonized on its own, but as a new ingredient allowed in twisted vertex algebras we consider another descendant of $\phi^D(z)$– the field $T\phi^D(z) = \phi^D(-z)$. In the twisted vertex algebra we now immediately have

$$\phi^D(z)T\phi^D(w) \sim \frac{1}{z+w},$$

which of course is not an OPE allowed in a super vertex algebra. Using the language of delta functions (see e.g. [21], [34], [2]) we get

$$[T\phi^D(z), T\phi^D(w)]_\dagger = -\delta(z-w), \quad [T\phi^D_m, T\phi^D_n]_\dagger = -\delta_{m,-n}1,$$

These mean that on its own each of the fields $\phi^D(z)$ and $T\phi^D(z)$ (without the other) will generate a super vertex algebra, but the two "glue together" to form a twisted vertex algebra. This situation resembles the gluing together of the two sheets of the square root Riemann surface.

Since in a **twisted** vertex algebra we have both the fields $\phi^D(z)$ and $T\phi^D(z)$ and their descendants, we have the following proposition:

Proposition 5.3. *(Proposition 3.13) The field*

$$h(z) = \frac{1}{2} : \phi^D(z)\phi^D(-z) := \frac{1}{2} : \phi^D(z)T\phi^D(z) :, \quad h = \frac{1}{2}\phi \cdot T\phi$$

is a Heisenberg field: it can be indexed as $h(z) = \sum_{n\in\mathbb{Z}} h_n z^{-2n-1}$, and has OPE

$$h(z)h(w) \sim \frac{zw}{(z^2-w^2)^2},$$

i.e., the commutation relations $[h_m, h_n] = m\delta_{m+n,0}1$ for the Heisenberg algebra $\mathcal{H}_\mathbb{Z}$ hold.

Proof. We use lemma 4.31:

$$: \phi^D(z)T\phi^D(z) := \sum(-1)^{\tilde{\phi''}\tilde{\phi'}} f^{1,-1}_{\phi'',\phi''} Y(\phi' \cdot (T\phi)', z) =$$
$$= (-1)^{\tilde{\phi}\tilde{1}} f^{1,-1}_{\phi,\phi} Y(1\cdot 1, z) + (-1)^{\tilde{\phi}\tilde{1}} f^{1,-1}_{1,1} Y(\phi\cdot T\phi, z) = 0\cdot 1_W + Y(\phi\cdot T\phi, z)$$

Thus we have that the field $h(z)$ from (3.12) is actually the vertex operator $Y(\frac{1}{2}\phi\cdot T\phi, z)$ corresponding to the element $\frac{1}{2}\phi\cdot T\phi$. To calculate the OPE that we listed in (3.13) we again use theorem 4.27. And again as in the previous section we notice that the OPE of $h(z)h(w)$ can potentially contain first and second order poles. The second order pole comes from

$$r_{z,w}(h_\phi \otimes h_\phi) = r_{z,w}(\phi\cdot T\phi \otimes \phi \cdot T\phi) = \tag{5.8}$$
$$= -r_{z,w}(\phi\otimes\phi)r_{z,w}(T\phi\otimes T\phi) + r_{z,w}(\phi\otimes T\phi)r_{z,w}(T\phi\otimes\phi) = \tag{5.9}$$
$$= +\frac{1}{(z-w)^2} - \frac{1}{(z+w)^2} = \frac{4zw}{(z^2-w^2)^2}. \tag{5.10}$$

Let us show that there are no first order poles in the OPE. The first order poles in the OPE come from the h'' terms of the coproduct which have first order poles in their bicharacter, namely from $r_{z,w}(\phi\otimes T\phi) = -r_{z,w}(T\phi\otimes\phi) = \frac{1}{z+w}$ and from $r_{z,w}(\phi\otimes\phi) = -r_{z,w}(T\phi\otimes T\phi) = \frac{1}{z-w}$. To use theorem 4.27 we we look at the coproduct :

$$\triangle h_\phi = h_\phi \otimes 1 + 1 \otimes h_\phi + \phi \otimes T\phi - T\phi \otimes \phi$$

The field-coefficients for the OPE coming in front of the first order poles then are:
for $\frac{1}{z-w}$ we get $Y(-\phi\cdot\phi + T\phi\cdot T\phi, w)$, which is zero as $\phi\cdot\phi = 0 = T\phi\cdot T\phi$, for $\frac{z-w}{z+w}$ we get $Y(-T\phi\cdot T\phi + T^2\phi\cdot\phi, w)$, which is zero as $T^2\phi = \phi$. Thus there are no first order poles in the OPEs of $h(z)h(w)$.

Further, we have $Th = \frac{1}{2}T\phi\cdot\phi = -h$, and since $Y(Th, z) = Y(h, -z)$ from the transfer of action axiom of twisted vertex algebras, we have $Y(h, -z) = -Y(h, z)$. Which means that we have only **odd** powers of z in $Y(h, z)$, and we can index it $h(z) = \sum_{n\in\mathbb{Z}} h_n z^{-2n-1}$. With this indexing, we have from the OPE:

$$[h(z), h(w)] = (i_{z,w} - i_{w,z})\frac{zw}{(z^2-w^2)^2} = \sum_{n\in\mathbb{Z}} n\frac{w^{2n-1}}{z^{2n+1}} = \frac{1}{4}\partial_w(\delta(z-w) + \delta(z+w)),$$

which gives us the required commutation relations for the Heisenberg algebra $\mathcal{H}_\mathbb{Z}$. □

Remark 5.4. The field $h(z)$ has a very special property, which is peculiar to twisted vertex algebras, and not possible in a super vertex algebra. This field corresponds to the nonzero element h in the space of fields of the twisted vertex algebra V. But its projection to the space of states is zero, $\pi_f(h) = \pi_T(h) = 0$, as

$$\pi_T(h) = \pi_T(\frac{1}{2}h_\phi) = \pi_T(\frac{1}{2}\phi\cdot T\phi) = \frac{1}{2}\phi\cdot\phi = 0. \tag{5.11}$$

This property was also true for the B case.

Proposition 5.5. *(Proposition 3.14) The space of states F_D can be decomposed as*

$$W = F_D \cong \oplus_{i\in\mathbb{Z}} B_i \cong \mathbb{C}[e^\alpha_\phi, e^{-\alpha}_\phi] \otimes \mathbb{C}[x_1, x_2, \dots, x_n, \dots] = B_D, \tag{5.12}$$

where $\mathbb{C}[e^\alpha_\phi, e^{-\alpha}_\phi]$ denotes the Laurent polynomials with one variable e^α_ϕ. The isomorphism above is as Heisenberg modules, where $e^{n\alpha}_\phi$ is identified as the highest weight vector for the irreducible Heisenberg module B_n with highest weight n.

