Communications in Contemporary Mathematics
Vol. 11, No. 6 (2009) 937–991

H_D-QUANTUM VERTEX ALGEBRAS AND BICHARACTERS

IANA I. ANGUELOVA

College of Charleston and Stony Brook University
Department of Mathematics, Stony Brook University
Stony Brook, NY 11794-3651, USA
anguelov@math.sunysb.edu

MAARTEN J. BERGVELT

Department of Mathematics, University of Illinois
Urbana-Champaign, Illinois 61801, USA
bergv@uiuc.edu

Received 12 December 2007
Revised 8 January 2009

We define a new class of quantum vertex algebras, based on the Hopf algebra $H_D = \mathbb{C}[D]$ of "infinitesimal translations" generated by D. Besides the braiding map describing the obstruction to commutativity of products of vertex operators, H_D-quantum vertex algebras have as a main new ingredient a "translation map" that describes the obstruction of vertex operators to satisfying translation covariance. The translation map also appears as obstruction to the state-field correspondence being a homomorphism.

We use a bicharacter construction of Borcherds to construct a large class of H_D-quantum vertex algebras. One particular example of this construction yields a quantum vertex algebra that contains the quantum vertex operators introduced by Jing in the theory of Hall–Littlewood polynomials.

Keywords: Quantum vertex algebra; bicharacter.

Mathematics Subject Classification 2000: 17B69, 22E46, 53C35, 57S20

1. Introduction

Vertex operators were introduced in the earliest days of string theory and now play an important role in such areas of mathematics as representation theory, algebraic topology and random matrices. Vertex algebras were introduced to axiomatize the properties of vertex operators.

Similarly, quantum vertex operators were discovered in integrable models in statistical mechanics and in connection with the theory of symmetric polynomials and the theory of quantum affine algebras. One would like to have a theory of quantum vertex algebras to axiomatize the properties of quantum vertex operators. In this paper, we introduce and study a class of quantum vertex algebras that produce the quantum vertex operators related to Hall–Littlewood polynomials.

Recall that a vertex operator on a space V is a series $a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$, $a_{(n)} \in \mathrm{End}(V)$, satisfying some extra conditions. We call vertex operators $a(z), b(z)$ *local* with respect to each other if the commutator is a sum of derivatives of delta distributions:

$$[a(z_1), b(z_2)] = \sum_{n=0}^{N} c_n(z_2)\partial_{z_2}\delta(z_1, z_2).$$

For *quantum vertex operators* this will no longer be true: one needs a *braiding* map $S^{(\tau)}_{z_1,z_2}\colon b(z_2)a(z_1) \mapsto ba(z_2, z_1)$, where $ba(z_2, z_1)$ is some other $\mathrm{End}(V)$-valued series. Then we should have S-locality, [5], i.e. we need that the *braided commutator*

$$[a(z), b(z_2)]_S = a(z_1)b(z_2) - ba(z_2, z_1)$$

is a sum of derivatives of delta distributions. One of the goals of this paper is to give explicit examples of quantum vertex algebras where one can easily calculate both the quantum vertex operators and their braiding.

There are several proposals for what a quantum vertex algebra should be. There are Borcherds' theory of (A, H, S)-vertex algebras, see [4], the Etingof–Kazhdan theory of quantum vertex operator algebras [5], and the Frenkel–Reshetikhin theory of deformed chiral algebras, see [7]. (Li has developed the Etingof–Kazhdan theory further, see for example [13, 12].)

The Borcherds theory is based on the observation that products and iterates of vertex operators in vertex algebras are expansions of rational functions in multiple variables. The idea then is to start with these rational objects instead of constructing them after the fact from the vertex operators. Instead of a single vector space V on which the vertex operators act, one has for any integer $n \geq 1$ the space $V(n)$ of "rational vertex operators" in n variables. This is a quite beautiful idea, and is easily adapted to include quantum vertex algebras of (A, H, S)-type. However, even for classical vertex algebras it seems not known how to include such basic examples as affine vertex algebras in the (A, H, S)-framework. In this paper, we therefore prefer to develop a theory that is closer to the usual theory, with a single underlying vector space V. We do take, however, from Borcherds' paper the idea of a *bicharacter* as a method to construct examples: we will use bicharacters both to produce the vertex operators and the braiding. (See [1] for more details.)

