

Symmetric Polynomials and H_D -Quantum Vertex Algebras

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ABSTRACT. In this paper we use a bicharacter construction to define an H_D -quantum vertex algebra structure corresponding to the quantum vertex operators describing classes of symmetric polynomials.

1. Introduction

Vertex operators were introduced in the earliest days of string theory and axioms for vertex algebras were developed to incorporate these examples (see for instance [FLM88]). Similarly, the definition of quantum vertex algebra should be such that it accommodates the existing examples of quantum vertex operators and their properties (see for instance [FJ88], [FR96], [BFJ98], [JM99] and many others).

In this series of papers we study the quantum vertex algebra structure corresponding to classes of symmetric polynomials (e.g., Hall-Littlewood or Macdonald polynomials). The vertex operators describing these polynomials were considered by N. Jing in a series of papers ([Jin91], [Jin95], [Jin94b]). As shown for instance by these examples a major difference between classical and quantum vertex algebras lies in the fact that two vertex operators (fields) $Y(z)$ and $Y(w)$ are no longer ‘almost’ commuting (i.e., commuting, except on the diagonal $z = w$), but instead there is a braiding map connecting the products $Y(z)Y(w)$ and $Y(w)Y(z)$ (see section 3). The goal in this series of papers is to incorporate these vertex operators in certain braided vertex algebra structures.

There are several proposals for the axioms of quantum vertex algebras (or deformed chiral algebras). We will consider throughout the papers three of these definitions, namely by Borcherds in [Bor01], Frenkel-Reshetikhin in [FR] and Etingof-Kazhdan in [EK00]). We will refer to them as (A, H, S) quantum vertex algebras, deformed chiral algebras (of FR type) or quantum vertex algebras of EK type. One of the goals of this series of papers is to show by examples that these axioms are surprisingly not equivalent. Also we show that the axioms of EK or FR are not sufficient to describe the vertex operators of the symmetric polynomials.

Our main goal in this paper will be to define an H_D -type of quantum vertex algebra. (Here $H_D = \mathbf{C}[D]$ is the Hopf algebra of infinitesimal translations, a

1991 *Mathematics Subject Classification*. 17B69, 33D80 (Primary); 81R50, 33D52 (Secondary).

fundamental ingredient in the classical vertex algebras.) We construct examples of such H_D -quantum vertex algebras incorporating the quantum vertex operators appearing in the theory of Hall-Littlewood and Macdonald symmetric polynomials.

We use extensively the bicharacter construction from [Bor01] as a help towards defining the quantum vertex algebra structures. We will state a technical theorem (see Theorem 5.3) which will allow us to use this construction on a large class of vertex operators corresponding to the general orthogonal polynomials as defined by Macdonald.

2. Symmetric polynomials and vertex operators

In this section we recall the Macdonald definition of general symmetric polynomials ([Mac95]). Also we recall the vertex operators associated to these general symmetric polynomials (as considered for instance by Jing, [Jin91], [Jin94b])

We work over a field k of characteristic zero containing the rationals, F is a field extension of k (for instance $k(t)$ or $k(q, t)$, where q, t are parameters).

Denote by Λ the ring of symmetric functions over \mathbf{Z} in countably many independent variables $x_i, i \geq 0$. Let $\Lambda_F = \Lambda \otimes_{\mathbf{Z}} F$. As usual denote by p_n the power symmetric functions. If λ is a partition, $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k, \dots)$, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq \dots$, we call a family (a_λ) of elements in a ring indexed by partitions multiplicative if $a_\lambda = \prod a_{\lambda_i}$. The family (p_λ) is a multiplicative basis for the symmetric functions. We will also use the basis (m_λ) of monomial symmetric functions (it is not multiplicative). Denote $z_\lambda = \prod_{i \geq 0} i^{m_i} \cdot m_i!$, where $m_i = m_i(\lambda)$ is the number of parts of λ equal to i .