Proof. In order to decompose the space of states $F_D = W = H_D(\mathbb{C}\{\phi\})$ into Heisenberg submodules we need to identify the highest weight vectors in W.

These are the elements a_λ of W such that $h(z)a_\lambda$ has only nonnegative powers of z. Let $a_n^{\text{even}} = \phi D^{(2)}\phi \cdots D^{(2n)}\phi$, and denote as usual $\phi^i = D^{(i)}\phi$. We claim a_n^{even} is a highest weight vector for $\mathcal{H}_{\mathbb{Z}}$. To prove that, we will be using the bicharacter formula (4.25) and (4.24). We can easily calculate the coproduct of a_n^{even}:

$$\triangle(a_n^{\text{even}}) = \sum_i (-1)^{i-1}\phi\phi^2\cdots\widehat{\phi^{2i}}\cdots\phi^{2n}\otimes\phi^{2i}+ \\ + \sum_{i,j}(-1)^{i+j}\phi\cdots\widehat{\phi^{2i}}\cdots\widehat{\phi^{2j}}\cdots\phi^{2n}\otimes\phi^{2i}\phi^{2j}+\dots$$

The only parts of the coproduct of consequence in this case are the parts with either single or quadratic terms in $(a_n^{\text{even}})''$, as the bicharacter with any term from h'' will be 0 otherwise.

We first consider the even $a_0^{\text{even}} = 1$: We have

$$h(z)1 = \mathcal{E}_z h = \frac{1}{2}e^{zD}\phi\cdot e^{-zD}\phi = \sum_{n\in\mathbb{Z}_{\geq 0}} z^{2n+1}\sum_{p+q=2n+1}(-1)^q D^{(p)}\phi D^{(q)}\phi = \\ = \sum_{n\in\mathbb{Z}_{\geq 0}} z^{2n+1}\sum_{p+q=2n+1}(-1)^q\phi^p\phi^q,$$

since we have $\sum_{p+q=\text{even}}(-1)^q D^{(p)}\phi D^{(q)}\phi = 0$. This of course implies that $a_0^{\text{even}} = 1$ is a highest weight vector, as we see that there are no negative powers of z in $h(z)1$, and also $h_0 1 = 0$, i.e., the highest weight of 1 is 0.

Further, we have:

$$r_{z,0}(\phi T\phi\otimes\phi^i\phi^j) = 0 \quad \text{if } i,j \text{ both even, or } \text{ if } i,j \text{ both odd,}$$
$$r_{z,0}(\phi T\phi\otimes\phi^i\phi^j) = -\frac{2}{z^{i+j+2}} \quad \text{if } i=\text{even},\ j=\text{odd},$$
$$r_{z,0}(\phi T\phi\otimes\phi^i\phi^j) = +\frac{2}{z^{i+j+2}} \quad \text{if } i=\text{odd},\ j=\text{even}.$$

$$h(z)a_n^{\text{even}} = \mathcal{E}_z h\cdot a_n^{\text{even}} + \frac{1}{2}e^{zD}\phi\cdot(\sum(-1)^i\phi\phi^2\phi^4\cdots\widehat{\phi^{2i}}\cdot\cdot\phi^{2n}r_{z,0}(T\phi\otimes\phi^{2i}))- \\ -\frac{1}{2}e^{-zD}\phi\cdot(\sum(-1)^i\phi\phi^2\phi^4\cdots\widehat{\phi^{2i}}\cdot\cdot\phi^{2n})r_{z,0}(\phi\otimes\phi^{2i})+ \\ +\sum_{i,j}(-1)^{2i+2j}1\cdot\phi\phi^2\phi^4\cdots\widehat{\phi^{2i}}\cdots\widehat{\phi^{2j}}\cdots\phi^{2n}r_{z,0}(\phi T\phi\otimes\phi^{2i}\phi^{2j}).$$

Hence

$$h(z)a_n^{\text{even}} = \mathcal{E}_z h\cdot a_n^{\text{even}} - \frac{1}{2}\sum_i(-1)^i(e^{zD}\phi + e^{-zD}\phi)\cdot\left(\phi\phi^2\phi^4\cdots\widehat{\phi^{2i}}\cdot\cdot\phi^{2n}\frac{1}{z^{2i+1}}\right) \tag{5.13}$$

Thus, there will be no nonzero contribution to any power of z less than -1 in (5.13), as we are multiplying by one of the ϕ^{2l} that is already in the product $\phi\phi^2\phi^4\cdots\widehat{\phi^{2i}}\cdot\cdot\phi^{2n}$, and thus getting 0. On the other hand, the contribution to the coefficient in front of z^{-1} is $-2na_n^{\mathrm{even}}$, as we get a z^{-1} precisely when from $\frac{1}{2}(e^{zD}\phi+(-1)^i e^{-zD}\phi)$ we are multiplying by the $\phi^{2i}z^{2i}$ term which exactly complements the $\phi\phi^2\phi^4\cdots\widehat{\phi^{2i}}\cdot\cdot\phi^{2n}$. Also, the minus sign in $-2na_n^{\mathrm{even}}$ is due to the fact that when multiplying $(-1)^i\phi^{2i}\cdot\phi\phi^2\phi^4\cdots\widehat{\phi^{2i}}\cdot\cdot\phi^{2n}$ we get $+a_n^{\mathrm{even}}$. These considerations mean that h_n annihilate a_n^{even} for $n>0$, and $h_0a_n^{\mathrm{even}}=-2na_n^{\mathrm{even}}$. Hence a_n^{even} is a highest weight vector with weight $-2n$.

Also closer observation of the positive powers of z in (5.13) shows which elements of $W=F_D$ can be generated from the highest weight vector a_n^{even}: the elements with $n+2m$ factors, $m\geq 0$, with m factors ϕ^p where p is odd. We see that $W=F_D$ is bi-graded: first by the number n of factors in an element $a=\phi^{k_1}\cdot\phi^{k_2}\cdots\phi^{k_n}$, $k_1<k_2<\cdots<k_n$, and second by the difference between how many of these k_i are odd minus how many of them are even (we will call the second grading "derivative grading"). For example the element $\phi D\phi$ is in the highest weight module generated by the highest vector 1, as it has derivative grading 0 equal to the highest weight of 1.