The Etingof–Kazhdan theory is very close to the classical theory, in fact so close that it is not suitable to describe quantum vertex operators related to symmetric polynomials. Briefly, in the usual theory (and in [5]) vertex operators $a(z)$ satisfy translation covariance of the form

$$e^{\gamma D} a(z_1) e^{-\gamma D} = a(z_1 + \gamma), \tag{1.1}$$

where $D\colon V \to V$ is the infinitesimal translation operator and we expand in positive powers of γ. If we introduce the notation $a(z)b = Y_z(a \otimes b)$, we can write this, since

$\partial_z a(z) = (Da)(z)$, as

$$e^{\gamma D} Y_z = Y_z \circ (e^{\gamma D} \otimes e^{\gamma D}),$$

making clear the similarity of a vertex algebra with an associative ring M with a group action (where $gm(a \otimes b) = m(ga \otimes gb)$ if m is the multiplication in M, and $g \in G$, $a, b \in M$, see Appendix A).

It was shown in the thesis [1] that (1.1) cannot hold in the case of quantum vertex operators related to symmetric polynomials. Also, and this is not unrelated to the translation covariance in Etingof–Kazhdan, the braiding map S_{z_1,z_2} in [5] is in fact assumed to be of the form $S_{z_1,z_2} = \tilde{S}_{z_1-z_2}$, where $\tilde{S}_z$ is a function of a single variable. It was also shown in [1] that this does not hold for symmetric polynomials.

In this paper we introduce the notion of an H_D-quantum vertex algebra (where $H_D = \mathbb{C}[D]$ is the Hopf algebra of infinitesimal translations), generalizing [5] in various ways. First we need to relax the translations covariance (1.1). We introduce, besides the braiding map $S^{(\tau)}_{z_1,z_2}$, also another map $S^{(\gamma)}_{z_1,z_2}$ on $V \otimes V$ such that we get instead

$$e^{\gamma D} Y_{z_1} \circ S^{(\gamma)}_{z_1,z_2} = Y_{z_1} \circ (e^{\gamma D} \otimes e^{\gamma D}).$$

Both $S^{(\gamma)}_{z_1,z_2}$ and $S^{(\tau)}_{z_1,z_2}$ are rational functions of both z_1 and z_2, not just of the difference $z_1 - z_2$ as in [5]. Another difference is that in [5] vertex operators satisfy a braided version of skew-symmetry:

$$Y_z \circ S^{(\tau)}_{z,0}(a \otimes b) = e^{zD} Y(b, -z)a. \tag{1.2}$$

This relation does not make sense for quantum vertex operators coming from symmetric polynomials: the braiding $S^{(\tau)}_{z_1,z_2}$ is in general *singular* for $z_2 = 0$. This motivates us to take as basic building block of the theory not the vertex operator Y_z, but the *two-variable* vertex operators $X_{z_1,z_2} \colon V \otimes V \to V[[z_1, z_2]][z_1^{-1}, (z_1 - z_2)^{-1}]$. We can define then Y by $Y(a, z)b = X_{z,0}(a \otimes b)$, but such a vertex operator Y no longer needs to satisfy (1.2). See Corollary 9.1 for the version of skew-symmetry that holds for H_D-quantum vertex algebras.

Conversely, if we start with Y, we can introduce X_{z_1,z_2} by analytic continuation: we have the expansion

$$i_{z_1;z_2} X_{z_1,z_2}(a \otimes b) = Y(a, z_1)Y(b, z_2)1, \tag{1.3}$$

where $i_{z;w}$ is the expansion in the region $|z| > |w|$. See Sec. 8 for details and an alternative definition of X_{z_1,z_2}.

Note that for a classical vertex algebra (and also for an Etingof–Kazhdan quantum vertex operator algebra) the translation map $S^{(\gamma)}_{z_1,z_2}$ is the identity, so that in this case

$$X_{z_1,z_2}(a \otimes b) = e^{z_2} Y(a, z_1 - z_2)b \in V[[z_1, z_2]][(z_1 - z_2)^{-1}].$$

In particular in these cases $X_{z_1,z_2}(a \otimes b)$ is not singular for $z_1 = 0$. We consider a more general theory where in $X_{z_1,z_2}(a \otimes b)$ poles in z_1 are allowed (and in fact are necessary to be able to treat the quantum vertex operators associated with the Hall–Littlewood polynomials).