Let (v_λ) be a multiplicative family in F^\times . We want to define for each such family a set of symmetric polynomials $\{P_\lambda\}$ indexed by partitions. First, we define a scalar product $\langle \cdot, \cdot \rangle_v$ on Λ_F (depending on the multiplicative family v_λ) by

$$\langle p_\lambda, p_\mu \rangle_v = \delta_{\lambda\mu} z_\lambda v_\lambda,$$

for any partitions λ, μ . The set $\{P_\lambda\}$ associated to v_λ should satisfy the following two (over-determining) conditions:

$$\langle P_\lambda, P_\mu \rangle_v = 0 \quad \text{for } \lambda \neq \mu,$$

$$P_\lambda = m_\lambda + \sum_{\mu < \lambda} u_{\lambda\mu} m_\mu, \quad u_{\lambda\mu} \in F.$$

Here $\mu < \lambda$ is with respect to the usual partial order on partitions. In the cases where $\{P_\lambda\}$ exist denote also by Q_λ the dual of P_λ , i.e., $\langle P_\lambda, Q_\mu \rangle_v = \delta_{\lambda,\mu}$.

For general (v_λ) the corresponding $\{P_\lambda\}$ might not exist, but the following important examples are known:

- If $v_n = 1$ the family $\{P_\lambda\}$ is in fact $\{S_\lambda\}$, the Schur symmetric functions; Schur functions are self-dual.
- If $v_n = \frac{1}{1-t^n}$ the $\{P_\lambda\}$ are the Hall-Littlewood functions.
- If $v_n = \frac{1-q^n}{1-t^n}$ the family $\{P_\lambda\}$ consists of the Macdonald symmetric functions.
- Jack symmetric functions and the zonal polynomials can also be obtained by picking appropriate v_n , but we will not consider them in this paper.

We can view p_n as an operator acting on Λ_F by multiplication. Define also the operators p_n^\perp by requiring

$$\langle p_n^\perp f, g \rangle_v = \langle f, p_n g \rangle_v,$$

for any $f, g \in \Lambda_F$.

Given a multiplicative family (v_λ) , let $\mathcal{H} = \mathcal{H}_\square$ be the algebra generated by the operators $h_n = -v_{-n}^{-1}p_{-n}^\perp$, $h_{-n} = v_n^{-1}p_n$ for $n \in \mathbf{N}$.

LEMMA 2.1. *For each multiplicative family (v_λ) the operators $\{h_n | n \in \mathbf{Z}\}$ generate a representation of a deformed Heisenberg algebra on Λ_F , i.e.,*

$$(2.1) \quad [h_m, h_n] = mv_{|m|}^{-1} \delta_{m+n, 0} 1.$$

Define the following vertex operator

$$(2.2) \quad \Phi(z) = \exp\left(\sum_{n \geq 1} \frac{h_{-n}}{n} z^n\right) \exp\left(-\sum_{n \geq 1} \frac{h_n}{n} z^{-n}\right) = \sum \Phi_n z^n,$$

where $\Phi_n \in \text{End}(\Lambda_F)$ are the Fourier coefficients of the field $\Phi(z)$. (Note that in fact we have a different vertex operator for each multiplicative family (v_λ)). The usefulness of these vertex operators is due to the following facts. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ be a partition.

- In the case $v_n = 1$,

$$\Phi_{\lambda_1} \Phi_{\lambda_2} \dots \Phi_{\lambda_k} 1 = S_\lambda.$$

Here S_λ is the Schur function corresponding to the partition λ (recall that Schur functions are self-dual).

- In the case $v_n = \frac{1}{1-t^n}$,

$$\Phi_{\lambda_1} \Phi_{\lambda_2} \dots \Phi_{\lambda_k} 1 = Q_\lambda.$$

Here Q_λ is the dual to the Hall-Littlewood polynomial corresponding to the partition λ . (This fact was proved by Jing in [Jin91]).

- In the case of $v_n = \frac{1-q^n}{1-t^n}$ the symmetric function $\Phi_{\lambda_1} \Phi_{\lambda_2} \dots \Phi_{\lambda_k} 1$ is connected to the Macdonald symmetric function, but not in general equal to the dual Macdonald function as in the previous two cases. For a single row partition they are indeed equal, for two-row partitions Jing proves that the Macdonald polynomials are related via a q -hypergeometric function (see [Jin94a]).