Now let $a_n^{\mathrm{odd}}=\phi^1\phi^3\cdots\phi^{2n-1}$, $a_1^{\mathrm{odd}}=\phi^1=D\phi$. We claim a_n^{odd} is a highest weight vector for $\mathcal{H}_{\mathbb{Z}}$. Similar calculations as for a_n^{even} show that

$$h(z)a_n^{\mathrm{even}}=\mathcal{E}_zh\cdot a_n^{\mathrm{odd}}+\frac{1}{2}\sum_i(-1)^i(e^{zD}\phi-e^{-zD}\phi)\cdot\Big(\phi^1\phi^3\cdots\widehat{\phi^{2i-1}}\cdot\cdot\phi^{2n-1}\frac{1}{z^{2i}}\Big) \tag{5.14}$$

Similar considerations as for a_n^{even} hold for the a_n^{odd}, the difference is in the minus sign, i.e., a_n^{odd} is a highest weight vector with highest weight n for the Heisenberg algebra $\mathcal{H}_{\mathbb{Z}}$. We also see that any element with derivative grading $n\in\mathbb{Z}$ will be in the highest module with highest weight n.

From the above consideration it follows that the following decomposition of the space of states $F_D=W$ holds:

$$W=F_D\cong\oplus_{i\in\mathbb{Z}}B_i. \tag{5.15}$$

Any highest weight module for $\mathcal{H}_{\mathbb{Z}}$ with highest weight k is isomorphic to $\mathbb{C}[x_1,x_2,\ldots,x_n,\ldots]$ via

$$h_n=n\partial_{x_n}\quad\text{for}\quad n>0;\quad h_{-n}=x_n\cdot\quad\text{for}\quad n>0;\quad h_0=k\cdot\ .$$

Thus we can rewrite

$$W=F_D\cong\oplus_{i\in\mathbb{Z}}B_i\cong\mathbb{C}[e_\phi^{\alpha},e_\phi^{-\alpha}]\otimes\mathbb{C}[x_1,x_2,\ldots,x_n,\ldots]=B_D, \tag{5.16}$$

where $e_\phi^{n\alpha}$, $e_\phi^{-n\alpha}$ are for now just labels for the highest weight vectors, but as we will see in the next section the notation is used because it is coming from a twisted vertex algebra based on a Leibnitz module over a rank one abelian group (see example 4.9). Denote the right hand side of the isomorphism above by B_D, this is the bosonic space of states for the boson-fermion correspondence of type D-A. □

We modify the labeling of the highest weight vectors, as formulas (3.16) and (3.17) of Proposition 3.15 look simpler with such identification. Let

$$e_{\phi}^{n\alpha} \cong \phi^{2n-1}\cdot\phi^{2n-3}\cdots\phi^{3}\cdot\phi^{1} = (-1)^{n-1}a_n^{\text{odd}} \quad n>0 \tag{5.17}$$

$$e_{\phi}^{-n\alpha} \cong \phi^{2n}\cdot\phi^{2n-2}\cdots\phi^{2}\cdot\phi = (-1)^{n}a_n^{\text{even}} \quad n\geq 0 \tag{5.18}$$

To prove Proposition (3.15) we first calculate the OPE $h(z)\phi^D(w)$. We use corollary 4.29, as all the poles are simple here. The coefficients coming in front of the first order poles then are:
for $\frac{1}{z-w}$ we get $-Y(T\phi\cdot 1,w) = -Y(T\phi,w)$; for $\frac{1}{z+w}$ we get $-Y(T\phi\cdot 1,w) = -Y(T\phi,w)$. Thus

$$h(z)\phi^D(w) \sim \frac{1}{2}\Big(-\frac{Y(T\phi,w)}{z-w} - \frac{Y(T\phi,w)}{z+w}\Big) \sim \frac{-z}{z^2-w^2}(T\phi)^D(w),$$
$$h(z)(T\phi)^D(w) \sim \frac{1}{2}\Big(-\frac{Y(\phi,w)}{z-w} - \frac{Y(\phi,w)}{z+w}\Big) \sim \frac{-z}{z^2-w^2}\phi^D(w),$$

and

$$h(z)\frac{1}{2}(\phi^D(w)+T\phi^D(w)) \sim -\frac{z}{z^2-w^2}\frac{1}{2}(\phi^D(w)+T\phi^D(w)),$$
$$h(z)\frac{1}{2}(\phi^D(w)-T\phi^D(w)) \sim \frac{z}{z^2-w^2}\frac{1}{2}(\phi^D(w)-T\phi^D(w)).$$

Denote $e_{\phi}^{-\alpha}(w) = \frac{1}{2}(\phi^D(w)+T\phi^D(w))$, $e_{\phi}^{\alpha}(w) = \frac{1}{2}(\phi^D(w)-T\phi^D(w))$, we have

$$h(z)e_{\phi}^{-\alpha}(w) \sim -\frac{z}{z^2-w^2}e_{\phi}^{-\alpha}(w), \quad h(z)e_{\phi}^{\alpha}(w) \sim \frac{z}{z^2-w^2}e_{\phi}^{\alpha}(w). \tag{5.19}$$

These OPEs immediately imply the commutation relations

$$[h(z),e_{\phi}^{-\alpha}(w)] = -(i_{z,w}-i_{w,z})\frac{z}{z^2-w^2}e_{\phi}^{-\alpha}(w) = -\frac{1}{2}(\delta(z-w)+\delta(z+w))e_{\phi}^{-\alpha}(w),$$
$$[h(z),e_{\phi}^{\alpha}(w)] = (i_{z,w}-i_{w,z})\frac{z}{z^2-w^2}e_{\phi}^{\alpha}(w) = \frac{1}{2}(\delta(z-w)+\delta(z+w))e_{\phi}^{\alpha}(w).$$

From these commutation relations and the exact description of the split of $W=F_D$ into Heisenberg submodules, standard calculational lemmas (see for example [23], [34]) will give us

$$\frac{1}{2}(\phi^D(w)+\phi^D(-w)) = e_\phi^{-\alpha}(z) = \exp(-\sum_{n\geq 1}\frac{h_{-n}}{n}z^{2n})\exp(\sum_{n\geq 1}\frac{h_n}{n}z^{-2n})e_\phi^{-\alpha}z^{-2\partial_\alpha}, \tag{5.20}$$

$$\frac{1}{2}(\phi^D(w)-\phi^D(-w)) = e_\phi^{\alpha}(z) = \exp(\sum_{n\geq 1}\frac{h_{-n}}{n}z^{2n})\exp(-\sum_{n\geq 1}\frac{h_n}{n}z^{-2n})e_\phi^{\alpha}z^{2\partial_\alpha+1}, \tag{5.21}$$

where the operators e_ϕ^α, $e_\phi^{-\alpha}$, z^{∂_α} and $z^{-\partial_\alpha}$ act in an obvious way on the space $F_D \cong B_D$ as in (5.16). This proves Proposition 3.15.

Formulas (5.20) and (5.21) alone completely determine the twisted vertex algebra isomorphism between the two twisted vertex algebras: the fermionic with space of states F_D and the bosonic with space of states B_D. But we can go one step further: in the section 5.5 we will present a bicharacter construction of the bosonic side of the boson-fermion correspondences.

