QUADRATIC DIFFERENTIAL OPERATORS, BICHARACTERS AND • PRODUCTS

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Abstract. For a commutative cocommutative Hopf algebra we study the relationship between a certain linear map defined via a bicharacter, an exponential of a quadratic differential operator and a $\bullet$ product obtained via twisting by a bicharacter. This new relationship between $\bullet$ products and exponentials of quadratic differential operators was inspired by studying the exponential of a particular quadratic differential operator introduced in [FLM88] and used in the theory of twisted modules of lattice vertex algebras.

Contents

1. Introduction

1.1. **Motivation From Vertex Algebras.** We would like to start with a few words about our motivation. This discussion uses some basic facts about vertex algebras and their twisted modules. The reader who is not interested in vertex algebras can skip to the next subsection 1.2. In the main part of the paper we will not use vertex algebras at all.

In the theory of vertex algebras (see for example [Kac98], [LL04] for background and more details) there is a notion of state-field correspondence: if V is a vertex algebra and $a \in V$ is a state, then the theory produces a field $a(z) = Y(a, z) \colon V \to V((z))$. A vertex algebra carries infinitely many products for the states: if $a, b \in V$ then we define $a_{(n)}b = \mathrm{Res}_z(z^n a(z)b)$.

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Of particular importance is the (-1) product: On the one hand it often happens that V is the span of the (-1) products of a number of generating states $v^{[i]}$, i.e., span of products $v^{[i_1]}_{(-1)}v^{[i_2]}_{(-1)}\cdots_{(-1)}v^{[i_n]}$. On the other hand the (-1)-product of states corresponds to the normal ordered product of the fields of these states, i.e., the field corresponding to a state $a_{(-1)}b$ is the normal ordered product $:a(z)b(z):$ of the fields $a(z)$ and $b(z)$. (See Appendix A for an introduction to normal ordered products.)

Vertex algebras have (twisted) modules M (see [FLM88], [Don94], [DL96], [Li96], [Roi03], [BK04]); in this situation we still have a state-field correspondence Y_M, now from states in V to (twisted) fields $a_M(z)$ on M. In many cases one can still define the normal ordered product of twisted fields. But, for twisted modules, it is no longer true that in general the normal ordered product of the twisted fields corresponds to the (-1)-product of states in V. And vice versa, it is not true that the field on a twisted module corresponding to a state $a_{(-1)}b$ is the normal ordered product of the fields $a_M(z)$ and $b_M(z)$.

To rectify this I. Frenkel, Lepowsky and Meurman introduced in [FLM88] a very clever and unexpected modification of the construction, in the case of a lattice vertex algebra. They introduced a specific quadratic differential operator Δ_z acting on the states, such that the field of $a_{(-1)}b$ on a twisted module M is given by

$$Y_M(a_{(-1)}b,z)=e^{\Delta_z}:a_M(z)b_M(z): .$$

(Here the meaning of the right-hand side is that one decomposes $e^{\Delta_z}(a_{(-1)}b)$ in a sum of (-1) products of generating states and takes the sum of the normal ordered product of the corresponding fields on M.)

This leads to the following question: does the normal ordered product of fields on a twisted module correspond to a new, modified, product of states, just as the normal ordered product on the vertex algebra itself corresponds to the (-1) product? Or equivalently: is there a modified product of states, call it $a\bullet b$, on V, such that the state field correspondence $Y_M\colon a\mapsto a_M(z)$ is a homomorphism from V with the $\bullet$ product to the space of fields on M with normal ordered product as operation? The answer turns out to be yes: the operator e^{Δ_z} indeed leads to a new product $\bullet$ on V satisfying this property.

In this paper we abstract this situation, and show that in fairly general context we have a similar relation between exponentials of quadratic differential operators and $\bullet$ products.

1.2. **Overview of the Paper.** The definition of the $\bullet$ product was motivated by the theory of twisted modules of vertex algebras. We abstract and simplify the situation, and consider the following problem:
Given a commutative algebra $(M,\cdot)$ and a new commutative product $\bullet$ on M can we find a map $\mathrm{EQ}\colon M\to M$ such that

$$\mathrm{EQ}(a\cdot b)=\mathrm{EQ}(a)\bullet\mathrm{EQ}(b),\quad a,b\in M?$$

Of course, in general such a map EQ will not exist. Therefore we study this problem under the assumption of an extra structure on M: we will assume that M is a commutative and cocommutative Hopf algebra with a bicharacter r.

This allows us to define, using the bicharacter r, a general $\bullet$ product[1]. Next we define a map EQ, again depending on r, which acts as a homomorphism between the ordinary product and the $\bullet$ product (see Section 2).

Further, we prove that we can actually find a logarithm of the map EQ — a quadratic operator $\Delta_z = \mathbf{Q}$ acting on the commutative cocommutative Hopf algebra, such that we have $\mathrm{EQ} = e^{\mathbf{Q}}$. (In the paper we will use the notation $\mathbf{Q}$ instead of Δ_z in order to avoid confusion with the notation for the coproduct in the Hopf algebra.)[2]

In Section 3 we first prove the existence of the logarithm for the case where M is a polynomial algebra. In the next section 4 we continue with the general case of a commutative cocommutative Hopf algebra (i.e., when grouplike elements are present). With some restrictions on the bicharacter r we again establish the relation between the $\bullet$ product and the maps EQ and $e^{\mathbf{Q}}$. The result is summarized in Theorem 4.7. In section 5 we show that the operator e^{Δ_z} of [FLM88] is a special case of what we call $e^{Q} = \mathrm{EQ}$.

In Appendix A we make the connection between our construction for a commutative and cocommutative Hopf algebra and the case of Heisenberg algebra studied by I. Frenkel, Lepowsky and Meurman. We discuss the definition of normal ordered product for (twisted) Heisenberg fields, and the relation with a $\bullet$ product of the space of states. In Appendix B we discuss an alternative operator description of the coproduct and some of its uses.

2. Bicharacters and Deformations of Commutative Algebras

For a Hopf algebra M we will denote the coproduct and the counit by $\triangle$ and η, the antipode by S. If a is an element of a Hopf algebra we will use Sweedler's notation and write $\triangle(a) = \sum a' \otimes a''$. We often will omit the summation sign. (For more details on Hopf algebras see for example [Kas95]).

Recall that a Hopf algebra M is cocommutative if for any $m \in M$ we have $\sum m' \otimes m'' = \sum m'' \otimes m'$. A primitive element $m \in M$ is such that we have $\triangle(m) = m \otimes 1 + 1 \otimes m$, $\eta(m) = 0$, and $S(m) = -m$. A grouplike element $g \in M$ is such that $\triangle(g) = g \otimes g,\ \eta(g) = 1,\ S(g) = g^{-1}$.

Definition 2.1 (**Bicharacter**([Bor01]))**.** *Let M be a commutative and cocommutative Hopf algebra over $\mathbb{C}$ and A a commutative $\mathbb{C}$ algebra. An A-valued*

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[1] This is a construction similar to the one first introduced in the context of vertex algebras by Borcherds in [Bor01], but also used in a more general context in the theory of Hopf algebras and quantum groups.

[2] The $\bullet$ product in this sense resembles the Moyal star product, see e.g. [FS94]. The Moyal star product is of course noncommutative, but it too can be defined via an exponential of a bi-differential operator.

bicharacter on M is a linear map $r : M\otimes M \to A$, such that for any $a, b, c \in M$

$$r(1 \otimes a) = \eta(a) = r(a \otimes 1),$$
$$r(ab \otimes c) = \sum r(a \otimes c')r(b \otimes c''),$$
$$r(a \otimes bc) = \sum r(a' \otimes b)r(a'' \otimes c).$$

If r, s are bicharacters as above, we can define their product $r \circ s$ by

$$r \circ s(a \otimes b) = r(a' \otimes b')s(a'' \otimes b''). \tag{2.1}$$

We refer to $\circ$ as the convolution product of bicharacters. It is easy to see that the product of two bicharacters is a bicharacter. By cocommutativity of M and commutativity of A the convolution product is commutative. The formula

$$\epsilon(a \otimes b) = \eta(a)\eta(b) \tag{2.2}$$

defines the identity bicharacter ϵ. If S is the antipode of M then the formula

$$r^{-1}(a \otimes b) = r(S(a) \otimes b) \tag{2.3}$$

defines the inverse bicharacter with respect to convolution. The bicharacters on M therefore form an Abelian group with respect to convolution.

The group of bicharacters carries an involution, $r \mapsto r^t$ where

$$r^t(a \otimes b) = r(b \otimes a).$$

A bicharacter s is called symmetric if it is invariant under the involution: $s = s^t$.

One point of bicharacters on M is that they allow us to deform the multiplication of M. So let M be a commutative and cocommutative Hopf algebra over $\mathbb{C}$ as above, A a $\mathbb{C}$-algebra and let $r : M \otimes M \to A$ be an A-valued bicharacter on M. We define a new product on $M_A = M \otimes A$ by

$$m_r(a \otimes b) = \sum a'b'r(a'' \otimes b''), \tag{2.4}$$

for any $a, b \in M$, where ab is the initial algebra product of a and b in M.

Lemma 2.2 (**Twisting by a bicharacter** [Bor01]). *The product m_r is associative and unital (with same unit 1_M). If the bicharacter is symmetric then the new multiplication on M_A is commutative.*

In general the new multiplication on M_A is not commutative.

Definition 2.3. (Symmetrization of a bicharacter and $\bullet$ product) *Starting with the bicharacter r (which might not be symmetric), define a new, symmetric, bicharacter s by $s = r \circ r^t$, or explicitly*

$$s(a \otimes b) = r(a' \otimes b')r(b'' \otimes a''),$$

and define a (commutative) multiplication on M by the twisting with this symmetrized bicharacter:

$$a \bullet b = a'b's(a'' \otimes b'').$$

In general we will call a $\bullet$ product on M any twisting m_s of the standard product by a symmetric bicharacter.

We write $\bullet_s$ when we want to emphasize the dependence on the symmetric bicharacter s.

Note that if r happens to be symmetric, the new bicharacter s obtained from r is *not* identical to r: for instance if $g, \bar{g}$ are grouplike, then $s(g \otimes \bar{g}) = r(g \otimes \bar{g})^2$, and if x, y are primitive $s(x \otimes y) = 2r(x \otimes y)$.

Now we can ask what the relation is between the two multiplications on M_A: the original multiplication, and the $\bullet$ multiplication obtained by twisting with s. To answer this question we begin with the following definition:

Definition 2.4. (The linear map EQ_r) *Let M be a commutative and cocommutative Hopf algebra over $\mathbb{C}$, A a commutative $\mathbb{C}$-algebra and let $r : M \otimes M \to A$ be an A-valued bicharacter on M. Define a linear map*

$$\mathrm{EQ}_r \colon M \to M_A, \quad m \mapsto r(m' \otimes m'')m'''. \tag{2.5}$$

Here we write $\Delta^2(m) = \sum m' \otimes m'' \otimes m'''$.