Define the space $V = \Lambda_F \otimes \mathbf{C}[\mathbf{Z}\alpha]$, where $\mathbf{C}[\mathbf{Z}\alpha]$ is the group algebra of the rank-one lattice generated by e^α , $m \in \mathbf{Z}$ (such that $e^{m\alpha} e^{n\alpha} = e^{(m+n)\alpha}$). Define also the vertex operator on V

$$(2.3) \quad \Psi(z) = \exp\left(\sum_{n \geq 1} \frac{h_{-n}}{n} z^n\right) \exp\left(-\sum_{n \geq 1} \frac{h_n}{n} z^{-n}\right) e^\alpha z^{\partial_\alpha},$$

which is more convenient to work with.

In the case of Schur polynomials the vertex operator $\Psi(z)$ belongs to the classical vertex algebra of the odd rank-one lattice (the Heisenberg algebra in this case is the undeformed infinite-dimensional Heisenberg algebra). In the deformed case the vertex algebra can no longer be classical, i.e., the vertex operators will no longer “almost” commute, but instead will form a “braided singular ring”, i.e., they will belong to a braided vertex algebra structure. The goal of this paper will be to describe the quantum vertex algebra structure to which this vertex operator $\Psi(z)$ belong.

The simplest case of a braided vertex algebra structure—the H_D -type—is described below.

3. H_D -quantum vertex algebras

Let V be a vector space over F , \mathbf{t} be a (multi) parameter. A field $a(z)$ (on V) is a series of the form

$$a(z) = \sum_{n \in \mathbf{Z}} a_{(n)} z^{-n-1}, \quad a_{(n)} \in \text{End}(V), \quad a_{(n)} v = 0 \text{ for any } v \in V, \quad n \gg 0.$$

A state-field correspondence is a linear map from V to the space of fields that associates to any $a \in V$ a field $Y(a, z) = a(z)$. We will use also the following notation introduced in [EK00]

$$\begin{aligned} Y : V \otimes V &\rightarrow V((z)), \\ a \otimes b &\mapsto Y(z)(a \otimes b). \end{aligned}$$

Denote by \mathbf{K} the algebra $k[z^{\pm 1}, w^{\pm 1}, (z-w)^{\pm 1}][[\mathbf{t}]]$.

DEFINITION 3.1. An H_D -quantum vertex algebra over $k[[\mathbf{t}]]$ consists of the following data:

- the space of states—a topologically free module V over $k[[\mathbf{t}]]$.
- the vacuum vector—a vector $|0\rangle \in V$.
- the space of fields and state-field correspondence.
- a distinguished operator $D : V \rightarrow V$
- a braiding map—a linear map $R(z, w) : V \otimes V \rightarrow V \otimes V \otimes \mathbf{K}$, which satisfies the conditions:
 - Yang-Baxter equation:

$$R_{12}(z_1, z_2)R_{13}(z_1, z_3)R_{23}(z_2, z_3) = R_{23}(z_2, z_3)R_{13}(z_1, z_3)R_{12}(z_1, z_2)$$

- shift conditions

$$\begin{aligned} [D \otimes 1, R(z, w)] &= -\partial_z R(z, w), \\ [1 \otimes D, R(z, w)] &= -\partial_w R(z, w). \end{aligned}$$

- unitarity condition

$$\tau R_{w,z} \tau = R_{z,w}^{-1}, \quad \text{where } \tau \text{ is the flip, } \tau(a \otimes b) = b \otimes a.$$

Here we denote $R_{z,w} = R(z, w)$.

These data should satisfy the following two sets of axioms

- The axioms related to the action of the Hopf algebra $H_D = \mathbf{C}[D]$:
 - translation covariance: $Y(Da, z) = \partial_z Y(a, z)$.
 - vacuum axiom: $Y(|0\rangle, z) = Id_V$.
 - creation axiom: $Y(a, z)|0\rangle|_{z=0} = a$.
- Braided locality axiom: for all $a, b \in V$ there exist N such that

$$(z-w)^N Y(a, z)Y(b, w) = (w-z)^N \tilde{R}(Y(b, w), Y(a, z)),$$

where we denote

$$(3.1) \quad \tilde{R}(Y(b, w), Y(a, z))c = Y(w)(1 \otimes Y(z))(R(w, z)(b \otimes a) \otimes c),$$

for any $a, b, c \in V$.