5.5. **Twisted vertex algebras based on $\mathbb{C}[\mathbb{Z}\alpha]$ and a choice of a bicharacter.**

In this section we fix M to be the Hopf algebra $L_1 = \mathbb{C}[\mathbb{Z}\alpha]$, the group algebra of the rank-one free abelian group $\mathbb{Z}\alpha$, as in example 4.9. We constructed the free Leibnitz module $\tilde{V} = H_{T_\epsilon}^N(L_1)$, and its sub-Hopf algebra $\tilde{W} = H_D(L_1)$. If we want to define a $H_{T_\epsilon}^N \otimes H_{T_\epsilon}^N$-covariant bicharacter on $H_{T_\epsilon}^N(M)$ it is clear that we can only choose one bicharacter value, that of $r_{z,w}(e^\alpha \otimes e^\alpha)$, as the bicharacter values on all other elements of $\tilde{V}$ and $\tilde{W}$ are determined by the bicharacter properties and the $H_{T_\epsilon}^N \otimes H_{T_\epsilon}^N$-covariance. Recall (example 4.9) we defined an element $h = (De^\alpha)e^{-\alpha}$, which is primitive; note that $h \in \tilde{W} \subset \tilde{V}$.

Lemma 5.6. *The following hold for any covariant bicharacter on $\tilde{V}$:*

$$r_{z,w}(h\otimes e^{m\alpha}) = m\partial_z \log r_{z,w}(e^\alpha\otimes e^\alpha); \tag{5.22}$$

$$r_{z,w}(h\otimes h) = \partial_z\partial_w \log r_{z,w}(e^\alpha\otimes e^\alpha); \tag{5.23}$$

$$h(z)h(w) \sim 1\cdot r_{z,w}(h\otimes h); \tag{5.24}$$

$$h(z)e^{m\alpha}(w) \sim e^{m\alpha}(w)\cdot r_{z,w}(h\otimes e^{m\alpha}). \tag{5.25}$$

These give us for example the commutation relations:

$$[h(z), e^{m\alpha}(w)] = (i_{z,w}-i_{w,z})r_{z,w}(h\otimes e^{m\alpha})\cdot e^{m\alpha}(w). \tag{5.26}$$

One familiar with the standard vertex operator calculations would recognize that the Heisenberg relation (5.24) and the Exponential relation (5.25) are the reason for the exponential operator formulas of the type (3.10), (3.16) and (3.17) in the bosonic side of the boson-fermion correspondences.

5.6. **Twisted vertex algebras based on $\mathbb{C}[\mathbb{Z}\alpha]$: product vacuum expectation values.**

Let V be a twisted vertex algebra based on $L_1 = \mathbb{C}[\mathbb{Z}\alpha]$ and a supersymmetric bicharacter r (by use of Theorem 4.39) with space of states $W = H_D(L_1)$. Denote by $e^{m\alpha}(z)$ the field $Y(e^{m\alpha}, z)$ produced by definition 4.25, via (4.24). Let the projection map π_f from the space of fields to the space of states satisfy $\pi_f(e^{m\alpha}) \neq 0$ for any $m \in \mathbb{Z}$ (which holds for $\pi_f = \pi_T$).

Lemma 5.7. *The following formula for the vacuum expectation values holds:*

$$\langle 0 \mid e^{m_1\alpha}(z_1)e^{m_2\alpha}(z_2)\dots e^{m_n\alpha}(z_n)|0\rangle = i_z\delta_{m_1+m_2+\dots+m_n,0}\prod_{i<j=1}^{n} r_{z_i,z_j}(e^{m_i\alpha}\otimes e^{m_j\alpha}).$$

Here i_z stands for the expansion $i_{z_1,z_1,\dots,z_n}$.

Proof. From Lemma 4.37, from the fact that everything is even parity and the elements $e^{m_k\alpha}$ are grouplike we have

$$X_{z_1,z_2,\dots,z_n}(e^{m_1\alpha}\otimes e^{m_2\alpha}\otimes\dots\otimes e^{m_n\alpha}) = \\ = \mathcal{E}_{z_1}e^{m_1\alpha}\cdot\mathcal{E}_{z_2}e^{m_2\alpha}\cdots\mathcal{E}_{z_n}e^{m_n\alpha}\cdot r_{z_1,z_2,\dots,z_n}(e^{m_1\alpha}\otimes e^{m_2\alpha}\otimes\dots\otimes e^{m_n\alpha}).$$

We have

$$\langle 0 \mid \mathcal{E}_{z_1}e^{m_1\alpha}\cdot\mathcal{E}_{z_2}e^{m_2\alpha}\cdots\mathcal{E}_{z_n}e^{m_n\alpha}\rangle = \langle 0 \mid \pi_T(e^{(m_1+m_2+\dots+m_n)\alpha}) + O(z)\rangle.$$

Since we required that the bilinear form is such that the vacuum vector $|0\rangle$ is orthogonal to all $e^{m\alpha}$, except for the $m = 0$, then

$$\langle 0 \mid e^{(m_1+m_2+\dots+m_n)\alpha}\rangle = \delta_{m_1+m_2+\dots+m_n,0}.$$

Note also that the $O(z)$ terms contain non-vacuum descendants of the $e^{m_k\alpha}$, and so do not contribute to the vacuum expectation value. Thus

$$\langle 0 \mid X_{z_1,\dots,z_n}(e^{m_1\alpha}\otimes\dots\otimes e^{m_n\alpha})\rangle = \delta_{m_1+\dots+m_n,0}r_{z_1,z_2,\dots,z_n}(e^{m_1\alpha}\otimes e^{m_2\alpha}\otimes\dots\otimes e^{m_n\alpha}).$$

Now since the elements $e^{m_k\alpha}$ are grouplike, the n-character $r_{z_1,z_2,\dots,z_n}(e^{m_1\alpha}\otimes\dots\otimes e^{m_n\alpha})$ also has especially simple form:

$$r_{z_1,z_2,\dots,z_n}(e^{m_1\alpha}\otimes e^{m_2\alpha}\otimes\dots\otimes e^{m_n\alpha}) = \prod_{i<j=1}^{n} r_{z_i,z_j}(e^{m_i\alpha}\otimes e^{m_j\alpha}), \tag{5.27}$$

which concludes our proof. □

5.7. **Twisted vertex algebra based on $\mathbb{C}[\mathbb{Z}\alpha]$: the free boson of type B.**

We continue with the bosonic side of the boson-fermion correspondence of type B. This is the first example we will encounter where the spaces of states and fields are not free Leibnitz modules, but are quotients of a free Leibnitz module.