If the bicharacter r is clear from the context, we will just write EQ for EQ_r.

Example 2.5. For any bicharacter r if x is primitive we have

$$r(x \otimes 1) = r(1 \otimes x) = 0,$$

thus $\mathrm{EQ}_r(x) = x$ for any bicharacter r. If g is grouplike, then $\mathrm{EQ}_r(g) = gr(g \otimes g)$.

Lemma 2.6. *If $a, b \in M$, where M is a commutative and cocommutative Hopf algebra over $\mathbb{C}$ as above, then for any bicharacter r with symmetrization s we have*

$$\mathrm{EQ}_r(ab) = \mathrm{EQ}_r(a) \bullet_s \mathrm{EQ}_r(a).$$

We first recall some facts that will be used in the proof below. Coassociativity requires

$$\begin{aligned} (\triangle)^2(a) &= a' \otimes a'' \otimes a''' = \\ = (Id \otimes \triangle) \circ \triangle(a) &= a' \otimes (a'')' \otimes (a'')'' = \\ = (\triangle \otimes Id) \circ \triangle(a) &= (a')' \otimes (a')'' \otimes a'', \end{aligned}$$

for any element a of the Hopf algebra M, and similarly we can uniquely define

$$\triangle^{n-1}(a) = \sum a^{(1)} \otimes a^{(2)} \otimes \ldots \otimes a^{(n)} \tag{2.6}$$

(extended Sweedler notation). By cocommutativity of M the factors $a^{(i)}$ of $\Delta^{(n-1)}(a)$ are invariant under all permutations of n.

Proof.

$$\begin{aligned}
\mathrm{EQ}(ab) &= r((ab)'\otimes(ab)'')(ab)''' = r(a'b'\otimes a''b'')a'''b''' = \\
&= r(a'\otimes(a''b'')')r(b'\otimes(a''b'')'')a'''b''' = \\
&= r(a^{(1)}\otimes(a^{(2)})'(b^{(2)})')r(b^{(1)}\otimes(a^{(2)})''(b^{(2)})'')a^{(3)}b^{(3)} = \\
&= r(a^{(1)}\otimes a^{(2)}b^{(2)})r(b^{(1)}\otimes a^{(3)}b^{(3)})a^{(4)}b^{(4)} = \\
&= r((a^{(1)})'\otimes a^{(2)})r((a^{(1)})''\otimes b^{(2)})r((b^{(1)})'\otimes a^{(3)})\cdot \\
&\qquad \cdot r((b^{(1)})''\otimes b^{(3)})a^{(4)}b^{(4)} = \\
&= r(a^{(1)}\otimes a^{(3)})r(a^{(2)}\otimes b^{(3)})r(b^{(1)}\otimes a^{(4)})r(b^{(2)}\otimes b^{(4)})a^{(5)}b^{(5)}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathrm{EQ}(a)\bullet\mathrm{EQ}(b) &= \big(r(a'\otimes a'')a'''\big)\bullet\big(r(b'\otimes b'')b'''\big) = \\
&= r(a'\otimes a'')r(b'\otimes b'')(a''')\bullet(b''') = \\
&= r(a'\otimes a'')r(b'\otimes b'')(a''')'(b''')'s((a''')''\otimes(b''')'') = \\
&= r(a^{(1)}\otimes a^{(2)})r(b^{(1)}\otimes b^{(2)})a^{(3)}b^{(3)}s(a^{(4)}\otimes b^{(4)}) = \\
&= r(a^{(1)}\otimes a^{(2)})r(b^{(1)}\otimes b^{(2)})a^{(3)}b^{(3)}r(a^{(4)}\otimes b^{(4)})r(b^{(5)}\otimes a^{(5)})
\end{aligned}$$

The equality then follows using cocommutativity of the coproduct:

$$\begin{aligned}
\mathrm{EQ}(ab) &= r(a^{(1)}\otimes a^{(3)})r(a^{(2)}\otimes b^{(3)})r(b^{(1)}\otimes a^{(4)})r(b^{(2)}\otimes b^{(4)})a^{(5)}b^{(5)} = \\
&= r(a^{(1)}\otimes a^{(2)})r(a^{(3)}\otimes b^{(3)})r(b^{(4)}\otimes a^{(4)})r(b^{(1)}\otimes b^{(2)})a^{(5)}b^{(5)} = \\
&= r(a^{(1)}\otimes a^{(2)})r(b^{(1)}\otimes b^{(2)})r(a^{(4)}\otimes b^{(4)})r(b^{(5)}\otimes a^{(5)})a^{(3)}b^{(3)} = \\
&= \mathrm{EQ}(a)\bullet\mathrm{EQ}(b).
\end{aligned}$$

□

The conclusion is that the map EQ is a homomorphism from $(M_A,\cdot)$ to $(M_A,\bullet)$.

Notice that the algebra structure on $(M,\cdot)$ (and by extension of $(M_A,\cdot)$) can be considered to be in fact the twisting of M by the identity bicharacter ϵ (see (2.2)).

Thus $(M,\cdot)$ is in fact $(M,\bullet_\epsilon)$, and the map EQ_r is a homomorphism from $(M_A,\bullet_\epsilon)$ to $(M_A,\bullet_s)$. This leads to the question of the relation between the different multiplication structures on M given by symmetric bicharacters s. If s_1 and s_2 are the symmetrization of bicharacters r_1 and r_2 then there is a homomorphism EQ that intertwines them. Indeed, $\mathrm{EQ}_{\_}$ is a homomorphism from the Abelian group of bicharacters to linear maps on M:

Lemma 2.7. *For all bicharacters r_1, r_2 we have*

$$\mathrm{EQ}_r = \mathrm{EQ}_{r_1}\circ\mathrm{EQ}_{r_2}, \quad \textit{if } r = r_1\circ r_2.$$

Proof.

$$\mathrm{EQ}_{r_1} \circ \mathrm{EQ}_{r_2}(a) = \mathrm{EQ}_{r_1}(r_2(a^{(1)} \otimes a^{(2)})a^{(3)}) =$$
$$= r_1(a^{(3)\prime} \otimes a^{(3)\prime\prime})a^{(3)\prime\prime\prime} r_2(a^{(1)} \otimes a^{(2)}) = r_1(a^{(1)} \otimes a^{(2)}) r_2(a^{(3)} \otimes a^{(4)})a^{(5)} =$$
$$= r_1 \circ r_2(a' \otimes a'')a''' = r(a' \otimes a'')a''' = \mathrm{EQ}_r(a),$$

by coassociativity and cocommutativity. □

Corollary 2.8. *Each* EQ_r *is invertible, with inverse* $\mathrm{EQ}_{r^{-1}}$ *and if* r_i *are bicharacters with symmetrization* s_i*,* $i = 1, 2$ *then*

$$\mathrm{EQ}_{r_2} \circ \mathrm{EQ}_{r_1^{-1}} \colon (M, \bullet_{s_1}) \to (M, \bullet_{s_2})$$

gives the homomorphism from the multiplication $\bullet_{s_1}$ *to* $\bullet_{s_2}$*.*

More generally one can ask about the relation between symmetric bicharacters that are not necessarily symmetrizations.

Indeed, we have the following generalization of Lemma 2.6.

Theorem 2.9. *Let* r *be a bicharacter defined on a commutative and cocommutative Hopf algebra* M *and* s_1 *any symmetric bicharacter. We have*

$$\mathrm{EQ}_r(a \bullet_{s_1} b) = \mathrm{EQ}_r(a) \bullet_{s_2} \mathrm{EQ}_r(b), \tag{2.7}$$

for $s_2 = s \circ s_1$*, where* s *is the symmetrization of the bicharacter* r*.*

Proof. The proof is very similar to the proof of the Lemma 2.6 above. □

Observe that the maps EQ_r gives an action of the Abelian group of bicharacters on the space of $\bullet$ products. The theorem implies that the space of $\bullet$ products (parametrized by symmetric bicharacters) decomposes into orbits under the action of the Abelian group of all bicharacters.

This leads to the question of the orbit structure of the space of $\bullet$ products on M. In the next section we show that if M is generated by primitive elements s is always a symmetrization, and consequently there is only one orbit, and all $\bullet$ products are related to the trivial product $\cdot = \bullet_\epsilon$. In Section 4 we discuss the case where there are also nontrivial grouplike elements in M. Then there the orbit structure can be more complicated.

3. The Polynomial Algebra and Quadratic Differential Operators

Consider a cocommutative Hopf algebra M with no grouplike elements except the unit 1. Any such Hopf algebra over a field of characteristic 0 is the universal enveloping algebra of the Lie algebra of its primitive elements (for proof see for example [MM65]). Suppose that M is also commutative: then M is the universal enveloping algebra of the **abelian** Lie algebra of its primitive elements. Thus any such Hopf algebra over $\mathbb{C}$ is nothing else but the polynomial algebra over a basis for the primitive elements. The result described below works for any polynomial algebra M, but for the examples we are interested in (motivated from the theory of twisted modules of vertex algebras) we will only look at the case when the algebra M is generated by countably many primitive generators.

Thus, consider the polynomial algebra

$$V_0 = \mathbb{C}[x_1, x_2, \dots]$$

in variables $x_i, i \geq 1,\ i \in \mathbb{N}$. ($V_0$ is the universal enveloping algebra of the Abelian Lie algebra $\oplus_{i\geq 1}\mathbb{C}x_i$). As such, V_0 is a commutative and cocommutative Hopf algebra with primitive generators x_i.

First of all we can immediately answer the question whether a symmetric bicharacter gives a multiplication $\bullet_s$ related to the standard multiplication $\cdot$ of V_0 by an isomorphism EQ_r for some bicharacter r.

This is certainly the case if s is a symmetrization, as we argued above, but in our present case (V_0 generated by primitives) we see that we can define, given s, a bicharacter r by defining $r(x_i\otimes x_j) = \frac{1}{2}s(x_i\otimes x_j)$ on the generators. The symmetrization of the bicharacter r is then

$$r\circ r^t(x_i\otimes x_j) = r(x_i'\otimes x_j')r(x_j''\otimes x_i'') = r(x_i\otimes x_j) + r(x_j\otimes x_i) = s(x_i\otimes x_j).$$

Theorem 3.1. *Let $V_0 = \mathbb{C}[x_1, x_2, \dots]$. Then for any symmetric bicharacters s_1, s_2 exists an isomorphism* $\mathrm{EQ}\colon (V\otimes A, \bullet_{s_1}) \to (V\otimes A, \bullet_{s_2})$ *intertwining the $\bullet$ products. In particular, exists an isomorphism* $\mathrm{EQ}\colon (V\otimes A, \cdot) \to (V\otimes A, \bullet_s)$ *relating the standard product to any $\bullet_s$ product.*

Proof. Follows from Theorem 2.9 and Corollary 2.8. □

Now in the present case (V_0 generated by primitives) the map EQ, although defined by a bicharacter r, depends only on the symmetrization s of the bicharacter r. In order to prove this, see Corollary 3.6, we first proceed to give another description of the map EQ.