This definition is a generalization of the definition from [EK00] and therefore we will sometimes use the name vertex algebras of EK type for H_D -quantum vertex algebras. In order to accommodate the examples of the symmetric polynomials the following two important changes were made. First, in [EK00] the braiding $R(z, w)$ is in fact of the form $\hat{R}(z-w)$, where \hat{R} is a function of a single variable. This is no

longer possible for quantum vertex algebras describing the symmetric polynomials, where the braiding map is truly a function of two variables (see the appendix in [Ang]). Second, in [EK00] (as in the case of classical vertex algebras) the vertex operators satisfy

$$(3.2) \quad \frac{d}{dz}Y(z) = DY(z) - Y(z)(1 \otimes D).$$

The above equation is not valid for Hall-Littlewood vertex operators for instance, but the translation covariance as required in the axiom holds (see the appendix in [Ang], as well as [AB]). This last consideration in fact implies that our H_D -quantum vertex algebras are no longer field algebras (as defined for instance in [Kac97]).

The classical vertex (super)algebras are a particular case of H_D -quantum vertex algebras, with braiding map $R(z, w) = \pm Id_V \otimes Id_V$, and for them (3.2) is equivalent to the translation covariance axiom. In order to prove that the quantum vertex operators associated with the symmetric polynomials belong to the type of structure described above we have to follow a somewhat roundabout way.

4. Borcherds bicharacter construction of vertex algebras

This section recalls Borcherds bicharacter construction from [Bor01]. This construction was used there to construct examples of (A, H, S) (quantum) vertex algebra, but we will not discuss here (A, H, S) vertex algebras, as it is outside of the scope of this paper. We will adapt the bicharacter construction as a tool for recovering the braiding map between the quantum vertex operators. The idea is to define a braided vertex algebra using a bicharacter (it can be made into (A, H, S) quantum vertex algebra, see Theorem 4.2, [Bor01]), as such algebras come with a formula for the braiding map, then to prove that the vertex operators we are dealing with can be in fact identified with those defined via a bicharacter. We will provide in [AB] some of the details which are omitted here.

First let us recall some definitions.

4.1. Bicharacters. Let M be a commutative and cocommutative Hopf algebra. Denote the coproduct and the counit by Δ and η , the antipode by S

If a is an element of a coalgebra we will use Sweedler's notation and write $\Delta(a) = \sum a' \otimes a''$. We often will omit the summation sign.

DEFINITION 4.1. (Bicharacter) A bicharacter on M is a linear map r from $M \otimes M$ to T , where T is a commutative k algebra, such that for any $a, b, c \in M$

$$\begin{aligned} r(1 \otimes a) &= \eta(a) = r(a \otimes 1), \\ r(ab \otimes c) &= \sum r(a \otimes c')r(b \otimes c''), \\ r(a \otimes bc) &= \sum r(a' \otimes b)r(a'' \otimes c). \end{aligned}$$

We will be interested mostly in the cases where the target space for the bicharacter is \mathbf{K} .

Note that in fact if $T = \mathbf{K}$ the bicharacter $r(a \otimes b)$ is a function of z and w , and should be written as $r(a \otimes b)(z, w)$ but we will omit the (z, w) unless strictly necessary, in which case we will sometimes write it as $r(a_z \otimes b_w)$.

LEMMA/DEFINITION 4.2. Let r and s be bicharacters, where M is a commutative cocommutative Hopf algebra. We can define a convolution product $r * s$

by

$$r * s(a \otimes b) = r(a' \otimes b')s(a'' \otimes b'').$$

The identity bicharacter is given by $\epsilon(a \otimes b) = \eta(a)\eta(b)$.

The inverse bicharacter is defined by $r^{-1}(a \otimes b) = r(S(a) \otimes b)$.

The transpose bicharacter is defined to be $r^\tau(a_z \otimes b_w) = r(b_w \otimes a_z)$. (Here we assume that the target space $T = \mathbf{K}$.)

The bicharacters on M form a commutative group.

Note: In the above definition it is essential that M is a cocommutative Hopf algebra.

4.2. Free Leibnitz modules and extension of bicharacters.

Now we proceed to describe the type of vector spaces we will be using as underlying our braided vertex algebras. Denote by H_D the Hopf algebra $\mathbf{C}[D]$, generated by a primitive element D . As usual we write $D^{(n)} = \frac{D^n}{n!}$. We have then

$$(4.1) \quad \Delta D^{(n)} = \sum_{k+l=n} D^{(k)} \otimes D^{(l)}, \quad S(D^{(n)}) = (-1)^n D^{(n)}.$$

For any M a commutative, cocommutative Hopf algebra we want to construct a (universal) H_D -module algebra $H_D(M)$ containing M .