In the construction of quantum vertex algebras one or more quantum parameters will appear. They can usually be thought of as describing the deformation away from an ordinary vertex algebra. We should mention that, just as when quantizing universal enveloping algebras, there are two ways of interpreting the quantum parameters in quantum vertex algebras. Either the quantum parameters are independent formal variables or they are complex numbers. The theory of Etingof–Kazhdan follows the first approach, as opposed to the Frenkel–Reshetikhin definition of deformed chiral algebras, which considers the deformation parameter(s) to be complex number(s). In this paper, we also follow the first approach: we have an independent variable t and an H_D-quantum vertex algebra V is a (topologically free) module over the ring $\mathbb{C}[[t]]$ of formal power series in t. When putting $t = 0$ one gets in examples generally an ordinary vertex algebra, although we did not require this in our axioms. Note that Li in [13], for instance, studies a form of the Etingof–Kazhdan axioms where the quantum parameter is a complex number.

Maybe the most important difference between our H_D-quantum vertex algebras and classical vertex algebras (and the theory of Etingof–Kazhdan) is the following. We can define a product of states: $a_{(n)}b$, and for fields: $a(z)_{(n)}b(z)$, but it is no longer true that the state-field correspondence $a \mapsto Y(a, z)$ is a homomorphism of products: in general $Y(a_{(n)}b, z) \neq Y(a, z)_{(n)}Y(b, z)$, see Theorem 18.1 for an exact statement.

The outline of the paper is as follows. There are three parts. In the first part we define H_D-quantum vertex algebras in Sec. 4 and study their properties in the following sections. We derive in Secs. 15 and 16 fundamental identities in our quantum vertex algebras: the braided Jacobi identity and the braided Borcherds identity. These are used in Secs. 17–19 to study the S-commutator, (n)-products of states and of fields and normal ordered products. In Sec. 20, we derive a weak associativity relation. In the next part of the paper, we assume that our underlying vector space V is a commutative and cocommutative Hopf algebra, which allows us to define bicharacters on V in Sec. 22. Using bicharacters we construct a class of H_D-quantum vertex algebras in Sec. 23 and in the rest of this section we explore some of the properties of bicharacter H_D-quantum vertex algebras.

In last part of the paper, Sec. 26 and the following sections, we study in detail a single example of a Hopf algebra V with a fixed bicharacter on it. The resulting H_D-quantum vertex algebra is a deformation of the familiar lattice vertex algebra based on the lattice $L = \mathbb{Z}$ with pairing $(m, n) \mapsto mn$. Some of quantum vertex operators in this example were used by Jing, see [8], to study Hall–Littlewood symmetric polynomials. In Appendix A, we describe the "nonsingular" analog of H_D-quantum vertex algebras: braided algebras with group action. In Appendix B, we discuss the construction of braiding maps for H_D-quantum vertex algebras.

2. The Hopf Algebra H_D

Let $H_D = \mathbb{C}[D]$ be the universal enveloping algebra of the 1-dimensional Lie algebra generated by D. H_D is a Hopf-algebra, with coproduct $\Delta_{H_D} : D \mapsto D \otimes 1 + 1 \otimes D$, antipode $S\colon D \mapsto -D$ and counit $\epsilon_{H_D} : D \mapsto 0$. H_D is a fundamental ingredient in the construction of vertex algebras, where it appears as the symmetry algebra of infinitesimal translations in physical space. In this paper the full Hopf algebra structure of H_D will play only an explicit role when we discuss bicharacter constructions, in the definition of H_D-quantum vertex algebras in the next section only the algebra structure will be used. However, from Borcherds' papers [3, 4] it will be clear that in fact the Hopf algebra H_D underlies the whole theory of vertex algebras (and their quantum versions).

3. Topologically Free Modules and Series

Let t be a variable. We will use t to describe quantum deformations, the classical limit corresponding to $t \to 0$.