The coproduct on V_0 is conveniently described using infinite order differential operators. Write $x_i^{(1)} = x_i\otimes 1$ and $x_i^{(2)} = 1\otimes x_i$. Then we have

$$\Delta(f(x_i)) = f(x_i^{(1)} + x_i^{(2)}) = e^{\sum_{i\geq 1} x_i^{(1)}\frac{\partial}{\partial x_i^{(2)}}} f(x_i^{(2)}), \tag{3.1}$$

and similarly

$$\Delta^2(f(x_i)) = f(x_i^{(1)} + x_i^{(2)} + x_i^{(3)}) = e^{\sum_{i\geq 1}(x_i^{(1)}+x_i^{(2)})\frac{\partial}{\partial x_i^{(3)}}} f(x_i^{(3)}), \tag{3.2}$$

where $f(x_i)$ is any polynomial in the variables $x_i,\ i\geq 1$.

Next we fix a bicharacter on V_0. Since V_0 is generated by the variables x_i the bicharacter is completely determined by the elements

$$q_{mn} = r(x_m\otimes x_n) \in A, \quad m, n\geq 1.$$

Note also that we have the following simple properties of the bicharacter evaluated at powers of the variables:

$$r(x_m^s\otimes x_n^t) = s!\delta_{st} r(x_m\otimes x_n)^s. \tag{3.3}$$

The following lemma asserts that we can find the logarithm of the map EQ:

Lemma 3.2. *The map* EQ *defined in* (2.5) *is the exponential of the infinite order quadratic differential operator* $Q_p : V_0 \to V_0 \otimes A$*,*

$$\mathrm{EQ}(f) = e^{Q_p}(f), \quad \textit{where} \quad Q_p = \sum_{m,n\geq 1} q_{mn} \frac{\partial^2}{\partial x_m \partial x_n}.$$

Proof. We use the exponential form (3.2) of the coproduct:

$$\mathrm{EQ}(f) = r(f' \otimes f'')f''' = r(e^{\sum_m x_m \partial_{x_m}} \otimes e^{\sum_n x_n \partial_{x_n}})f(x_i),$$

where the exponentials are expanded as power series and the partial derivatives act on f. Next we use some simple properties of the bicharacter: we have

$$r(e^{\sum_m x_m \partial_{x_m}} \otimes e^{\sum_n x_n \partial_{x_n}}) = \prod_{m,n} r(e^{x_m \partial_{x_m}} \otimes e^{x_n \partial_{x_n}}),$$

and, using (3.3),

$$r(e^{x_m \partial_{x_m}} \otimes e^{x_n \partial_{x_n}}) = \sum_{s,t\geq 0} r(x_m^s \otimes x_n^t) \frac{\partial^{s+t}}{s!t!\partial^s x_m \partial^t x_n} =$$

$$= \sum_{s,} r(x_m \otimes x_n)^s \left(\frac{\partial^2}{\partial x_m \partial x_n} \right)^s /s! = e^{r(x_m \otimes x_n) \frac{\partial^2}{\partial x_m \partial x_n}}.$$

Combining the last two results gives the proof of the lemma. □

We can rephrase Lemmas 2.6 and 3.2 by introducing the notion of a $\bullet$ polynomial. If $P \in V_0$, then P is a linear combination of monomials $x_{i_1} x_{i_2} \dots x_{i_s}$. Define then $P^\bullet$, the $\bullet$ polynomial[3] of P, as the same linear combination of expressions $x_{i_1} \bullet x_{i_2} \bullet \cdots \bullet x_{i_s}$.

We get then the following result from Lemmas 2.6 and 3.2:

Theorem 3.3. *Let* $V_0 = \mathbb{C}[x_1, x_2, \dots]$*, with* r *a bicharacter with values in* A *on* V_0 *and let for* $q_{mn} = r(x_m \otimes x_n)$ *the quadratic differential operator* Q_p *be*

$$Q_p = \sum_{m,n\geq 1} q_{mn} \frac{\partial^2}{\partial x_m \partial x_n}$$

Let $s = r \circ r^t$ *be the symmetrization of* r *and* $\bullet = \bullet_s$ *the associated* $\bullet$ *product on* V_0*. Then for all* $P \in V_0$ *we have*

$$e^{Q_p}(P) = EQ_r(P) = r(P' \otimes P'')P'''.$$

Moreover we also have

$$e^{Q_p}(P) = P^\bullet.$$

and e^{Q_p} *is a homomorphism from* $(V_0 \otimes A, \cdot)$ *to* $(V_0 \otimes A, \bullet)$*.*

[3]The $\bullet$ polynomial of a monomial $x_{i_1} x_{i_2} \dots x_{i_s}$ can be thought of as a "normal ordered product of states"—the analog of the normal ordered product of fields $a_{i_1}(z), a_{i_2}(z), \dots, a_{i_s}(z)$ in the theory of vertex algebras. See Appendix A.

Example 3.4. For instance: the $\bullet$ monomial of a single variable is

$$(x_m)^\bullet = e^{Q_p}(x_m) = \mathrm{EQ}_p(x_m) = x_m,$$

and for two variables we have

$$(x_m x_n)^\bullet = x_m \bullet x_n = m_s(x_m \otimes x_n) = x_m x_n + s(x_m \otimes x_n) = x_m x_n + (q_{mn} + q_{nm}),$$

One should note the above is indeed true both when $m \neq n$ and when $m = n$.

One should also be careful when mixing the two products—we can write the twisting (it involves both products):

$$(x_m x_n) \bullet x_l = m_s(x_m x_n \otimes x_l) = x_n x_m x_m + (q_{ml} + q_{lm}) x_n + (q_{nl} + q_{ln}) x_m,$$

but the above is **not** the $\bullet$ monomial $P^\bullet = x_m \bullet x_n \bullet x_l = e_p^Q(P)$, corresponding to the monomial $P = x_m x_n x_l$; $P^\bullet$ is by definition the **successive** application of twisting by the bicharacter:

$$P^\bullet = x_m \bullet x_n \bullet x_l = x_n x_m x_m + (q_{mn} + q_{nm}) x_l + (q_{ml} + q_{lm}) x_n + (q_{nl} + q_{ln}) x_m.$$

Example 3.5. To get examples of bicharacters, and hence of maps EQ_r and quadratic differential operators one has great freedom. One first chooses a $\mathbb{C}$-algebra A, and then for each pair of primitive elements x_m, x_n an element q_{mn} in A. These can be conveniently encoded in a generating series

$$f(x, y) = \sum_{m,n=1}^{\infty} q_{mn} x^m y^n$$

so that

$$r(x_m \otimes x_n) = q_{mn} = \frac{\partial_x^m}{m!} \frac{\partial_x^n}{n!} f(x, y).$$

See Example 5.2 for an explicit choice of such generating series.

Corollary 3.6. (to Theorem 3.1) *Let $s_1, s_2 : V_0 \to A$ be two symmetric bicharacters defined on the polynomial algebra V_0. Then the map $\mathrm{EQ}_r \colon V_0 \to V_0 \otimes A$ intertwining the $\bullet$ products*

$$\mathrm{EQ}_r(a \bullet_{s_1} b) = \mathrm{EQ}_r(a) \bullet_{s_2} \mathrm{EQ}_r(b), \tag{3.4}$$

is unique among the maps EQ_r, for various bicharacters r, and depends only on the symmetrization bicharacter $s = r \circ r^t = s_2 \circ s_1^{-1}$.

Proof. From Lemma 2.8, we can determine the symmetrization bicharacter $s = s_1 \circ (s_2)^{-1}$ for the map EQ_r. It is obvious we cannot find the bicharacter r from it's symmetrization s uniquely, but on V_0 the map EQ_r doesn't actually depend on r, but only on its symmetrization bicharacter $s = r \circ r^t$: Since the map EQ_r on V_0 actually coincides with the map e^Q, and Q is a quadratic differential operator on a polynomial algebra, we can rewrite Q as

$$Q = \sum_{m,n \geq 1} q_{mn} \frac{\partial^2}{\partial x_m \partial x_n} = \sum_{m,n \geq 1} \frac{q_{mn} + q_{nm}}{2} \frac{\partial^2}{\partial x_m \partial x_n} =$$

$$= \sum_{m,n \geq 1} \frac{r(x_m \otimes x_n) + r(x_n \otimes x_m)}{2} \frac{\partial^2}{\partial x_m \partial x_n} = \sum_{m,n \geq 1} \frac{s(x_m \otimes x_n)}{2} \frac{\partial^2}{\partial x_m \partial x_n}$$

Thus, we see that the map e^Q, and so to the map EQ_r on V_0, depends only on the symmetrization bicharacter s. Thus, as a map, the intertwiner EQ of the bullet products $\bullet_{s_1}$ and $\bullet_{s_2}$ is unique. □

4. The case of a Commutative and Cocommutative Hopf Algebra.

In the previous section we saw that if M is generated by primitive elements any two $\bullet$ products can be intertwined (see Theorem 3.1). This depended on the fact that for a Hopf algebra M generated by primitive elements every symmetric bicharacter is a symmetrization. This is no longer true if M contains nontrivial grouplike elements (i.e., grouplike elements distinct from the unit element 1_M of M).

Example 4.1. Let M be the group algebra of the free rank 1 abelian group. M has a basis $e^{n\alpha}$, $n \in \mathbb{Z}$, with multiplication $e^{m\alpha}e^{n\alpha} = e^{(m+n)\alpha}$, and $e^0 = 1_M$. To define a bicharacter choose $A = \mathbb{C}[z]$, and define on the generator

$$s(e^\alpha \otimes e^\alpha) = z.$$

Clearly there is no bicharacter r (with values in A) with a symmetrization s: if such a bicharacter would exist we would have

$$r \circ r^t(e^\alpha \otimes e^\alpha) = r(e^{\alpha'} \otimes e^{\alpha'})r(e^{\alpha''} \otimes e^{\alpha''}) = r(e^\alpha \otimes e^\alpha)^2 = z$$

This has no solution in A.