Let $H_D(M) \equiv \text{Sym}(H_D \otimes M)$. Then $H_D(M)$ is naturally a module-algebra over H_D . We extend the comultiplication, the counit and the antipode from M to $H_D(M)$ as follows. We have for any $a \in H(M)$, $h \in H_D$

$$\begin{aligned} \Delta h.a &= \sum h'.a' \otimes h''.a'', \\ \eta(h.a) &= \eta(h)\eta(a), \\ S(h.a) &= S(h).S(a). \end{aligned}$$

It is easy to check that the comultiplication, the counit and the antipode defined as above will turn $H_D(M)$ into commutative cocommutative Hopf algebra. All of the vector spaces underlying our vertex algebras in this paper are going to be of the type $H_D(M)$ for some M . We call such module-algebras free H_D -Leibnitz modules.

If we have a bicharacter on M with target T which is an H_D -bimodule we can extend it to $H_D(M)$ by requiring

$$r(Da \otimes b)(z, w) = \partial_z r(a \otimes b)(z, w), \quad r(a \otimes Db)(z, w) = \partial_w r(a \otimes b)(z, w).$$

It is easy to check that this extension satisfies the axioms for a bicharacter on $H_D(M)$.

4.3. Fields defined via bicharacter.

Let $V = H_D(M)$ for some commutative cocommutative algebra M with a bicharacter r with target \mathbf{K} .

DEFINITION 4.3. For any $a, b \in V$ define $Y(a, z)b \in V((z))$ by

$$(4.2) \quad Y(a, z)b = \sum (e^{zD}(a'))b'r(a'' \otimes b'')(z, 0)$$

This definition gives us a state-field correspondence, even though the fields are not given by generating series as is usual. Since M and hence V is an algebra with unit element 1, define the vacuum vector for the quantum vertex algebra structure we are constructing to be the unit element in the algebra V . We have the following lemma:

LEMMA 4.4. *The fields $Y(a, z)b$ for $a, b \in V$ defined by (4.2) satisfy the vacuum, the creation and the translation covariance axioms for an H_D -quantum vertex algebra.*

PROOF. For all $a \in V$

$$\begin{aligned} Y(a, z)1 &= \sum e^{zD}(a')r(a'' \otimes 1)(z, 0) = \\ &= \sum e^{zD}(a')\eta(a'') = \sum a'\eta(a'') + z.O(z) = a + z.O(z), \end{aligned}$$

which proves the creation axiom. Similarly

$$\begin{aligned} Y(Da, z)b &= \sum e^{zD}((Da)')b'r((Da)'' \otimes b'')(z, 0) = \\ &= \sum e^{zD}(Da')b'r(a'' \otimes b'')(z, 0) + \sum e^{zD}(a')b'r(Da'' \otimes b'')(z, 0) = \\ &= \partial_z(\sum e^{zD}a'b'r(a'' \otimes b'')(z, 0)) = \partial_z Y(a, z)b. \end{aligned}$$

This proves the translation covariance, the vacuum axiom is proved similarly. \square

DEFINITION 4.5. Let $a, b \in V$. For given bicharacter we define

$$R(z, w) : V \otimes V \rightarrow V \otimes V \otimes \mathbf{K}$$

by

$$R(a \otimes b) = \Sigma a' \otimes b' r^t * r^{-1}(a'' \otimes b'')$$

The map R is going to be our braiding map. It is easy to prove that it satisfies the Yang-Baxter equation, the shift and the unitarity conditions (more details can be found in [AB]). We have the following

THEOREM 4.6. *The data defined as above (the space of states V , the vacuum vector 1 , the distinguished operator D , the state-field correspondence as defined by (4.2) above and the braiding map R) satisfies the axioms of H_D -quantum vertex algebra.*

5. Bicharacter construction for lattice (quantum) vertex algebras

In this section we connect the vertex operators as defined in the previous section to the vertex operator Ψ defined in the first section. We consider a specific case of the very general bicharacter construction described in the previous section.