Let $k = \mathbb{C}[[t]]$ and recall that a k-module V is called *topologically free* if it is torsion free and if it is separated and complete for the t-adic topology. Equivalently, V is topologically free iff it is of the form $V_0[[t]]$, where $V_0 = V/tV$, and $W[[t]]$ is the space of series $\sum_{k=0}^{\infty} w_k t^k$, $w_k \in W$. See [10] for more details.

In the sequel we will often have to deal with series in parameters $z, z_1, z_2, \ldots$, with coefficients depending on t. For instance, consider $k((z)) = \mathbb{C}[[t]]((z))$. This is a k-module, but not topologically free, as it is not complete. The completion is, of course, $\mathbb{C}((z))[[t]]$. More generally, if V is a topologically free k-module, then the completion of $V((z))$ is the topologically free $V_0((z))[[t]]$.

Convention: We will assume that all k-modules V in this paper are complete and we will not distinguish between V and its completion $\hat{V}$. So we write, to unclutter the notation, always $V((z))$ for its completion $V_0((z))[[t]]$, if $V = V_0[[t]]$, and similar for more complicated spaces of series.

We will also consider rational expressions in multiple variables and their expansions. For instance for a rational function in z_1, z_2 with only possibly poles at $z_1 = 0$, $z_2 = 0$ or $z_1 - z_2 = 0$ we can define expansion maps

$$
\begin{aligned}
i_{z_1;z_2} &: \frac{1}{z_1 - z_2} \mapsto \sum_{n\geq 0} z_1^{-n-1} z_2^n, \qquad \frac{1}{z_1} \mapsto \frac{1}{z_1}, \quad \frac{1}{z_2} \mapsto \frac{1}{z_2},\\
i_{z_2;z_1} &: \frac{1}{z_1 - z_2} \mapsto -\sum_{n\geq 0} z_2^{-n-1} z_1^n, \quad \frac{1}{z_1} \mapsto \frac{1}{z_1}, \quad \frac{1}{z_2} \mapsto \frac{1}{z_2},\\
i_{z_2;z_1-z_2} &: \frac{1}{z_1} \mapsto \sum_{n\geq 0} z_2^{-n-1}(z_1 - z_2)^n, \quad \frac{1}{z_2} \mapsto \frac{1}{z_2}, \quad \frac{1}{z_1 - z_2} \mapsto \frac{1}{z_1 - z_2}.
\end{aligned}
$$

We will write $i_{z_1,z_2;w_1}$ for $i_{z_1;w_1}i_{z_2;w_1}$, and $i_{z_1,z_2;w_1,w_2}$ for $i_{z_1,z_2;w_1}i_{z_1,z_2;w_2}$. We define $i_{z_1;z_2;\dots;z_n}$ to be the expansion in the region $|z_1|>|z_2|>\cdots>|z_n|$.

If $A\in V\otimes V$ then we define for instance $A^{23}, A^{13}\in V^{\otimes 3}$ by $A^{23}=1\otimes A$, and $A^{13}=a'\otimes 1\otimes a''$, if $A=a'\otimes a''$.

4. H_D-Quantum Vertex Algebras

Now we are ready to define the central concept of this paper. The definition is rather complicated, and in Appendix A we explain a simpler version of this notion, called a braided ring with symmetry, where the multiplication is nonsingular.

We emphasize that in the following definition we use the Convention on complete k-modules in Sec. 3.

Definition 4.1. Let V be a topologically free $k=\mathbb{C}[[t]]$-module and an H_D-module. An H_D-quantum vertex algebra structure on V consists of

- $1\in V$, the vacuum vector.
- a (singular) k-linear multiplication map

$$X_{z_1,z_2}\colon V^{\otimes 2}\to V[[z_1,z_2]][z_1^{-1},(z_1-z_2)^{-1}].$$

- A k-linear braiding map $S^{(\tau)}$ and a k-linear translation map $S^{(\gamma)}$ of the form

$$S^{(\tau)}_{z_1,z_2}: V^{\otimes 2}\to V^{\otimes 2}[z_1^{\pm 1},z_2^{\pm 1},(z_1-z_2)^{-1}],$$
$$S^{(\gamma)}_{z_1,z_2}: V^{\otimes 2}\to V^{\otimes 2}[z_1^{\pm 1},z_2,(z_1+\gamma)^{\pm 1},(z_2+\gamma),(z_1-z_2)^{-1}].$$

These objects satisfy the following axioms:

(Vacuum)

$$X_{z_1,z_2}(a\otimes 1)=e^{z_1D}a,\quad X_{z_1,z_2}(1\otimes a)=e^{z_2D}a, \tag{4.1}$$

$$S_{z_1,z_2}(a\otimes 1)=a\otimes 1,\quad S_{z_1,z_2}(1\otimes a)=1\otimes a. \tag{4.2}$$

Here and below we write generically S for both $S^{(\tau)}$ and $S^{(\gamma)}$.