Similarly, there is no isomorphism $\mathrm{EQ}_r\colon (M_A, \cdot) \to (M_A, \bullet_s)$: For we have

$$\mathrm{EQ}_r(e^{2\alpha}) = e^{2\alpha}r(e^{2\alpha} \otimes e^{2\alpha}) = e^{2\alpha}r(e^\alpha \otimes e^\alpha)^4$$
$$\mathrm{EQ}_r(e^\alpha) \bullet_s \mathrm{EQ}_r(e^\alpha) = e^{2\alpha}r(e^\alpha \otimes e^\alpha)^2 s(e^\alpha \otimes e^\alpha) = ze^{2\alpha}r(e^\alpha \otimes e^\alpha)^2.$$

Thus if such an isomorphism EQ_r would exist we would have

$$e^{2\alpha}r(e^\alpha \otimes e^\alpha)^4 = \mathrm{EQ}_r(e^{2\alpha}) = \mathrm{EQ}_r(e^\alpha) \bullet_s \mathrm{EQ}_r(e^\alpha) = ze^{2\alpha}r(e^\alpha \otimes e^\alpha)^2,$$

which is not possible for a bicharacter r taking values in $A = \mathbb{C}[z]$.

From now on in this paper we will assume that for all bicharacters r (symmetric or not) the values on grouplikes is a constant, i.e.,

$$r(g \otimes \tilde{g}) \in \mathbb{C} \subset A,$$

for any $g, \tilde{g}$ are grouplike. In this case it is immediate that all symmetric bicharacters are symmetrizations, so that all $\bullet$ product are related to $\cdot = \bullet_\epsilon$ by some map EQ_r. In fact, for this to be true, we only need to require that the values of the bicharacters on grouplikes are exact squares; but we need the "constant on grouplikes" condition if we are to find logarithm of the map EQ_r (see Lemma 4.4 and Remark 4.5). Even if this "constant on grouplikes" condition is imposed, in contrast to Corollary 3.6, it is clear that the intertwiner maps EQ_r are not unique when grouplike elements are present.

We require our Hopf algebras to be commutative and cocommutative. This implies (see [Swe67] for details) that M is the product of a group algebra $\mathbb{C}[G]$, and an universal enveloping algebra $\mathcal{U}(L)$, where G and L are Abelian. For simplicity assume that G is finitely generated. Then G decomposes as $G =$

$G_{\text{Tor}} \times G_{\text{Free}}$, with G_{Tor} the torsion part, a finite group consisting of elements δ of finite order, and G_{Free} the free part, generated by elements $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ of infinite order. Let L have basis $x_n, n \geq 1$. Then our Hopf algebra has the form

$$M = \mathbb{C}[\delta_1, \delta_2, \ldots, \delta_s, e^{\pm\alpha_1}, \ldots, e^{\pm\alpha_l}, x_1, x_2, \ldots],$$

where $\delta_t^{N_t} = 1$, $e^{\alpha_i} e^{-\alpha_i} = 1$. The Hopf structure is determined by declaring the elements $\delta_1, \ldots, \delta_s, e^{\alpha_1}, \ldots, e^{\alpha_\ell}$ to be grouplike, and the elements $x_1, x_2, \ldots$ to be primitive.

We would like to give a logarithm of the map EQ_r (similar to Lemma 3.2) in the more general case when grouplike elements are present in M.

But first consider the values of a bicharacter on torsion elements. If $\delta \in M$ is finite order grouplike element, $\delta^N = 1$, then we have for any grouplike g $r(\delta^N \otimes g) = r(\delta \otimes g)^N = 1$. So in this case $r(\delta \otimes g)$ is a root of unity. In the same way, if x is primitive we have $r(\delta \otimes x) = 0$. This implies that the torsion elements contribute to r and hence to EQ only root of unity factors. Therefore we will assume for simplicity's sake that M is in fact torsion free. (In Remark 4.9 we will discuss the impact of torsion elements.) From now on V will denote a commutative and cocommutative Hopf algebra without torsion elements, $V = \mathbb{C}[G] \otimes \mathcal{U}(L)$ with G a finitely generated free Abelian group, and L with countable basis.

This means that we assume V has the form

$$V = \mathbb{C}[e^{\pm\alpha_1}, \ldots, e^{\pm\alpha_l}, x_1, x_2, \ldots].$$

We also keep the condition that $r(e^{\alpha_i} \otimes e^{\alpha_j}) = e^{a_{ij}} \in \mathbb{C}$.

Then we will argue that we can find (in the torsion free case) a quadratic differential operator $\mathbf{Q}$ serving as a logarithm to the map EQ_r and establish a relation similar to that of Theorem 3.3 for more general Hopf algebras with grouplike elements (as above).

Definition 4.2. *Let V be a torsion free commutative cocommutative Hopf algebra as above, $V = \mathbb{C}(G) \otimes \mathbb{C}[x_1, x_2, \ldots]$, $G = \mathbb{Z}[\alpha_1, \alpha_2, \ldots, \alpha_k]$. Define derivations $\frac{\partial}{\partial \alpha_i} : V \to V$, $i = 1, \ldots, k$, by*

$$\frac{\partial}{\partial \alpha_i}(e^{\alpha} P(x)) = m_i e^{\alpha} P(x) \tag{4.1}$$

for any $\alpha = \sum_{i=1}^k m_i \alpha_i$, $m_i \in \mathbb{Z}$, $P(x) \in \mathbb{C}[x_1, x_2, \ldots]$.

Example 4.3. We have

$$\frac{\partial}{\partial \alpha_i}(e^{\alpha_j}) = \delta_{ij} e^{\alpha_j}.$$

Recall that on $V = \mathbb{C}(G) \otimes \mathbb{C}[x_1, x_2, \ldots]$ we also have the partial derivatives with respect to the variables x_n, with

$$\frac{\partial}{\partial x_n}(e^{\alpha} P(x)) = e^{\alpha}(\frac{\partial}{\partial x_n} P(x)), \quad \text{for any} \quad n \geq 1, \quad \alpha \quad \text{as above}.$$

Of course, $\frac{\partial}{\partial x_n}$ commutes with $\frac{\partial}{\partial \alpha_i}$.

Lemma 4.4. *Let* $V = \mathbb{C}[e^{\pm\alpha_1}, \ldots, e^{\pm\alpha_\ell}, x_1, x_2, \ldots]$ *as before. Let* A *be a commutative* $\mathbb{C}$*-algebra and* r *an* A*-valued bicharacter with values on generators*

$$
\begin{aligned}
r(e^{\alpha_i} \otimes e^{\alpha_j}) &= e^{a_{ij}} \in \mathbb{C}\\
r(e^{\alpha_i} \otimes x_m) &= b_{im} \in A\\
r(x_m \otimes e^{\alpha_i}) &= c_{mi} \in A\\
r(x_i \otimes x_m) &= q_{mn} \in A
\end{aligned}
\tag{4.2}
$$

Then the map $\mathrm{EQ}\colon V \to V_A, \quad a \mapsto r(a' \otimes a'')a'''$ *is the exponential of an infinite order quadratic operator* $\mathbf{Q}$

$$\mathrm{EQ}_r(m) = e^{\mathbf{Q}}(m), \quad m \in V, \tag{4.3}$$

where $\mathbf{Q}$ *is defined by*

$$
\begin{aligned}
\mathbf{Q} = \sum_{i,j=1}^{\ell} a_{ij} \frac{\partial}{\partial \alpha_i} \frac{\partial}{\partial \alpha_j} + \sum_{i=1}^{\ell} \sum_{m \geq 1} b_{im} \frac{\partial}{\partial \alpha_i} \frac{\partial}{\partial x_m} +\\
+ \sum_{m \geq 1} \sum_{i=1}^{\ell} c_{mi} \frac{\partial}{\partial x_m} \frac{\partial}{\partial \alpha_i} + \sum_{m,n \geq 1} q_{mn} \frac{\partial^2}{\partial x_m \partial x_n}.
\end{aligned}
\tag{4.4}
$$

Remark 4.5. The first equation of (4.2) is the reason that we require $r(e^{\alpha_i} \otimes e^{\alpha_j})$ to be a constant, as opposed to no such requirement for instance on the bicharacter $r(x_i \otimes x_m)$, which is allowed to be in a more general target algebra A. For example, when $A = \mathbf{C}[z]$, if $r(e^{\alpha_i} \otimes e^{\alpha_j}) \in A$ is not a constant, the "logarithm" a_{ij} is not an element of A.

Before giving the proof, we will start with some examples.

Example 4.6. In particular, for an element e^{α_i} we have:

$$e^{\mathbf{Q}}(e^{\alpha_i}) = e^{\alpha_i} e^{a_{ii}} = e^{\alpha_i} r(e^{\alpha_i} \otimes e^{\alpha_i}) = \mathrm{EQ}(e^{\alpha_i}) \tag{4.5}$$

For two independent grouplike elements $e^{\alpha_i}, \ e^{\alpha_j}$ we have:

$$
\begin{aligned}
&e^{\mathbf{Q}}(e^{\alpha_i} e^{\alpha_j}) = e^{\alpha_i} e^{\alpha_j} e^{a_{ii} + a_{ij} + a_{ji} + a_{jj}} =\\
&= e^{\alpha_i} e^{\alpha_j} r(e^{\alpha_i} \otimes e^{\alpha_i}) r(e^{\alpha_i} \otimes e^{\alpha_j}) r(e^{\alpha_j} \otimes e^{\alpha_i}) r(e^{\alpha_j} \otimes e^{\alpha_j}) =\\
&= e^{\alpha_i} e^{\alpha_j} r(e^{\alpha_i} e^{\alpha_j} \otimes e^{\alpha_i} e^{\alpha_j}) = \mathrm{EQ}(e^{\alpha_i} e^{\alpha_j}).
\end{aligned}
$$

We now proceed with the proof of the lemma.