Let $M = \mathbf{C}[\mathbf{Z}\alpha]$, the group algebra of the rank-one lattice generated by $e^{m\alpha}$ (such that $e^{m\alpha}e^{n\alpha} = e^{(m+n)\alpha}$). Then $V = H_D(M)$ can be written as $\mathbf{C}[h^{(n)}] \otimes \mathbf{C}[\mathbf{Z}\alpha]$, $n \geq 1$, where the new variables $h^{(n)}$ are defined by

$$(5.1) \quad h = D\alpha = (D.e^\alpha)e^{-\alpha}, \quad h^{(n)} = D^{(n)}h.$$

The bialgebra structure on V is determined by e^α being grouplike. For instance we have

$$\Delta(h) = h \otimes 1 + 1 \otimes h,$$

i.e., h is a primitive element. The antipode is determined by $S(e^\alpha) = e^{-\alpha}$. It follows then that

$$S(h) = -h.$$

To define a bicharacter on V we only need to define it on M . Moreover, since $r(e^\alpha \otimes e^{-\alpha})r(e^\alpha \otimes e^\alpha) = 1$ (follows directly from the properties of a bicharacter), we only need to define $\sigma(z, w) = r(e^\alpha \otimes e^\alpha)(z, w)$. Simple calculation shows that

$$\begin{aligned} r(e^\alpha \otimes h)(z, w) &= \frac{\partial_w \sigma(z, w)}{\sigma(z, w)} = \partial_w \ln \sigma(z, w), \\ r(h \otimes h)(z, w) &= \partial_z \partial_w \ln \sigma(z, w). \end{aligned}$$

For any function $f(x; \mathbf{t})$ depending on a (multi) parameter denote by $i_{\mathbf{t}} f(x; \mathbf{t})$ the power series expansion of $f(x; \mathbf{t})$ in terms of this parameter (i.e. consider the parameter(s) as very small). We will suppress the dependence on the parameter in the notation and will write just $f(x)$ instead of $f(x, \mathbf{t})$.

Let

$$(5.2) \quad f(x) = \exp\left(-\sum_{n \geq 1} \frac{v_n^{-1}}{n} x^n\right),$$

$$(5.3) \quad \sigma(z, w) = z \cdot i_{\mathbf{t}} f\left(\frac{w}{z}\right).$$

The choice of the (multi)parameter \mathbf{t} in the above expansion will depend on the multiplicative family (v_λ) . For instance, in the case of Hall-Littlewood polynomials we have

$$(5.4) \quad \mathbf{t} = t, \quad v_n^{-1} = 1 - t^n, \quad f(x) = \frac{1-x}{1-tx}, \quad i_{\mathbf{t}} f(x) = (1-x) \sum_{n=0}^{\infty} x^n t^n.$$

We have the following two lemmas which provide the connection between the bicharacter definition of fields and the expressions as generating series as for instance in Lemma (2.3).

LEMMA 5.1. *Let $\sigma(z, w)$ be as above. The field $h(z) := Y(D\alpha, z)$ defined by (4.2) is a Heisenberg field, i.e., if we write $h(z)$ as $h(z) = \sum_{n \in \mathbf{Z}} h_n z^{-n-1}$, then the operators $h_n \in \text{End}(V)$, $n \in \mathbf{Z}$ define a deformed Heisenberg algebra (as in Lemma 2.1).*

LEMMA 5.2. *Let $\sigma(z, w)$ be as above. The field $Y(e^\alpha, z)$ as defined in (4.2) can be written as*

$$Y(e^\alpha, z) = \exp\left(\sum_{n \geq 1} \frac{h_{-n}}{n} z^n\right) \exp\left(-\sum_{n \geq 1} \frac{h_n}{n} z^{-n}\right) e^\alpha z^{\partial_\alpha},$$

where the operators h_n are the generators of the Heisenberg algebra.

For proofs, see [Ang] or [AB]. Hence we have the following theorem:

THEOREM 5.3. *Let $\sigma(z, w)$ be defined as in (5.3). The field $Y(e^\alpha, z)$ as defined in (4.2) equals the field $\Psi(z)$ as defined in (2.3). Moreover, the field $\Psi(z)$ belongs to an H_D -quantum vertex algebra structure on $V = \text{Sym}(D^{(n)}\alpha) \otimes \mathbf{C}[\mathbf{Z}\alpha]$, with braiding map given by*

$$(5.5) \quad R(a \otimes b) = \Sigma a' \otimes b' r^t * r^{-1}(a'' \otimes b''),$$

for any $a, b \in V$.