(H_D-Covariance)

$$X_{z_1,z_2}(a\otimes Db)=\partial_{z_2}X_{z_1,z_2}(a\otimes b), \tag{4.3}$$

$$(1\otimes e^{\gamma D})i_{z_1-z_2,z_2;\gamma}S_{z_1,z_2+\gamma}=S_{z_1,z_2}(1\otimes e^{\gamma D}), \tag{4.4}$$

$$e^{\gamma D}X_{z_1,z_2}S^{(\gamma)}_{z_1,z_2}=X_{z_1+\gamma,z_2+\gamma}. \tag{4.5}$$

(Yang–Baxter)

$$S^{12}_{z_1,z_2}S^{13}_{z_1,z_3}S^{23}_{z_2,z_3}=S^{23}_{z_2,z_3}S^{13}_{z_1,z_3}S^{12}_{z_1,z_2}. \tag{4.6}$$

(Compatibility with Multiplication)

$$S_{z_1,z_2}(X_{w_1,w_2}\otimes 1)=(X_{w_1,w_2}\otimes 1)i_{z_1,z_1-z_2;w_1,w_2}S^{23}_{z_1+w_1,z_2}S^{13}_{z_1+w_2,z_2}, \tag{4.7}$$

$$S_{z_1,z_2}(1\otimes X_{w_1,w_2})=(1\otimes X_{w_1,w_2})i_{z_1-z_2,z_2;w_1,w_2}S^{12}_{z_1,z_2+w_1}S^{13}_{z_1,z_2+w_2}. \tag{4.8}$$

(Group Properties)

$$S^{(\tau)}_{z_1,z_2} \circ \tau \circ S^{(\tau)}_{z_2,z_1} \circ \tau = 1_{V^{\otimes 2}}, \tag{4.9}$$

$$S^{(\gamma_1)}_{z_1,z_2} S^{(\gamma_2)}_{z_1+\gamma_1,z_2+\gamma_1} = S^{(\gamma_1+\gamma_2)}_{z_1,z_2}, \tag{4.10}$$

$$S^{(\gamma=0)}_{z_1,z_2} = 1_{V^{\otimes 2}}. \tag{4.11}$$

(Locality) For every $a, b \in V$ and $k \geq 0$ there is $N \geq 0$ such that for all $c \in V$

$$\begin{aligned}(z_1 - z_2)^N X_{z_1,0}&(1 \otimes X_{z_2,0})(a \otimes b \otimes c) \\ &\equiv (z_1 - z_2)^N X_{z_2,0}(1 \otimes X_{z_1,0})(i_{z_2;z_1} S^{(\tau)}_{z_2,z_1}(b \otimes a) \otimes c) \mod t^k. \end{aligned}\tag{4.12}$$

Remark 4.1. In the above definition $z_1, z_2, w_1, w_2, \gamma$ are independent commuting variables. In general one should be careful with specializing these variables. For instance, we can evaluate X_{z_1,z_2} at $z_2 = 0$ but not at $z_1 = 0$, in general. For this reason one cannot put $\gamma = -z_1$ in (4.5).

Remark 4.2. The vacuum axioms (4.1) for $z_2 = 0$ are

$$X_{z_1,0}(a \otimes 1) = e^{z_1 D} a, \quad X_{z_1,0}(1 \otimes a) = a.$$

In the literature on vertex algebras the first equation is called the *creation axiom*, and the second the vacuum axiom. In our formalism it seems unnatural to give different names to very similar statements, so we call in (4.1) both vacuum axioms, as they involve the vacuum vector 1.