Recall the free Leibnitz module $\tilde{V} = H^N_{T_\epsilon}(L_1)$, and its sub-Hopf algebra $\tilde{W} = H_D(L_1)$. We again take $\epsilon = -1$ and write just T instead of T_ϵ. Let

$$V = \tilde{V}/\{Te^\alpha = e^{-\alpha}\}, \tag{5.28}$$

Denote the quotient relations generated from $\{Te^\alpha = e^{-\alpha}\}$ by $\mathcal{R}_\mathcal{B}$.

If we want to define a $H^2_{T_\epsilon} \otimes H^2_T$-covariant bicharacter on V we can only choose the bicharacter value $r_{z,w}(e^\alpha \otimes e^\alpha)$, as the bicharacter values on all other elements of $\tilde{V}$ and $\tilde{W}$ are determined by the bicharacter properties and the covariance. In order to restrict this bicharacter to a bicharacter on $V = \tilde{V}/\mathcal{R}_\mathcal{B}$, it needs to be consistent with the relations $\mathcal{R}_\mathcal{B}$, thus

$$r_{-z,w}(e^\alpha \otimes e^\alpha) = r_{z,-w}(e^\alpha \otimes e^\alpha) = \frac{1}{r_{z,w}(e^\alpha \otimes e^\alpha)}. \tag{5.29}$$

If we choose a bicharacter value $r_{z,w}(e^\alpha \otimes e^\alpha)$ that satisfies the above relations (5.29), it will extend to an $H^2_T \otimes H^2_T$-covariant bicharacter on V, and we will be able to use Theorem 4.39. We choose

$$r_{z,w}(e^\alpha \otimes e^\alpha) = \frac{z-w}{z+w}. \tag{5.30}$$

Now we turn to the exact description of the space of fields V and the space of states W of this twisted vertex algebra. First, from example 4.9, the free Leibnitz module $H^2_T(L_1)$ is isomorphic to $L_2 \otimes H^2_T(\mathbb{C}[h])$, where L_2 is the group algebra $L_2 = \mathbb{C}[\mathbb{Z}\alpha, \mathbb{Z}\alpha_1]$ of the free abelian group of rank 2 (we identify Te^α, which is grouplike, with e^{α_1}). Denote by h^B_α the element $h^B_\alpha = \frac{1}{2}(De^\alpha)Te^\alpha \in V$, which coincides with $\frac{1}{2}(De^\alpha)e^{-\alpha} \in V$ under the relation $\mathcal{R}_\mathcal{B}$. It follows then that $TDe^\alpha = -DTe^\alpha = -De^{-\alpha}$. Therefore

$$Th^B_\alpha = h^B_\alpha. \tag{5.31}$$

Thus under the imposed relations $\mathcal{R}_\mathcal{B}$ in V we have $H^2_T(\mathbb{C}[h])/\mathcal{R}_\mathcal{B} = H_D(\mathbb{C}[h^B_\alpha])$ and

$$V = L_1 \otimes H_D(\mathbb{C}[h^B_\alpha]). \tag{5.32}$$

The space of states W is defined via the projection map $\pi_f : V \to W$, and in order to apply Theorem 4.39 we use as projection map the map from definition 4.13 adapted to the relations $\mathcal{R}_\mathcal{B}$. More precisely, define $\pi_f : V \to W$ to be the linear map defined by

$$\pi_f(H^2_T(\mathbb{C}[h^B_\alpha])/\mathcal{R}_\mathcal{B}) = Id, \quad \pi_f(Te^{n\alpha}) = e^{n\alpha}, \quad \pi_f(e^{n\alpha}) = e^{n\alpha}, \ n \in \mathbb{Z}. \quad (5.33)$$

Denote by $\bar{v}$ the element of W that is the projection of the element $v \in V$. We have

$$\bar{1} = \overline{e^\alpha}\overline{e^{-\alpha}} = \pi_f(e^\alpha e^{-\alpha}) = \pi_f(e^\alpha Te^\alpha) = \pi_f(e^\alpha)\pi_f(Te^\alpha) = \pi_f(e^{2\alpha}) = \overline{e^{2\alpha}},$$

thus we have in W

$$\overline{e^{2\alpha}} = 1, \quad \overline{e^\alpha} = \overline{e^{-\alpha}}. \quad (5.34)$$

Thus $W = H_D(\mathbb{C}[h^B_\alpha]) \oplus e^\alpha H_D(\mathbb{C}[h^B_\alpha])$ and as expected, as vector spaces $W = B_B$ (as in Lemma 3.9). Moreover, from Lemma 5.6 h^B_α is a Heisenberg element: we use (5.24), which in this case specializes from (5.23) to

$$h^B_\alpha(z)h^B_\alpha(w) \sim 1 \cdot r_{z,w}(h^B_\alpha \otimes h^B_\alpha) \sim 1 \cdot \frac{1}{4}\partial_w\partial_z \log\frac{z-w}{z+w} \sim 1 \cdot \frac{z^2+w^2}{2(z^2-w^2)^2}.$$

Now the unexpected twist here is that $Th^B_\alpha = h^B_\alpha$, hence the field $h^B_\alpha(z)$ has only **even** powers of z, and we can write it as $h^B_\alpha(z) = \sum_{n\in\mathbb{Z}} h_{2n+1}z^{-2n}$. From the OPE above we immediately get the commutation relations

$$[h_m, h_n] = \frac{m}{2}\delta_{m+n,0}, \quad m, n \ \text{odd integers}, \quad (5.35)$$

which are precisely the commutation relations of the Heisenberg algebra that we had in the fermion of type B. Note that we can reindex the field as $h^B_\alpha(z) = \sum_{n\in\mathbb{Z}+1/2} h_n z^{-2n-1}$. which translates to

$$[h_m, h_n] = m\delta_{m+n,0}, \quad m, n \in \mathbb{Z} + 1/2, \quad (5.36)$$

and explains the name $\mathcal{H}_{\mathbb{Z}+1/2}$ for this Heisenberg algebra.

Remark 5.8. Note that

$$h^B_\alpha(z) = z \cdot h^B_\phi(z), \quad (5.37)$$

which is allowed in an isomorphism of twisted vertex algebras: the doubly infinite sequences of the modes of the two fields $h^B_\alpha(z)$ and $h^B(z)$ are the identical, but there is the **shift** in the indexing (multiplication by z).

For the OPEs of the fields $e^{m\alpha}(w)$ with $h^B_\alpha(z)$, from (5.25) we get

$$h^B_\alpha(z)e^{m\alpha}(w) \sim me^{m\alpha}(w) \cdot \frac{w}{z^2-w^2}. \quad (5.38)$$

We see that if we identify $e^\alpha(z) = \phi^B(z)$, this OPE implies the exponential operator formula (3.10) in the bosonic side of the boson-fermion correspondences.