For instance, in the simplest case of $a = e^\alpha, b = e^\alpha$, the braiding map looks as follows:

$$(5.6) \quad \tilde{R}(Y(e^\alpha, z), Y(e^\alpha, w)) = i_{\mathbf{t}} \left(\frac{w \cdot f\left(\frac{z}{w}\right)}{z \cdot f\left(\frac{w}{z}\right)} \right) Y(e^\alpha, z) Y(e^\alpha, w)$$

where \tilde{R} is defined as in (3.1).

The function $f(x)$ as defined in (5.2) is precisely the one considered by Jing in [Jin94b], where the above equation (5.6) is also derived. The use of $i_{\mathbf{t}}f(x)$ instead of $f(x)$ is dictated by the choice of \mathbf{K} as the target space for the bicharacters, and that in turn is chosen so that we can view our quantum vertex algebras as a kind of “deformation quantization” of the classical ones. It is notable that although the braiding map $R(z, w)$ is an expansion in power series in the small parameter, it is also a double-sided series in z and w . Consider for instance in the case of Hall-Littlewood polynomials (see (5.4)) the braiding $R(h \otimes h)(z, w)$ (where $h = D\alpha$). We have from (5.5)

$$\begin{aligned} R(h \otimes h)(z, w) &= h \otimes h + 1 \otimes 1 i_{\mathbf{t}} \left(-\frac{t}{(z-tw)^2} + \frac{t}{(w-zt)^2} \right) = \\ &= h \otimes h + 1 \otimes 1 \left(-i_{z,w} \frac{t}{(z-tw)^2} + i_{w,z} \frac{t}{(w-zt)^2} \right), \end{aligned}$$

where for a meromorphic function $f(z, w)$ we denote by $i_{z,w}f(z, w)$ the expansion of $f(z, w)$ in the region $z \gg w$, and similarly for $i_{w,z}f(z, w)$. More details are provided in [AB].

For the case of Hall-Littlewood polynomials the vertex operator Ψ_t

$$\Psi_t(z) = \exp\left(\sum_{n \geq 1} \frac{1-t^n}{n} p_n z^n\right) \exp\left(-\sum_{n \geq 1} \frac{1-t^n}{n} p_n^\perp z^{-n}\right) e^{\alpha z^{\partial_\alpha}}$$

belongs to an H_D -quantum vertex algebra over $k[[t]]$. If $t = 0$ we get back the classical vertex superalgebra describing the Schur polynomials.

In the case corresponding to Macdonald’s polynomials we have

$$\mathbf{t} = (t, q), \quad v_n^{-1} = \frac{1-t^n}{1-q^n}, \quad f(x) = \frac{(x; q)_\infty}{(tx; q)_\infty},$$

where $(x; q)_\infty = \prod_{n \geq 0} (1 - xq^n)$. The vertex operator

$$\Psi_{q,t}(z) = \exp\left(\sum_{n \geq 1} \frac{1-t^n}{n(1-q^n)} p_n z^n\right) \exp\left(-\sum_{n \geq 1} \frac{1-t^n}{n(1-q^n)} p_n^\perp z^{-n}\right) e^{\alpha z^{\partial_\alpha}}$$

belongs to H_D -quantum vertex algebra structure over $k[[t, q]]$. If $q = 0$ we get back the operator Ψ_t .

We want to make a note that all the bosonic vertex operators (i.e., vertex operators of exponential form similar to $\Psi(z)$) can be written in terms of bicharacters. For instance the bosonic vertex representation of the quantum affine algebras (as considered for instance by I. Frenkel and Jing in [FJ88]) can be written using a bicharacter. That is a help toward defining the vertex algebra structure, as we have a formula for the braiding map, but is only part of the picture, because as it turns out these vertex representations cannot be accommodated by the axioms of H_D -quantum vertex algebras. The H_D -quantum vertex algebras are part of bigger structures—FR type of deformed chiral algebras, that involve bigger Hopf algebras, containing H_D . This will be discussed in [Ang], where we also extend the definition of “fields” so that we can incorporate more general Hopf algebras. Thus the H_D -quantum vertex algebras are, if not the complete answer, at least a necessary ingredient in the description of quantum vertex algebras.

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