5. Intermezzo on Expansions

Let W be a vector space (over $\mathbb{C}$) and $A(z_1, z_2) \in W((z_1))((z_2))$. It is well known[a] that if there is an $N \geq 0$ such that

$$A_N = (z_1 - z_2)^N A(z_1, z_2) \in W[[z_1, z_2]][z_1^{-1}, z_2^{-1}], \tag{5.1}$$

then $A(z_1, z_2)$ is in the image of the (injective) map $i_{z_1;z_2}$, i.e. there is a (unique) $X(z_1, z_2) \in W[[z_1, z_2]][z_1^{-1}, z_2^{-1}, (z_1 - z_2)^{-1}]$ such that we have the expansion

$$A(z_1, z_2) = i_{z_1;z_2} X(z_1, z_2). \tag{5.2}$$

In fact, we can take $X(z_1, z_2) = (z_1 - z_2)^{-N} A_N$. In this case we have also

$$(z_1 - z_2)^N A(z_1, z_2) = (z_1 - z_2)^N X(z_1, z_2). \tag{5.3}$$

(Note that although A_N depends on N, we obtain the same X for all N that make (5.1) true, cf. [11].)

[a] See for instance the notion of compatible fields in [3, Definition 7.3], [17], and the reformulation of compatibility in [11, 14].

One way to check (5.1) is by finding $B(z_2, z_1) \in W((z_2))(z_1))$ such that

$$(z_1 - z_2)^N A(z_1, z_2) = (z_1 - z_2)^N B(z_2, z_1). \tag{5.4}$$

Indeed, the left-hand side shows that (5.4) has at worst a finite order pole in z_2 (by assumption on $A(z_1, z_2)$) and the right-hand side that at worst it has a finite order pole in z_1 (by assumption on $B(z_2, z_1)$). This means that (5.4) belongs to $W[[z_1, z_2]][z_1^{-1}, z_2^{-1}]$, as we wanted to show. In this case we have not only that A is the expansion (5.2) of X, but also that B is the "opposite" expansion:

$$B(z_1, z_2) = i_{z_2;z_1} X(z_1, z_2).$$

There are generalizations to more variables $z_1, z_2, \ldots, z_n$, and to various expansion maps.

We will need slight refinements of these phenomena in case there is a quantum parameter t present. For example:

Lemma 5.1. *Let $W = W_0[[t]]$ be a topologically free k-module and*

$$A(z_1, z_2; t) \in W((z_1))((z_2)),$$

and supppose that for every $k \geq 0$ there is an $N \geq 0$ such that

$$A_N^k \equiv (z_1 - z_2)^N A(z_1, z_2; t) \mod t^k \in W_0[[z_1, z_2]][z_1^{-1}, z_2^{-1}][[t]]/\langle t^k \rangle. \tag{5.5}$$

Then there is a $X(z_1, z_2) \in W[[z_1, z_2]][z_1^{-1}, z_2^{-1}, (z_1 - z_2)^{-1}]$ such that

$$i_{z_1;z_2} X(z_1, z_2) = A(z_1, z_2). \tag{5.6}$$

Proof. If (5.5) holds for some N we can define

$$X^k = (z_1 - z_2)^{-N} A_N^k \in W_0[[z_1, z_2]][z_1^{-1}, z_2^{-1}, (z_1 - z_2)^{-1}][[t]]/\langle t^k \rangle,$$

and we have

$$i_{z_1;z_2} X^k(z_1, z_2) = A_N^k(z_1, z_2; t).$$

Then the X^ks fit together to define a (unique) $X(z_1, z_2; t) \in W_0[[z_1, z_2]][z_1^{-1}, z_2^{-1}, (z_1 - z_2)^{-1}][[t]] = W[[z_1, z_2]][z_1^{-1}, z_2^{-1}, (z_1 - z_2)^{-1}]$ such that (5.6) holds (see the convention on k-modules in Sec. 3). □

Note that there need not to be a uniform N that makes (5.5) true for all k; consider for instance the case $A(z_1, z_2; t) = i_{z_1;z_2} e^{t/(z_1 - z_2)}$.