Proof. We can spit the quadratic operator $\mathbf{Q}$ in 3 parts:

$$\mathbf{Q} = Q_0 + Q_1 + Q_p, \tag{4.6}$$

where

$$Q_0 = \sum_{i,j=1}^{k} a_{ij}\frac{\partial}{\partial\alpha_i}\frac{\partial}{\partial\alpha_j} \tag{4.7}$$

$$Q_1 = \sum_{i=1}^{k}\sum_{m\geq 1} b_{im}\frac{\partial}{\partial\alpha_i}\frac{\partial}{\partial x_m} + \sum_{m\geq 1}\sum_{i=1}^{k} c_{mi}\frac{\partial}{\partial x_m}\frac{\partial}{\partial\alpha_i} \tag{4.8}$$

$$Q_p = \sum_{m,n\geq 1} q_{mn}\frac{\partial^2}{\partial x_m\partial x_n} \tag{4.9}$$

Notice that the third part, the operator Q_p, coincides with the operator Q_p used in section 3. These three differential operators commute among each other, thus we can write

$$e^{\mathbf{Q}} = e^{Q_1}e^{Q_p}e^{Q_0} \tag{4.10}$$

Moreover, if $e^{\alpha},\ \alpha = \sum_{i=1}^{k} m_i\alpha_i,\ m_i\in\mathbb{Z}$ is a grouplike element, and $P(x)\in\mathbb{C}[x_1,x_2,\dots]$, we have

$$e^{Q_p}(e^{\alpha}) = e^{\alpha},\quad e^{Q_p}(e^{\alpha}P(x)) = e^{\alpha}e^{Q_p}(P(x)) \tag{4.11}$$

On the other hand, we have

$$e^{Q_0}(P(x)) = P(x),\quad e^{Q_0}(e^{\alpha}P(x)) = e^{Q_0}(e^{\alpha})P(x) \tag{4.12}$$

Thus

$$e^{\mathbf{Q}}(e^{\alpha}P(x)) = e^{Q_1}e^{Q_p}e^{Q_0}(e^{\alpha}P(x)) = e^{Q_1}\big(e^{Q_0}(e^{\alpha})e^{Q_p}(P(x))\big) \tag{4.13}$$

We know from Theorem 3.3 that $e^{Q_p}(P(x))$ is the $\bullet$ polynomial $P^{\bullet}(x)$, which equals $\mathrm{EQ}(P(x))$. It is not hard to show, similar to example 4.6, that for purely grouplike elements

$$e^{Q_0}(e^{\alpha}) = \mathrm{EQ}(e^{\alpha}). \tag{4.14}$$

Thus we have

$$e^{\mathbf{Q}}(e^{\alpha}P(x)) = e^{Q_1}\big(\,\mathrm{EQ}(e^{\alpha})\,\mathrm{EQ}(P(x))\big). \tag{4.15}$$

We are preparing to use Lemma 2.6, and to do that we need to show that for a grouplike element e^{α}, and any polynomial $P(x)\in\mathbb{C}[x_1,x_2,\dots]$ we have

$$e^{Q_1}(e^{\alpha}P(x)) = e^{\alpha}\bullet P(x). \tag{4.16}$$

To that end,

$$Q_1(e^{\alpha}P(x)) = \Big(\sum_{i=1}^{k}\sum_{m\geq 1} b_{im}\frac{\partial}{\partial\alpha_i}\frac{\partial}{\partial x_m} + \sum_{m\geq 1}\sum_{i=1}^{k} c_{mi}\frac{\partial}{\partial x_m}\frac{\partial}{\partial\alpha_i}\Big)(e^{\alpha}P(x)) =$$

$$= e^{\alpha}\Big(\sum_{i=1}^{k} m_i\big(\sum_{m\geq 1} b_{im}\frac{\partial}{\partial x_m} + \sum_{m\geq 1} c_{mi}\frac{\partial}{\partial x_m}\big)\Big)(P(x)),$$

and so

$$e^{Q_1}(e^{\alpha}P(x)) = e^{\sum_{i=1}^k m_i\alpha_i} e^{\left(\sum_{i=1}^k m_i(\sum_{m\geq 1} b_{im}\frac{\partial}{\partial x_m}+\sum_{m\geq 1} c_{mi}\frac{\partial}{\partial x_m})\right)}(P(x)) =$$

$$= \prod_{i=1}^k \left((e^{\alpha_i})^{m_i}(e^{\sum_{m\geq 1} b_{im}\frac{\partial}{\partial x_m}})^{m_i}(e^{\sum_{m\geq 1} c_{mi}\frac{\partial}{\partial x_m}})^{m_i}P(x),\right.$$

which, using (3.1), is precisely the $\bullet$ product

$$e^{\alpha}(P(x))'s(e^{\alpha}\otimes(P(x))''),$$

where s is the symmetrization bicharacter

$$s(e^{\alpha}\otimes P(x)) = r(e^{\alpha}\otimes(P(x))')r((P(x))''\otimes e^{\alpha}).$$

Thus we have according to Lemma 2.6

$$e^{\mathbf{Q}}(e^{\alpha}P(x)) = e^{Q_1}\left(\mathrm{EQ}(e^{\alpha})\,\mathrm{EQ}(P(x))\right) = \mathrm{EQ}(e^{\alpha})\bullet\mathrm{EQ}(P(x)) = \mathrm{EQ}(e^{\alpha}P(x)).$$

□

In the paragraph before Theorem 3.3 we introduced the notion of a $\bullet$ polynomial $P^{\bullet}$ corresponding to $P\in V_0$. In a similar way we define the $\bullet$ element $a^{\bullet}$ for $a\in V$ as follows. We write $a = e^{\alpha}P$, and put

$$a^{\bullet} = (e^{\alpha})^{\bullet}\bullet P^{\bullet},$$

where $(e^{\alpha})^{\bullet} = e^{\alpha}r(e^{\alpha}\otimes e^{\alpha})$. Here we have fixed a bicharacter r with symmetrization s, and the corresponding product $\bullet = \bullet_s$.

Thus we obtain the following generalization of Theorem 3.3:

Theorem 4.7. *Let $V = \mathbb{C}(G)\otimes\mathbb{C}[x_1, x_2, \dots]$, $G = \mathbb{Z}[\alpha_1, \alpha_2, \dots, \alpha_k]$, with bicharacter r, taking complex values on grouplike elements, with symmetrization $s = r\circ r^t$ and associated $\bullet = \bullet_s$. Let $\mathbf{Q}$ be the quadratic operator* (4.4). *Then for $a\in V$ we have*

$$e^{\mathbf{Q}}(a) = \mathrm{EQ}_r(a) = r(a'\otimes a'')a'''.$$

Moreover

$$e^{\mathbf{Q}}(a) = a^{\bullet},$$

and $e^{\mathbf{Q}}$ is a homomorphism from $(V\otimes A, \cdot)\to(V\otimes A, \bullet)$.

Example 4.8. In particular, from an element e^{α_i} in V we obtain $(e^{\alpha_i})^{\bullet}$:

$$(e^{\alpha_i})^{\bullet} = e^{\alpha_i}r(e^{\alpha_i}\otimes e^{\alpha_i}) = e^{\alpha_i}e^{a_{ii}} = e^{\mathbf{Q}}(e^{\alpha_i}) \tag{4.17}$$

For two independent grouplike elements e^{α_i}, e^{α_j} we have:

$$(e^{\alpha_i}e^{\alpha_j})^{\bullet} = \mathrm{EQ}(e^{\alpha_i}e^{\alpha_j}) = e^{\alpha_i}e^{\alpha_j}r(e^{\alpha_i}e^{\alpha_j}\otimes e^{\alpha_i}e^{\alpha_j}) =$$

$$= (e^{\alpha_i})^{\bullet}(e^{\alpha_j})^{\bullet}s(e^{\alpha_i}\otimes e^{\alpha_j}) = (e^{\alpha_i})^{\bullet}\bullet(e^{\alpha_j})^{\bullet} = e^{\alpha_i}e^{\alpha_j}e^{a_{ii}+a_{ij}+a_{ji}+a_{jj}} =$$

$$= e^{\mathbf{Q}}(e^{\alpha_i}e^{\alpha_j}).$$

Remark 4.9. We now discuss briefly the effect of torsion elements on the above results. Let M be a Hopf algebra with torsion elements:

$$M = \mathbb{C}[G_{\mathrm{Tor}}] \otimes V = \mathbb{C}[\delta_1, \delta_2, \dots, \delta_s, e^{\pm\alpha_1}, \dots, e^{\pm\alpha_l}, x_1, x_2, \dots],$$

with δ_i torsion elements. An element a of M is then of the form

$$a = \delta e^{\alpha} P(x).$$

We fix a symmetric bicharacter on M (taking complex values on grouplikes), and let r be some bicharacter with s as symmetrization. We define the $\bullet$ element of a as

$$a^{\bullet} = (\delta^{\bullet}) \bullet (e^{\alpha})^{\bullet} \bullet P^{\bullet},$$

where $\delta^{\bullet} = \delta r(\delta \otimes \delta)$. (So $\delta^{\bullet}$ differs from δ by a root of unity.)

We have then, just as before,

$$\mathrm{EQ}_r(a) = a^{\bullet}, \quad \mathrm{EQ}_r(ab) = (a^{\bullet}) \bullet (b^{\bullet}).$$

If we want to write EQ in terms of a quadratic differential operator we have, for $a = \delta e^{\alpha} P(x)$

$$\mathrm{EQ}_r(a) = (\delta^{\bullet}) \bullet e^{Q_V}(e^{\alpha} P(x)).$$

Here $\mathbf{Q}_V$ is the differential operator (4.4) constructed from the restriction of the bicharacter r defined on $M = \mathbb{C}[G_{\mathrm{Tor}}] \otimes V$ to V.

5. The Frenkel-Lepowsky-Meurman Example

For the reader's convenience we explain in this last section why the operator e^{Δ_z} of [FLM88], which plays a prominent role in the construction of twisted modules over a lattice vertex algebra, is a special case of the operator we call $e^{\mathbf{Q}}$.

Recall the torsion free commutative and cocommutative Hopf algebra V, where

$$V = \mathbb{C}[G] \otimes V_0 = \mathbb{C}[G] \otimes \mathcal{U}(L) = \mathbb{C}[e^{\pm\alpha_1} \dots, e^{\pm\alpha_\ell}] \otimes \mathbb{C}[x_1, \dots, x_n, \dots],$$

studied in Section 4. Until now the Abelian group G and the Abelian Lie algebra L were independent. In the application to vertex algebras this is no longer true: L is in fact constructed from the group G, by use of an extra structure.

One starts with a lattice Q, i.e., a free Abelian group $Q = \oplus_{i=1}^{\ell} \mathbb{Z}\alpha_i$ equipped with a symmetric bilinear form $(\alpha, \beta) \mapsto \langle \alpha \mid \beta \rangle \in \mathbb{C}$. We will assume that the bilinear form is nondegenerate. Then, in order to construct L (and hence V_0), we complexify the lattice: Define

$$\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} Q.$$

Choose an orthonormal basis $h^s, s = 1, 2, \dots, \ell$ for $\mathfrak{h}$,

$$\langle h^s \mid h^t \rangle = \delta_{st}.$$

Then we let L be the Abelian Lie algebra with basis $h^s(-n)$, $1 \le s \le \ell$, $n \in \mathbb{N}$, and $V_0 = \mathcal{U}(L) = \mathbb{C}[h^s(-n)]$. (So V_0 has now the $h^s(-n)$, $1 \le s \le \ell$, $n \in \mathbb{N}$

as generating countable set of primitive elements, instead of the x_n as we used before.) We then let $V = \mathbb{C}[Q] \otimes V_0$.