Thus we have shown that the pair $(L_1/\mathcal{R}_\mathcal{B}, r_{z,w}(e^\alpha \otimes e^\alpha) = \frac{z-w}{z+w})$ generates the twisted vertex algebra which is the bosonic side of the boson-fermion correspondence of type B. To summarize all these considerations:

Theorem 5.9. *The boson-fermion correspondence of type B is the isomorphism between two twisted vertex algebras: the fermionic side, which is the vertex algebra based on the pair* $(\mathbb{C}\{\phi\}, r^{B_f})$*; and the bosonic side, which is the twisted vertex algebra based on the pair* $(\mathbb{C}[\mathbb{Z}\alpha]/\mathcal{R}_{\mathcal{B}}, r^{B_b})$*.*

Lemma 3.12 follows directly: Since the boson-fermion correspondence identifies the fields $e^{\alpha}(z) = \phi^B(z)$, we can directly calculate the vacuum expectation values on each side of the correspondence of type B and equate. As a special case of lemma 5.7 we have for the bosonic side:

$$\langle 0|e^{\alpha}(z_1)e^{\alpha}(z_2)\dots e^{\alpha}(z_{2n})|0\rangle = i_z \prod_{i<j}^{2n} \frac{z_i - z_j}{z_i + z_j} \tag{5.39}$$

Similarly, the vacuum expectation values on the fermionic side follow directly from Proposition 5.1.

5.8. **Twisted vertex algebra based on $\mathbb{C}[\mathbb{Z}\alpha]$: the free boson of type D-A.**

We continue with the bosonic side of the boson-fermion correspondence of type D-A. This is another example where the spaces of states and fields are not free Leibnitz modules, but quotients of a free Leibnitz module.

We are again working with the free Leibnitz module $\tilde{V} = H^N_{T_\epsilon}(L_1)$, and its sub-Hopf algebra $\tilde{W} = H_D(L_1)$, $T = T_\epsilon$. For the bosonic space of type D-A we let

$$V = \tilde{V}/\{Te^{\alpha} = e^{\alpha}\}, \tag{5.40}$$

i.e., V is the quotient Leibnitz module modulo the relations $\mathcal{R}_{\mathcal{D}}$ generated by $\{Te^{\alpha} = e^{\alpha}\}$.

As we did in the previous section, if we want to define a $H^2_T \otimes H^2_T$-covariant bicharacter on V, we need to choose a bicharacter value $r_{z,w}(e^{\alpha} \otimes e^{\alpha})$ which is consistent with the relations $\mathcal{R}_{\mathcal{D}}$, i.e.,

$$r_{-z,w}(e^{\alpha} \otimes e^{\alpha}) = r_{z,w}(Te^{\alpha} \otimes e^{\alpha}) = r_{z,w}(e^{\alpha} \otimes e^{\alpha}), \tag{5.41}$$

$$r_{z,-w}(e^{\alpha} \otimes e^{\alpha}) = r_{z,w}(e^{\alpha} \otimes Te^{\alpha}) = r_{z,w}(e^{\alpha} \otimes e^{\alpha}), \tag{5.42}$$

i.e., $r_{z,w}(e^{\alpha} \otimes e^{\alpha})$ needs to be even as a function of both z and w, as well as supersymmetric with exchange of z and w. Thus we can choose

$$r_{z,w}(e^{\alpha} \otimes e^{\alpha}) = z^2 - w^2, \tag{5.43}$$

which bicharacter value will generate a bicharacter r^{D_b} on V by covariance.

Now we turn to the exact description of the space of fields V and the space of states W of this twisted vertex algebra. Denote by h^D_{α} the element

$\frac{1}{2}(De^{\alpha})e^{-\alpha} \in V$, which we know is a Heisenberg element. In V due to the relations $\mathcal{R}_{\mathcal{D}}$ we have $TDe^{\alpha} = -DTe^{\alpha} = -De^{\alpha}$, and $Te^{-\alpha} = e^{-\alpha}$. Thus

$$Th^D_{\alpha} = \frac{1}{2}(TDe^{\alpha})Te^{-\alpha} = -\frac{1}{2}DTe^{\alpha}e^{-\alpha} = -\frac{1}{2}De^{\alpha}e^{-\alpha} = -h^D_{\alpha}.$$

Hence

$$Th^D_{\alpha} = -h^D_{\alpha}, \tag{5.44}$$

thus under the imposed relations $\mathcal{R}_{\mathcal{D}}$ in V we again have $H^2_T(\mathbb{C}[h])/\mathcal{R}_{\mathcal{D}} = H_D(\mathbb{C}[h^D_{\alpha}])$, although now h^D_{α} is odd under T. Hence

$$V = L_1 \otimes H_D(\mathbb{C}[h^D_{\alpha}]). \tag{5.45}$$

We define the space of states W to be equal to V, i.e., we take as projection map $\pi_f : V \to W$ the identity map on V, which is consistent with the projection map π_T modulo $\mathcal{R}_{\mathcal{D}}$. Hence we can again use Theorem 4.39 to get a twisted vertex algebra.

From (5.24) and (5.23) we have

$$h^D_{\alpha}(z)h^D_{\alpha}(w) \sim 1 \cdot \frac{zw}{(z^2 - w^2)^2}. \tag{5.46}$$

Here $Th^D_{\alpha} = -h^D_{\alpha}$, hence the field $h^D_{\alpha}(z)$ has only **odd** powers of z, and we can write it as $h^D_{\alpha}(z) = \sum_{n\in\mathbb{Z}} h_n z^{-2n-1}$. Note that the field $h^D_{\alpha}(z)$ has the same OPE as the Heisenberg field $h^D(z)$, (3.13), as it should.

On the bosonic side it is easy to identify the split into irreducible Heisenberg modules, as the highest weight vectors are precisely the elements $e^{n\alpha} \in V \equiv W,\ n \in \mathbb{Z}$. Hence as Heisenberg modules

$$V \equiv W \equiv \mathbb{C}[e^{\alpha}, e^{-\alpha}] \otimes \mathbb{C}[h^D_{\alpha}, Dh^D_{\alpha}, \dots, D^{(n)}h^D_{\alpha}, \dots] \cong \tag{5.47}$$

$$\cong \mathbb{C}[e^{\alpha}, e^{-\alpha}] \otimes \mathbb{C}[x_1, x_2, \dots, x_n, \dots] \cong \oplus_{i\in\mathbb{Z}} B_i = B_D. \tag{5.48}$$

Now it remains to calculate the OPEs of the fields $e^{m\alpha}(w)$ with $h^D_{\alpha}(z)$, from (5.25) we get

$$h^D_{\alpha}(z)e^{m\alpha}(w) \sim me^{m\alpha}(w) \cdot \frac{z}{z^2 - w^2}. \tag{5.49}$$