Lemma 5.2. *Let W be a topologically free k-module. If there are*

$$A(z_1, z_2; t) \in W((z_1))((z_2)), \quad B(z_2, z_1; t) \in W((z_2))((z_1))$$

such that for every $k \geq 0$ there is an $N \geq 0$ such that

$$(z_1 - z_2)^N A(z_1 z_2; t) \equiv (z_1 - z_2)^N B(z_2, z_1; t) \mod t^k, \tag{5.7}$$

then there is $X(z_1, z_2) \in W[[z_1, z_2]][z_1^{-1}, z_2^{-1}, (z_1 - z_2)^{-1}]$ such that

$$i_{z_1;z_2} X(z_1, z_2) = A(z_1, z_2), \quad i_{z_2;z_1} X(z_1, z_1) = B(z_2, z_1),$$

and in this case we have

$$(z_1 - z_2)^N A(z_1, z_2) = (z_1 - z_2)^N B(z_2, z_1) = (z_1 - z_2)^N X(z_1, z_2).$$

6. First Consequences of the Definition

Lemma 6.1.

$$D1 = 0.$$

Proof. By the vacuum axiom (4.1) we have

$$X_{z_1,z_2}(1 \otimes 1) = e^{z_1 D} 1 = e^{z_2 D} 1 \in V[[z_1]] \cap V[[z_2]].$$

This implies $D1 = 0$. □

We emphasize that X_{z_1,z_2} is assumed to be nonsingular in the z_2 variable at zero, so that $X_{z_1,0}$ is defined. (We used this already in the locality axiom, (4.12).) Define

$$X_z : V \to V[[z]], \quad a \mapsto X_z(a) = e^{zD} a. \tag{6.1}$$

We think of X_z as the "singular multiplication of 1 element of V", which happens to be nonsingular, just as X_{z_1,z_2} is the singular multiplication of 2 elements. Later, in Theorem 11.1, we will define a singular multiplication $X_{z_1,\ldots,z_n}$ of n elements of V.

Then we have

$$X_z(a) = X_{z,0}(a \otimes 1), \tag{6.2}$$

by the vacuum axiom (4.1).

Lemma 6.2. *For all $a, b \in V$ we have the following expansion*:

$$i_{z_1;z_2} X_{z_1,z_2}(a \otimes b) = X_{z_1,0}(1 \otimes X_{z_2})(a \otimes b).$$

Proof. Since X_{z_1,z_2} is regular at $z_2 = 0$ we have

$$\begin{aligned} i_{z_1;z_2} X_{z_1,z_2}(a \otimes b) &= e^{z_2 \partial_w} X_{z_1,w}|_{w=0} \\ &= X_{z_1,0}(a \otimes e^{z_2 D} b) \qquad \text{by (4.3)}, \\ &= X_{z_1,0}(1 \otimes X_{z_2})(a \otimes b) \quad \text{by (6.1)}. \end{aligned}$$

□

Remark 6.1. We derived the expansion of Lemma 6.2 from the covariance axiom (4.3). Conversely, if we know that X_{z_1,z_2} has this expansion we see that $\partial_{z_2} X_{z_1,z_2}(a \otimes b)$ and $X_{z_1,z_2}(a \otimes Db)$ both have the same image under $i_{z_1;z_2}$. So, $i_{z_1;z_2}$ being injective, we can derive the covariance axiom (4.3) from the existence of the expansion in Lemma 6.2.

7. Analytic Continuation for $n = 2$

To make contact with the usual notation and terminology in the theory of vertex algebras we introduce some definitions.

Definition 7.1 (Field). Let V be a topologically free k-module. A *field* on V is an element of $\mathrm{Hom}(V, V((z)))$.

We use here the Convention in Sec. 3. So if $a(z)$ is a field, we have for all $b \in V$

$$a(z)b \in V_0((z))[[t]].$$

Definition 7.2 (Vertex Operator). If V is an H_D-quantum vertex algebra we define the vertex operator $Y(a, z)$ associated to $a \in V$ by

$$Y(a, z)b = X_{z,0}(a \otimes b), \tag{7.1}$$

for $b \in V$. We will also use the notation $Y_z : a \otimes b \mapsto Y(a, z)b$, so that $Y_z = X_{z,0}$.

Note that the vertex operator $a(z) = Y(a, z)$ for an H_D-quantum vertex algebra is a field, for every $a \in V$.