On V we have operators $\frac{\partial}{\partial h^s(-n)}$ and $\frac{\partial}{\partial \alpha_i}$, the ingredients of the quadratic differential operator (4.4). In the vertex algebra literature it is usual to introduce notations

$$h^s(n) = n\frac{\partial}{\partial h^s(-n)}, \quad 1 \leq s \leq \ell, n > 0,\ n \in \mathbb{N}, \tag{5.1}$$

and for $n = 0$

$$h^s(0) = \sum_{i=0}^{\ell} \langle h^s \mid \alpha_i \rangle \frac{\partial}{\partial \alpha_i}. \tag{5.2}$$

Clearly $\{h^s(0)\}$ is another basis of the space of derivations of V spanned by the $\frac{\partial}{\partial \alpha_i}$ (here we use the assumption that the bilinear form is nondegenerate). Hence we can compactly write an alternative "Heisenberg" form (see Appendix A for the reason for this name) of a quadratic differential operator (4.4) as

$$\mathbf{Q} = \sum_{s,t=1}^{\ell} \sum_{m,n=0}^{\infty} c^{st}_{mn} h^s(m) h^t(n), \tag{5.3}$$

where the coefficients c^{st}_{mn} are expressed (invertibly) in terms of values of some bicharacter as in (4.2), $r(e^{\alpha_i} \otimes e^{\alpha_j}), r(e^{\alpha_i} \otimes h^s(-n))$, etc. We will give explicit formulas in a special case below.

So it remains to choose a bicharacter, or, equivalently, to choose the constants c^{st}_{mn}. Particularly nice formulas arise when the constants c^{st}_{mn} are independent of s and t (as is the case in [FLM88]), so that we obtain

$$Q = \Delta_z = \sum_{s,t=1}^{\ell} \sum_{m,n=0}^{\infty} c_{mn} h^s(m) h^t(n), \tag{5.4}$$

and the quadratic differential operator of interest in the theory of twisted modules is specified by choosing a generating series for the c_{mn} as

$$\sum_{m,n=0}^{\infty} c_{mn} x^m y^n = -\log \frac{\sqrt{1+\frac{x}{z}} + \sqrt{1+\frac{y}{z}}}{2}. \tag{5.5}$$

Here one expands the right-hand-side as a Maclaurin power series in the variables x and y, treating z as a parameter. Note that in this example the choice of the target algebra for the bicharacter $r : V \otimes V \to A$ is $A = \mathbb{C}[\frac{1}{z}]$, hence the notation Δ_z.

Of course, the substitutions (5.1), (5.2) and the choice of generating series (5.5) are not well motivated from the Hopf algebraic point of view. See the original [FLM88] for the vertex algebraic context, and [Doy10] for another approach.

If one wants to have explicit formulas for the bicharacter that corresponds to the quadratic differential operator we need to compare the two forms of the operator $\mathbf{Q}$: the (5.4) and (4.4). We are now using as primitive elements, instead of x_n, the elements $h^s(-n)$, so that instead of the form (4.4) we get the expression

$$\mathbf{Q} = \sum_{i,j=1}^{\ell} a_{ij}\frac{\partial}{\partial\alpha_i}\frac{\partial}{\partial\alpha_j} + \sum_{i=1}^{\ell}\sum_{m\geq 1} b^j_{im}\frac{\partial}{\partial\alpha_i}\frac{\partial}{\partial h^j(-m)} + + \sum_{m\geq 1}\sum_{i=1}^{\ell} c^j_{mi}\frac{\partial}{\partial h^j(-m)}\frac{\partial}{\partial\alpha_i} + \sum_{i,j=1}^{\ell}\sum_{m,n\geq 1} q^{ij}_{mn}\frac{\partial^2}{\partial h^i(-m)\partial h^j(-n)}, \tag{5.6}$$

By comparing the two forms of $\mathbf{Q}$, (5.4) and (5.6), and using the orthonormality of the basis h^i, we have

$$a_{ij} = c_{00}\langle\alpha_i|\alpha_j\rangle, \quad \text{for any} \quad i,j = 1,\dots k$$
$$mb^j_{im} = c_{0m}\sum_{s=1}^{k}\langle h^s|\alpha_i\rangle, \quad \text{for any} \quad j = 1,\dots k, \quad m\in\mathbb{N},$$
$$mc^j_{mi} = c_{m0}\sum_{s=1}^{k}\langle\alpha_i|h^s\rangle, \quad \text{for any} \quad j = 1,\dots k \quad m\in\mathbb{N}.$$

Thus the corresponding bicharacter on the grouplike elements from Lemma 4.4 simplifies to:

$$r(e^\alpha\otimes e^\beta) = (e^{c_{00}})^{\langle\alpha|\beta\rangle}. \tag{5.7}$$

In order to find similar "lattice" formulas for the rest of the bicharacters from Lemma 4.4, we turn to a basis typically used in the theory of vertex algebras. We consider the following degree 1 elements in the polynomial algebra V_0. For each $\alpha_i\in L$, let

$$\alpha_i(-m) = m\sum_{s=1}^{\ell}\langle h^s|\alpha_i\rangle h^s(-m), \quad i = 1,\dots,k; \quad m\in\mathbb{N}, \tag{5.8}$$

Lemma 5.1. *The elements $\alpha_i(-m)$ are primitive for any $i = 1,\dots,k;\ m\in\mathbb{N}$. For these primitive elements the bicharacters from Lemma 4.4 assume the form:*

$$r(e^\alpha\otimes\alpha_i(-m)) = \langle\alpha|\alpha_i\rangle c_{0m} \tag{5.9}$$
$$r(\alpha_i(-m)\otimes e^\alpha) = \langle\alpha_i|\alpha\rangle c_{m0} \tag{5.10}$$
$$r(\alpha_i(-m)\otimes\alpha_j(-n)) = \langle\alpha_i|\alpha^j\rangle c_{mn} \tag{5.11}$$

Proof. Since $h^s(-m)$ are primitive elements for any $s = 1,\dots,k,\ m\in\mathbb{N}$, it follows that $\alpha_i(-m)$ are primitive for any $i = 1,\dots,k,\ m\in\mathbb{N}$. Using the property

$$r(e^{\alpha_j}\otimes\alpha_i(-m))=\sum_{s=1}^{\ell}\langle h^s|\alpha_i\rangle r(e^{\alpha_j}\otimes mh^s(-m))=\sum_{s=1}^{\ell}\langle h^s|\alpha_i\rangle mb^s_{jm}=$$

$$=c_{0m}\sum_{s=1}^{\ell}\sum_{l=1}^{k}\langle h^s|\alpha_i\rangle\langle h^l|\alpha_j\rangle=c_{0m}\langle\alpha_j|\alpha_i\rangle. \tag{5.12}$$

For any primitive element x and grouplike elements g_1, g_2 we have

$$r(g_1g_2\otimes x)=r(g_1\otimes x)+r(g_2\otimes x). \tag{5.13}$$

Hence from (5.12) the equality (5.9) follows immediately. Equalities (5.10) and (5.11) follow similarly. □

As was mentioned above, I. Frenkel, J. Lepowsky and A. Meurman defined the quadratic differential operator Δ_z by choosing the generating function (5.5). As shown above in equation (5.7) and lemma 5.1, in the natural basis $\alpha_i(-m),\ i=1,\dots,k;\ m\in\mathbb{N}$ the bicharacter r is explicit and simple. Thus we can complete the picture and define a $\bullet_s$-product, together with the bicharacter map EQ_r (the bicharacter s being the symmetrization of r), so that the map $e^{\Delta_z}=e^{\mathbf{Q}}=\mathrm{EQ}_r$.

Example 5.2. Let us finish with calculating some examples of the $\bullet$ product on $V=\mathbb{C}(L)\otimes V_0$ in the specific case outlined above.

We have the generating series (5.5) for the coefficients c_{mn} of the quadratic differential operator $\mathbf{Q}$. The first few terms of the series expansion are

$$c_{00}+c_{01}y+c_{10}x+c_{11}xy+\cdots=-\frac{1}{4z}(x+y)+\frac{3}{32z^2}(x^2+y^2)+\frac{1}{16z^2}xy+\dots. \tag{5.14}$$

In particular note that $c_{00}=0$ in this case. Thus we have

$$\begin{aligned}
r(e^{\alpha}\otimes e^{\beta})&=(e^{c_{00}})^{\langle\alpha|\beta\rangle}=1,\\
r(e^{\alpha}\otimes\alpha_i(-1))&=\langle\alpha|\alpha_i\rangle c_{01}=-\frac{\langle\alpha|\alpha_i\rangle}{4z},\\
r(\alpha_i(-1)\otimes e^{\alpha})&=\langle\alpha_i|\alpha\rangle c_{10}=-\frac{\langle\alpha_i|\alpha\rangle}{4z},\\
r(\alpha_i(-1)\otimes\alpha_j(-1))&=\langle\alpha_i|\alpha_j\rangle c_{11}=\frac{\langle\alpha_i|\alpha_j\rangle}{16z^2}.
\end{aligned}$$

Hence for the symmetrization bicharacters we have

$$\begin{aligned}
&s(e^{\alpha}\otimes e^{\beta})=r(e^{\alpha}\otimes e^{\beta})r(e^{\beta}\otimes e^{\alpha})=1,\\
&s(e^{\alpha}\otimes\alpha_i(-1))=r(e^{\alpha}\otimes\alpha_i(-1))+r(\alpha_i(-1)\otimes e^{\alpha})=-\frac{\langle\alpha|\alpha_i\rangle}{2z},\\
&s(\alpha_i(-1)\otimes\alpha_j(-1))=r(\alpha_i(-1)\otimes\alpha_j(-1))+r(\alpha_i(-1)\otimes\alpha_j(-1))=\\
&\qquad=\frac{\langle\alpha_i|\alpha_j\rangle}{8z^2}.
\end{aligned}$$

We have then

$$\begin{aligned} e^{\mathbf{Q}}(e^{\alpha_i}) &= \mathrm{EQ}_r(e^{\alpha_i}) = (e^{\alpha_i})^\bullet = e^{\alpha_i} \\ e^{\mathbf{Q}}(e^{\alpha_i}e^{\alpha_j}) &= \mathrm{EQ}_r(e^{\alpha_i}e^{\alpha_j}) = (e^{\alpha_i}e^{\alpha_j})^\bullet = (e^{\alpha_i})^\bullet \bullet (e^{\alpha_j})^\bullet = e^{\alpha_i} \bullet e^{\alpha_j} = \\ &= e^{\alpha_i}e^{\alpha_j}s(e^{\alpha_i} \otimes e^{\alpha_j}) = e^{\alpha_i}e^{\alpha_j}. \end{aligned}$$