The commutation relations for $e^{\alpha}(z)$ and $e^{-\alpha}(z)$ thus are:

$$[h(z), e^{\pm\alpha}(w)] = \pm(i_{z,w} - i_{w,z})\frac{z}{z^2 - w^2} \cdot e^{\pm\alpha}(w) = \pm\frac{1}{2}(\delta(z-w) + \delta(z+w))e^{\pm\alpha}(w).$$

From the standard vertex operator calculations this commutation relations immediately imply the exponential operator formulas

$$e^{-\alpha}_D(z) = e^{-\alpha}(z) = \exp(-\sum_{n\geq 1}\frac{h_{-n}}{n}z^{2n})\exp(\sum_{n\geq 1}\frac{h_n}{n}z^{-2n})e^{-\alpha}z^{-2\partial_{\alpha}}, \tag{5.50}$$

$$e^{\alpha}_D(z) = e^{\alpha}(z) = \exp(\sum_{n\geq 1}\frac{h_{-n}}{n}z^{2n})\exp(-\sum_{n\geq 1}\frac{h_n}{n}z^{-2n})e^{-\alpha}z^{2\partial_{\alpha}}. \tag{5.51}$$

Note that both of these are entirely even in the variable z operators, i.e., $e_D^{-\alpha}(z) = e_D^{-\alpha}(-z)$ and $e_D^{\alpha}(z) = e_D^{\alpha}(-z)$, which is of course consistent with the relations $\mathcal{R}_{\mathcal{D}}$ ($Te^{-\alpha} = e^{-\alpha}$ and $Te^{\alpha} = e^{\alpha}$). Note that

$$e_D^{\alpha}(z) = e_A^{\alpha}(z^2), \quad e_D^{-\alpha}(z) = e_A^{-\alpha}(z^2), \tag{5.52}$$

where the operators $e_A^{\alpha}(z)$ and $e_A^{-\alpha}(z)$ describe the boson-fermion correspondence of type A. This is a very interesting occurrence, and is discussed also [31] from the point of view of the fermion side. One should note though that the isomorphism of the spaces of states as Heisenberg modules and the "change of variables" formula (5.52) does not mean isomorphism as twisted vertex algebras, as these two vertex algebras have a different set of singularities in the OPEs. The equivalence as [31] notes is as CAR algebras.

We see that the boson-fermion correspondence of type D identifies the pair of fields $e_D^{-\alpha}(z) = e_{\phi}^{-\alpha}(z)$ and $e_D^{\alpha}(z) = ze_{\phi}^{\alpha}(z)$, i.e., we have

$$\phi^D(z) = e_D^{-\alpha}(z) + ze_D^{\alpha}(z), \quad (T\phi)^D(z) = e_D^{-\alpha}(z) - ze_D^{\alpha}(z). \tag{5.53}$$

Here again we see the "shifts" that are allowed in an isomorphism of vertex algebras.

Thus we have shown that the pair $(L_1/\mathcal{R}_{\mathcal{D}}, r_{z,w}(e^{\alpha}\otimes e^{\alpha}) = z^2 - w^2)$ describes the twisted vertex algebra which is the bosonic side of the boson-fermion correspondence of type D-A. To summarize all these considerations:

Theorem 5.10. *The boson-fermion correspondence of type D is the isomorphism between two twisted vertex algebras: the fermionic side, which is the twisted vertex algebra based on the pair* $(\mathbb{C}\{\phi\}, r^{D_f})$*; and the bosonic side, which is the twisted vertex algebra based on the pair* $(\mathbb{C}[\mathbb{Z}\alpha]/\mathcal{R}_{\mathcal{D}}, r^{D_b})$.

In order to prove Lemma 3.18 we need to compare the vacuum expectation values on the bosonic side with those on the fermionic side. We need to take into account the isomorphism formula (5.53) if we are to apply lemma 5.7.

$$\begin{aligned}
&\langle 0|\phi^D(z_1)\phi^D(z_2)\dots\phi^D(z_{2n})|0\rangle = \\
&= \langle 0|(e_D^{-\alpha}(z_1) + z_1 e_D^{\alpha}(z_1))(e_D^{-\alpha}(z_2) + z_2 e_D^{\alpha}(z_2))\dots(e_D^{-\alpha}(z_{2n}) + z_{2n} e_D^{\alpha}(z_{2n}))|0\rangle \\
&= \langle 0|e^{-\alpha}(z_1)e^{-\alpha}(z_2)\dots e^{-\alpha}(z_{2n})|0\rangle + \sum_{i=1}^{2n} z_i\langle 0|e^{-\alpha}(z_1)\dots e^{\alpha}(z_i)\dots e^{-\alpha}(z_{2n})|0\rangle \\
&\qquad + \sum_{i<j}^{2n} z_i z_j\langle 0|e^{-\alpha}(z_1)\dots e^{\alpha}(z_i)\dots e^{\alpha}(z_j)\dots e^{-\alpha}(z_{2n})|0\rangle + \dots \\
&+ \sum_{i_1<i_2\cdots<i_k}^{2n} z_{i_1}z_{i_2}\cdots z_{i_k}\langle 0|e^{-\alpha}(z_1)\dots e^{\alpha}(z_{i_1})\dots e^{\alpha}(z_{i_k})\dots e^{-\alpha}(z_{2n})|0\rangle + \dots
\end{aligned}$$

Recall the factor of $\delta_{m_1+m_2+\dots+m_n,0}$ in the right-hand side of the lemma 5.7. That factor forces all the sums but one to vanish: the only sum that will not vanish is the sum with the product of exactly n factors of z_{i_k} in it, as it will have exactly n e^{α}'s in it and as many $e^{-\alpha}$'s. Denote in this non-vanishing sum the indexes corresponding to $e^{-\alpha}$ by z_{j_k}. Thus we have the disjoint split $\{1,2,\dots 2n\} = \{i_1,i_2,\dots,i_n\}\sqcup\{j_1,j_2,\dots,j_n\}$ and we can write the non-vanishing sum as

$$\sum_{i_1<i_2\dots<i_n}^{2n} z_{i_1}z_{i_2}\cdots z_{i_n}\langle 0|e^{-\alpha}(z_{j_1})e^{\alpha}(z_{i_1})e^{-\alpha}(z_{j_2})e^{\alpha}(z_{i_2})\dots e^{-\alpha}(z_{j_n})e^{\alpha}(z_{i_n})|0\rangle.$$

We have

$$r_{z_i,z_j}(e^{\alpha}\otimes e^{-\alpha}) = r_{z_i,z_j}(e^{-\alpha}\otimes e^{\alpha}) = \frac{1}{z_i^2 - z_j^2},$$

thus from lemma 5.7

$$\langle 0|\phi^D(z_1)\phi^D(z_2)\dots\phi^D(z_{2n})|0\rangle = \\ = i_z\frac{\sum_{i_1<i_2\dots<i_n}^{2n} z_{i_1}z_{i_2}\cdots z_{i_n}\prod_{k<l}^n(z_{i_k}^2 - z_{i_l}^2)\prod_{p<q}^n(z_{j_p}^2 - z_{j_q}^2)}{\prod_{k,p}^n z_{i_k}^2 - z_{j_p}^2}.$$

Thus since the boson-fermion correspondence identifies the fields $\phi^D(z) = e_D^{-\alpha}(z) + ze_D^{\alpha}(z)$, lemma 3.18 follows directly as the Pfaffian equality for the vacuum expectation values on the fermionic side is a special case of Proposition 5.1). It is important that this Pfaffian identity follows directly from the correspondence of type D, and is a representative "imprint" of the boson-fermion correspondence.