Thus in this case we have for any $\alpha = \sum_{i=1}^k m_i\alpha_i,\ m_i \in \mathbb{Z}$

$$e^{\mathbf{Q}}(e^{\alpha}) = \mathrm{EQ}_r(e^{\alpha}) = (e^{\alpha})^\bullet = e^{\alpha},$$

Further,

$$\begin{aligned} e^{\mathbf{Q}}(\alpha_i(-1)) &= \mathrm{EQ}_r(\alpha_i(-1)) = (\alpha_i(-1))^\bullet = \alpha_i(-1) \\ e^{\mathbf{Q}}(e^{\alpha}\alpha_i(-1)) &= \mathrm{EQ}_r(e^{\alpha}\alpha_i(-1)) = (e^{\alpha}\alpha_i(-1))^\bullet = (e^{\alpha})^\bullet \bullet (\alpha_i(-1))^\bullet = \\ &= (e^{\alpha}) \bullet (\alpha_i(-1)) = e^{\alpha}\alpha_i(-1) + s(e^{\alpha} \otimes \alpha_i(-1)) = \\ &= e^{\alpha}\alpha_i(-1) - \frac{\langle\alpha|\alpha_i\rangle}{2z} \\ e^{\mathbf{Q}}((\alpha_i(-1))^2) &= \mathrm{EQ}_r((\alpha_i(-1))^2) = (\alpha_i(-1))^\bullet \bullet (\alpha_i(-1))^\bullet = \\ &= (\alpha_i(-1)) \bullet (\alpha_i(-1)) = (\alpha_i(-1))^2 + s(\alpha_i(-1) \otimes \alpha_j(-1)) = \\ &= (\alpha_i(-1))^2 + \frac{\langle\alpha_i|\alpha_j\rangle}{8z^2}. \end{aligned}$$

Even though on lower degree products it is about as easy to calculate the $\bullet$ product or the action of $e^{\mathbf{Q}}$, on higher products it is much easier to calculate the $\bullet$ products (of course, Theorem 4.7 assures that we will get the same result). For instance:

$$\begin{aligned} e^{\mathbf{Q}}(e^{\alpha}(\alpha_i(-1))^2) =&(e^{\alpha}(\alpha_i(-1))^2)^\bullet = (e^{\alpha})^\bullet \bullet ((\alpha_i(-1))^2)^\bullet = \\ &= e^{\alpha} \bullet \big((\alpha_i(-1))^2 + \frac{\langle\alpha_i|\alpha_j\rangle}{8z^2}\big) = e^{\alpha} \bullet (\alpha_i(-1))^2 + e^{\alpha}\frac{\langle\alpha_i|\alpha_j\rangle}{8z^2}. \end{aligned}$$

Since e^{α} is grouplike, we have

$$s(e^{\alpha} \otimes (\alpha_i(-1))^2) = (s(e^{\alpha} \otimes (\alpha_i(-1))))^2.$$

From

$$\triangle((\alpha_i(-1))^2) = (\alpha_i(-1))^2 \otimes 1 + 2\alpha_i(-1) \otimes \alpha_i(-1) + 1 \otimes (\alpha_i(-1))^2$$

we have

$$\begin{aligned} e^{\alpha} \bullet (\alpha_i(-1))^2 = e^{\alpha}(\alpha_i(-1))^2 &+ 2e^{\alpha}(\alpha_i(-1))s(e^{\alpha} \otimes \alpha_i(-1)) + \\ &+ e^{\alpha}s(e^{\alpha} \otimes (\alpha_i(-1))^2) = \\ &= e^{\alpha}\Big((\alpha_i(-1))^2 - 2\alpha_i(-1)\frac{\langle\alpha|\alpha_i\rangle}{2z} + \frac{\langle\alpha|\alpha_i\rangle^2}{4z^2}\Big). \end{aligned}$$

So

$$e^{\mathbf{Q}}(e^{\alpha}(\alpha_i(-1))^2) = e^{\alpha}\Big((\alpha_i(-1))^2 - 2\alpha_i(-1)\frac{\langle\alpha|\alpha_i\rangle}{2z} + \frac{\langle\alpha|\alpha_i\rangle^2}{4z^2} + \frac{\langle\alpha_i|\alpha_j\rangle}{8z^2}\Big)$$

The rest of the $\bullet$ products are similarly obtained from the Taylor expansion.

Appendix A. Normal ordered products for the Heisenberg algebra

In this Appendix we recall the notion of normal ordered products of fields in the case of the Heisenberg algebra. The notion of normal ordered products has long been very common in the physics literature on conformal field theory, and has been introduced in the mathematical literature (in greater generality than presented here) by works like [FLM88], [KR87] and others.

We start with the setup of Section 5, where we have a lattice Q with complexification $\mathfrak{h}$, but with the simplification that $Q = \mathbb{Z}\alpha$ is rank 1, so that $\mathfrak{h}$ has dimension 1. We fix a basis element $h \in \mathfrak{h}$ (say of unit length) and simplify the notation and write in this case $x_n = h(-n)$, so that we deal with the polynomial algebra

$$V_0 = \mathbb{C}[x_1, x_2, \dots].$$

On V_0 we have *creation operators*

$$h_{-n} = \text{multiplication by } x_n,$$

and *annihilation operators* (cf., (5.1))

$$h_n = n\frac{\partial}{\partial x_n}.$$

We extend the action of creation and annihilation operators to all of
$V = \mathbb{C}[Q] \otimes V_0 = \oplus_{n\in\mathbb{Z}} V_0 e^{n\alpha}$. We also define (cf., (5.2))

$$h_0 = \frac{\partial}{\partial \alpha}.$$

Let $\mathcal{H}$ be the infinite dimensional Lie algebra generated by the operators h_n for $n \in \mathbb{Z}$, and c–a central element, satisfying the relations

$$[h_m, h_n] = m\delta_{m+n,0}c, \quad m, n \in \mathbb{Z}. \tag{A.1}$$

$\mathcal{H}$ is called the Heisenberg algebra. It is clear that V is a representation of $\mathcal{H}$, with the central element c acting as multiplication by 1.

We can organize the Heisenberg operators from $\mathcal{H}$ in a formal series, called *Heisenberg field*:

$$h(z) = \sum_{n\in\mathbb{Z}} h_n z^{-n-1}. \tag{A.2}$$

The indexing is due to the fact that we want the annihilation operators to be indexed by negative powers of the formal variable z, and creation operators to be indexed by non negative powers of z:

$$h(z) = h_-(z) + h_+(z), \tag{A.3}$$

where $h_-(z) = \sum_{n\geq 0} h_n z^{-n-1}$ is called *annihilation part* of the Heisenberg field and $h_+(z) = \sum_{n\geq 0} h_{-n-1} z^n$—*creation part* of the Heisenberg field.

The product of two Heisenberg fields with the same formal variable does not make sense, even when it acts on the element $1_{V_0} \in V_0$: If one naively was to multiply

$$\text{"}h(z)h(z) = \sum_{m\in\mathbb{Z}} h_m z^{-m-1} \sum_{n\in\mathbb{Z}} h_{-n} z^{n-1} = \sum_{k\in\mathbb{Z}} z^{-k-2}\Big(\sum_{m-n=k} h_m h_{-n}\Big)\text{"},$$

one has infinite sums as coefficients, for example, the coefficient in front of z^{-2} for $h(z)h(z)1_{V_0}$ is $\sum_{m>0} m$. To rectify this, following physicists, one introduces the notion of normal ordered products:

Definition A.1. *(*[FLM88]*,* [KR87]*,* [Kac98]*)* ***(Normal ordered products)*** *First, let*

$$\begin{aligned} :h_n h_m: &= h_n h_m \quad \text{if } m < 0 \\ :h_n h_m: &= h_m h_n \quad \text{if } m \geq 0. \end{aligned}$$

Then define

$$:h(z)h(z): = \sum_{k\in\mathbb{Z}} z^{-k-2}\Big(\sum_{m-n=k} :h_m h_{-n}:\Big), \tag{A.4}$$

called ***normal ordered products of fields****.*

The normal ordered product of the Heisenberg fields has a well defined action on any element of V.

Similarly to the $\bullet$ products, see Example 3.4, one can define normal ordered products of arbitrary number of fields by a consecutive application from the right:

$$:h(z)h(z)h(z): =: h(z) : h(z)h(z) :: . \tag{A.5}$$

Besides the Heisenberg field $h(z)$ one also considers derivative fields $\partial^i h(z)$, where $\partial = \partial_z$, and define similarly normal ordered product of those fields.

Now we are ready to define the **vertex algebra state-field correspondence**, which is a map from V_0 to the space of fields on V_0 (or on V). It is given by

(A.6)

$$x_{n_1} x_{n_2} \dots x_{n_k} = h_{-n_1} h_{-n_2} \dots h_{-n_k} 1_{V_0} \mapsto : \frac{\partial^{n_1-1}h(z)}{(n_1-1)!} \frac{\partial^{n_2-1}h(z)}{(n_2-1)!} \cdots \frac{\partial^{n_k-1}h(z)}{(n_k-1)!} :$$

This is in fact a one-to-one map, the inverse being given as following:

Fact A.2. *(*[FLM88]*,* [Kac98]*)* ***(Field-state correspondence)*** *To a normal product of Heisenberg fields one associates back the product of states given by:*

$$\begin{aligned} &: \frac{\partial^{n_1-1}h(z)}{(n_1-1)!} \frac{\partial^{n_2-1}h(z)}{(n_2-1)!} \cdots \frac{\partial^{n_k-1}h(z)}{(n_k-1)!} : \ \mapsto \\ &: \frac{\partial^{n_1-1}h(z)}{(n_1-1)!} \frac{\partial^{n_2-1}h(z)}{(n_2-1)!} \cdots \frac{\partial^{n_k-1}h(z)}{(n_k-1)!} : 1_{V_0}|_{z=0} = h_{-n_1} h_{-n_2} \dots h_{-n_k} 1_{V_0} = \\ &= x_{n_1} x_{n_1} \dots x_{n_k}. \end{aligned}$$

The fact that the evaluation of the normal product of the fields at $z = 0$ makes sense, and gives back the product of the states, is proved in many books, see for example [FLM88], [Kac98], [LL04].

Similar to the Heisenberg algebra there is the twisted Heisenberg algebra:

Definition A.3. *(*[FLM88]*,* [Don94]*,* [BK04]*)* ***(Twisted Heisenberg algebra)*** *Let $\mathcal{H}_{1/2}$ be the infinite dimensional Lie algebra generated by the operators h_n for $n \in \mathbb{Z} + 1/2$, and $\tilde{c}$–a central element, satisfying the relations*

$$[h_m, h_n] = m\delta_{m+n,0}\tilde{c}, \quad m, n \in \mathbb{Z} + 1/2. \tag{A.7}$$

The generators are organized in the ***twisted Heisenberg field****:*

$$\tilde{h}(z) = \sum_{n \in \mathbb{Z}+1/2} h_n z^{-n-1}. \tag{A.8}$$

Just as the Heisenberg algebra acts on V_0 one can define also a module, say $\tilde{V}_0$, for the twisted Heisenberg algebra, such that the annihilation operators are still $h_n, n > 0$ and creation operators $h_n, n < 0$. Since we have the notion of creation and annihilation operators we can define normal ordered products of (derivatives of) the twisted Heisenberg fields. Then we can also define a state-field correspondence, from states in the (untwisted) space V_0 to the twisted fields that act on $\tilde{V}_0$.

One of the questions we started in this paper is the following:

For a twisted Heisenberg algebra, normal ordered products of twisted fields correspond to what products of states (under the twisted state field correspondence)?

I.e., what is the equivalent of the field-state correspondence A.2 on the module $\tilde{V}_0$ for the twisted Heisenberg algebra $\mathcal{H}_{1/2}$? It is obvious one can no longer apply the evaluation at $z = 0$ as in A.2.

Nevertheless, we can now formulate an answer to this question, and we leave the proof to the reader familiar with twisted modules of vertex algebras:
Let the bicharacter r and its symmetrization s be defined as in Section 5, see Example 5.2, and consider their inverses as in (2.3); then

$$: \frac{\partial^{n_1-1}\tilde{h}(z)}{(n_1-1)!}\frac{\partial^{n_2-1}\tilde{h}(z)}{(n_2-1)!}\dots\frac{\partial^{n_k-1}\tilde{h}(z)}{(n_k-1)!} : \; \mapsto e^{-Q_p}(x_{n_1}x_{n_2}\dots x_{n_k}) =$$
$$= \mathrm{EQ}_{r^{-1}}(x_{n_1}x_{n_2}\dots x_{n_k}) = x_{n_1} \bullet_{s^{-1}} x_{n_2} \bullet_{s^{-1}} \cdots \bullet_{s^{-1}} x_{n_k}.$$

So the normal ordered product of twisted fields corresponds to the $\bullet$ product on V_0.

Appendix B. Operator description of the coproduct

We want to introduce an alternative operator description of the coproduct involving grouplike elements, similar to the well known description we used in Equation (3.1). To do that we need expressions involving exponentials of $\alpha \in G$ and $\frac{\partial}{\partial \alpha_i}$. Recall that in Section 3 we used expressions $e^{x_n \frac{\partial}{\partial x_n}}$ for primitive elements x_n; such an expression was interpreted as a power series $\sum \frac{1}{s!}(x_n \frac{\partial}{\partial x_n})^s$.

(It is a locally finite infinite order differential operator on $\mathbb{C}[x_1,\dots]$.) Now we want to consider the expression $e^{\alpha_i\frac{\partial}{\partial\alpha_i}}$ as an operator on $\mathbb{C}[G]$. This can not be interpreted as a power series, as the powers of α_i do not belong to $\mathbb{C}[G]$. However, $\frac{\partial}{\partial\alpha_i}$ is diagonalizable on $\mathbb{C}[G]$ (and on $\mathbb{C}[G]\otimes\mathbb{C}[x_1,\dots]$). So on the eigenspace for $\frac{\partial}{\partial\alpha_i}$ with eigenvalue m_i the exponential operator $e^{\alpha_i\frac{\partial}{\partial\alpha_i}}$ is just multiplication by $e^{m_i\alpha_i}=(e^{\alpha_i})^{m_i}$.

As in section 3 we can use the operators $e^{\alpha_i\frac{\partial}{\partial\alpha_i}}$ to give a convenient description of the coproduct of V. An element of V is a linear combination of elements of the form $e^\alpha P(x)$, $\alpha=\sum m_i\alpha_i$, $P(x)\in V_0$. The coproduct of $e^\alpha P(x)$ is

$$\begin{aligned}\Delta(e^\alpha P(x)) &= e^\alpha\otimes e^\alpha P(x^{(1)}+x^{(2)})=\\ &= e^{\alpha^{(1)}+\alpha^{(2)}}P(x^{(1)}+x^{(2)})=\\ &= e^{\sum_i\alpha_i^{(1)}\frac{\partial}{\partial\alpha_i^{(2)}}+\sum_n x_n^{(1)}\frac{\partial}{\partial x_n^{(2)}}}[e^{\alpha^{(2)}}P(x^{(2)})].\end{aligned}$$

Here we write $e^{\alpha^{(1)}}$ for $e^\alpha\otimes 1$, etc. In the same way for the square of the coproduct:

$$\Delta^2(e^\alpha P(x)) = e^{\sum_i[\alpha_i^{(1)}+\alpha_i^{(2)}]\frac{\partial}{\partial\alpha_i^{(3)}}+\sum_n[x_n^{(1)}+x_n^{(2)}]\frac{\partial}{\partial x_n^{(3)}}}[e^{\alpha^{(3)}}P(x^{(3)})]. \tag{B.1}$$

Now we want to show how we can use this alternative operator description of the coproduct to give an alternative proof of Lemma 4.4: We recall it for convenience:

Lemma B.1. *Let $V=\mathbb{C}[e^{\pm\alpha_1},\dots,e^{\pm\alpha_\ell},x_1,x_2,\dots]$ as before. Let A be a commutative $\mathbb{C}$-algebra and r an A-valued bicharacter with values on generators*

$$\begin{aligned} r(e^{\alpha_i}\otimes e^{\alpha_j}) &= e^{a_{ij}}\in\mathbb{C}\\ r(e^{\alpha_i}\otimes x_m) &= b_{im}\in A\\ r(x_m\otimes e^{\alpha_i}) &= c_{mi}\in A\\ r(x_i\otimes x_m) &= q_{mn}\in A\end{aligned} \tag{B.2}$$

Then the map $\mathrm{EQ}\colon V\to V_A,\quad a\mapsto r(a'\otimes a'')a'''$ *is the exponential of an infinite order quadratic operator* $\mathbf{Q}$

$$\mathrm{EQ}_r(m)=e^{\mathbf{Q}}(m),\quad m\in V, \tag{B.3}$$

where $\mathbf{Q}$ *is defined by*

$$\mathbf{Q}=\sum_{i,j=1}^{\ell}a_{ij}\frac{\partial}{\partial\alpha_i}\frac{\partial}{\partial\alpha_j}+\sum_{i=1}^{\ell}\sum_{m\geq1}b_{im}\frac{\partial}{\partial\alpha_i}\frac{\partial}{\partial x_m}+ \\ +\sum_{m\geq1}\sum_{i=1}^{\ell}c_{mi}\frac{\partial}{\partial x_m}\frac{\partial}{\partial\alpha_i}+\sum_{m,n\geq1}q_{mn}\frac{\partial^2}{\partial x_m\partial x_n}. \tag{B.4}$$

Proof. Just as in the proof of Lemma 3.2 we use the operator description of the coproduct by exponential operators. So we have for $a \in V$

$$\mathrm{EQ}(a) = r(e^{\sum_i \alpha_i \frac{\partial}{\partial \alpha_i} + \sum_m x_m \frac{\partial}{\partial x_m}} \otimes e^{\sum_j \alpha_j \frac{\partial}{\partial \alpha_j} + \sum_n x_n \frac{\partial}{\partial x_n}})(a),$$

Now the basic point is that $e^{\alpha_i \partial_{\alpha_i}}$ and $e^{x_m \partial_{x_m}}$ behave like grouplike elements in bicharacters: we have

$$r(e^{\alpha_i \partial_{\alpha_i}} \otimes ab) = r(e^{\alpha_i \partial_{\alpha_i}} \otimes a) r(e^{\alpha_i \partial_{\alpha_i}} \otimes b),$$
$$r(e^{x_m \partial_{x_m}} \otimes ab) = r(e^{x_m \partial_{x_m}} \otimes a) r(e^{x_m \partial_{x_m}} \otimes b),$$

and similar for the second argument of the bicharacter. This implies that

$$\mathrm{EQ}(a) = \prod_{i,j,n,m} r(e^{\alpha_i \partial_{\alpha_i}} \otimes e^{\alpha_j \partial_{\alpha_j}}) r(e^{\alpha_i \partial_{\alpha_i}} \otimes e^{x_n \partial_{x_n}}) \cdot$$
$$\cdot\, r(e^{x_m \partial_{x_m}} \otimes e^{\alpha_j \partial_{\alpha_j}}) r(e^{x_m \partial_m} \otimes e^{x_n \partial_{x_n}})(a).$$

Now one easily checks, using the values of the bicharacter on generators, see (B.2), that

$$r(e^{\alpha_i \partial_{\alpha_i}} \otimes e^{\alpha_j \partial_{\alpha_j}}) = e^{a_{ij} \partial_{\alpha_i} \partial_{\alpha_j}},$$
$$r(e^{\alpha_i \partial_{\alpha_i}} \otimes e^{x_n \partial_{x_n}}) = e^{b_{in} \partial_{\alpha_i} \partial_{x_n}},$$
$$r(e^{x_m \partial_{x_m}} \otimes e^{\alpha_j \partial_{\alpha_j}}) = e^{c_{mj} \partial_{x_m} \partial_{\alpha_j}},$$
$$r(e^{x_m \partial_{x_m}} \otimes e^{x_n \partial_n}) = e^{q_{mn} \partial_{x_m} \partial_{x_n}}.$$

For instance, to check the first equality consider a joint eigenspace of ∂_{α_i} and ∂_{α_j}. It consists of elements $e^{\alpha} P(x)$, with $\alpha = m_i \alpha_i + m_j \alpha_j + \sum_{k \neq i,j} m_k \alpha_k$, for fixed m_i, m_j. Then

$$r(e^{\alpha_i \partial_{\alpha_i}} \otimes e^{\alpha_j \partial_{\alpha_j}}) e^{\alpha} P(x) = r(e^{m_i \alpha_i} \otimes e^{m_j \alpha_j}) e^{\alpha} P(x) = e^{m_i m_j a_{ij}} e^{\alpha} P(x).$$

On the other hand

$$e^{a_{ij} \partial_i \partial_j} e^{\alpha} P(x) = e^{a_{ij} m_i m_j} e^{\alpha} P(x),$$

proving the first equality. The other equalities are proved similarly, which proves the lemma. □

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