5.9. **Boson-fermion correspondence of type D-A and order N.**

In this section we briefly show how the boson-fermion correspondence of type D-A extends to order N (and thereby give two examples of twisted vertex algebra of order N). Similarly to section 5.4 we are working with space of fields $V = H_T^N(\mathbb{C}\{\phi\})$, and space of states $W = H_D(\mathbb{C}\{\phi\})$, but now we allow N to be any natural number, $N \geq 2$. The projection map (recall definition 4.13) is again the algebra homomorphism defined by $\pi_T(T^i\phi) = \phi$, for any $0 \leq i \leq N-1$. As in section 5.4 the bicharacter
$r^D : H_T^N(\mathbb{C}\{\phi\})\otimes H_T^N(\mathbb{C}\{\phi\}) \to \mathbf{F}_\epsilon(z,w)^{+,w}$ is defined by

$$r_{z,w}^D(\phi\otimes\phi) = \frac{1}{z-w}$$

From theorem 4.39 it follows that the pair $(\mathbb{C}\{\phi\}, r_{z,w}^D)$ will generate a twisted vertex algebra of order N. In the twisted vertex algebra we have the following descendant fields $T^i\phi^D(z) = \phi^D(\epsilon^i z)$, for any $0 \leq i \leq N-1$, with OPEs

$$T^i\phi^D(z)T^j\phi^D(w) \sim \frac{1}{\epsilon^i z - \epsilon^j w}.$$

The boson-fermion correspondence is determined via the Heisenberg element

$$h = \frac{1}{N}\sum_{i=0}^{N-1} \epsilon^{i-1}(T^{i-1}\phi)(T^i\phi) \tag{5.54}$$

We have $Th = \epsilon^{-1}h$, hence we index the field $h(z) = \sum_{n\in\mathbb{Z}} h_n z^{-Nn-1}$. The following OPE holds:

$$h(z)h(w) \sim \frac{z^{N-1}w^{N-1}}{(z^N - w^N)^2}, \tag{5.55}$$

and thus the commutation relations $[h_m, h_n] = m\delta_{m+n,0}1$ for the Heisenberg algebra $\mathcal{H}_{\mathbb{Z}}$ again hold. Denote

$$e^{\alpha}_{\phi}(w) = \frac{1}{N}(\sum_{i=0}^{N-1} \epsilon^{-i}T^i\phi^D(w)), \quad e^{\epsilon^k\alpha}_{\phi}(w) = \frac{1}{N}(\sum_{i=0}^{N-1} \epsilon^{(k-1)i}T^i\phi^D(w)).$$

We get

$$e^{\epsilon^k\alpha}_{\phi}(z) = \exp(\epsilon^{-k}\sum_{n\geq 1}\frac{h_{-n}}{n}z^{Nn})\exp(\epsilon^k\sum_{n\geq 1}\frac{h_n}{n}z^{-Nn})U_{\epsilon^k\alpha}(z), \tag{5.56}$$

where $U_\alpha(z)$ acts as a constant on each Heisenberg submodule, $U_{\epsilon^k\alpha}(z) = e^{\epsilon^k\alpha}_{\phi}z^{1-k+N\partial_\alpha}$, and $e^{\epsilon^k\alpha}_{\phi}$ identifies the highest weight vector of the Heisenberg submodule. The last formula establishes the boson-fermion correspondence of order N.

5.10. **Other examples of boson-fermion correspondences and twisted vertex algebras.**

There are other important examples in the literature of boson-fermion (or boson-boson) correspondences which can be shown to be isomorphisms of twisted vertex algebras: in particular the CKP correspondence (also called correspondence of type C, [9] and [32]), and the "super boson-fermion correspondence of type B" ([20]), which are both isomorphisms of twisted vertex algebras.

Lemma 5.11. *The correspondence of type C is an isomorphism of twisted vertex algebras, where one of the sides is a twisted vertex algebra based on the pair* $(\mathbb{C}[h], r^C)$ *via Theorem 4.39.*

Proof. The "left-hand side" of the CKP correspondence has a space of states $V = H^2_T(\mathbb{C}[h])$ (recall the free Leibnitz module $H^2_T(\mathbb{C}[h])$ of example 4.8), and the twisted vertex algebra is generated by the bicharacter

$$r^C_{z,w}(h\otimes h) = \frac{1}{z+w}. \tag{5.57}$$

We only have to give the bicharacter value on the element h, which in this example will be denoted by h_ϕ. The OPE for the corresponding field $h_\phi(z)$ directly follows from the fact that h_ϕ is primitive:

$$h_\phi(z)h_\phi(w) \sim \frac{1}{z+w} \sim h_\phi(w)h_\phi(z). \tag{5.58}$$

We use ϕ_j as notation for the modes of $h_\phi(z)$ to follow [32]. The field $h_\phi(z)$ is indexed as follows: $h_\phi(z) = \sum_{j\in\mathbb{Z}+1/2} \phi_j z^{j-1/2}$. From theorem 4.39 it follows that the pair $(\mathbb{C}[h], r^C_{z,w})$ will generate a twisted vertex algebra of order 2. □

The paper [32] gives many details on this correspondence.

We also want to mention that other examples of twisted vertex algebras are supplied by the representation theory of affine Lie algebras and affine Lie super algebras, see for example [17], [11], [12] and [24].

6. Summary

There are three main results of this paper: first, we derived the new boson-fermion correspondence of type D-A. Second, we defined the new concept of a twisted vertex algebra of order N, which generalize super-vertex algebras (in the sense that a super vertex algebra is a twisted vertex algebra of order 1). This new concept of twisted vertex algebra was designed to answer the following question "What is a boson-fermion correspondence— isomorphism of what mathematical structures?". As a technical set of tools we developed the bicharacter construction which provided us with a general way of producing examples of twisted vertex algebras. We proved formulas for the OPEs, analytic continuations, normal ordered products and vacuum expectation values using the underlying Hope algebra structure and the bicharacter construction. Finally, we proved that the correspondences of types B, C and D-A are isomorphisms of twisted vertex algebras.

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