We can rewrite Lemma 6.2 as follows:

Corollary 7.1 (Analytic Continuation). *The singular multiplication $X_{z_1,z_2}(a\otimes b)$ is the analytic continuation of the product of vertex operators $Y(a, z_1)Y(b, z_2)1$, i.e.*

$$i_{z_1;z_2} X_{z_1,z_2}(a \otimes b) = Y(a, z_1)Y(b, z_2)1.$$

Remark 7.1. In Theorem 11.1 we construct an n-variable version $X_{z_1,z_2,\ldots,z_n}$ of the singular multiplication satisfying

$$i_{z_1;z_2;\ldots;z_n} X_{z_1,z_2,\ldots,z_n}(a_1 \otimes a_2 \otimes \cdots \otimes a_n) = Y(a_1, z_1)Y(a_2, z_2)\cdots Y(a_n, z_n)1,$$

i.e. we construct the analytic continuation of arbitrary product of vertex operators.

Remark 7.2. At this point we would like to emphasize that the axioms we use are much weaker than those of Frenkel–Reshetikhin, [7]. Indeed, one of their axioms not only requires that the product of (quantum) vertex operators can be analytically continued, but also that the resulting function is meromorphic in the variables. This is not always the case in our H_D-quantum vertex algebras. For instance, we allow a singular multiplication $X_{z_1,z_2}(a\otimes b)$ with a singularity of the form $e^{t/(z_1-z_2)}$, but this does not satisfy the Frenkel–Reshetikhin axioms, as there is an essential singularity at $z_1 = z_2$. In our setup the quantum parameter t is an independent variable (and we always expand in positive powers of t), whereas in Frenkel–Reshetikhin t is a complex number.

8. Alternative Axioms

We have formulated the axioms of an H_D-quantum vertex algebra in terms of the rational singular multiplication X_{z_1,z_2}. Traditionally the axioms of a vertex algebra have been formulated in terms of the 1-variable vertex operator Y_z. Let us briefly indicate how this would work for H_D-quantum vertex algebras. Our axioms from

Definition 4.1 would change slightly. We start out with assuming the existence of a map

$$Y_z\colon V\otimes V\to V((z)),$$

instead of the singular multiplication X_{z_1,z_2}. The braiding and translation maps $S^{(\tau)}$ and $S^{(\gamma)}$ are as before. The vertex operator satisfies the following axioms:

(Vacuum)

$$Y_z(1\otimes a)=a,\quad Y_z(a\otimes 1)=e^{zD}a$$

(H_D-Covariance)

$$i_{z;\gamma}Y(a,z+\gamma)e^{\gamma D}b=i_{z;\gamma}e^{\gamma D}Y_z\circ S^{(\gamma)}_{z,0}(a\otimes b). \tag{8.1}$$

(Compatibility with Multiplication)

$$S_{z_1,z_2}(Y_w\otimes 1)=(Y_w\otimes 1)i_{z_1,z_1-z_2;w}S^{23}_{z_1+w,z_2}S^{13}_{z_1,z_2}, \tag{8.2}$$

$$S_{z_1,z_2}(1\otimes Y_w)=(1\otimes Y_w)i_{z_1-z_2,z_2;w}S^{12}_{z_1,z_2+w}S^{13}_{z_1,z_2}. \tag{8.3}$$

(Locality) For all $a,b\in V$ and $k\geq 0$ there exists $N\geq 0$ such that for all $c\in V$

$$(z-w)^NY(a,z)Y(b,w)c\equiv(z-w)^NY_w(1\otimes Y_z)(S_{w,z}(b\otimes a)\otimes c)\mod t^k. \tag{8.4}$$

Given these axioms we can reconstruct X_{z_1,z_2}.

Lemma 8.1. *There exists a map*

$$X_{z_1,z_2}\colon V\otimes V\to V[[z_1,z_2]][z_1^{-1},(z_1-z_2)^{-1}]$$

such that

$$i_{z_1;z_2}X_{z_1,z_2}(a\otimes b)=Y(a,z_1)Y(b,z_2)1.$$

Proof. Let $A(z_1,z_2)=Y(a,z_1)Y(b,z_2)1$. By definition of the braiding $S^{(\tau)}$ and the locality (8.4) for every $k\geq 0$ we have an $N\geq 0$ such that

$$(z_1-z_2)^NA(z_1,z_2)\in V_0[[z_1,z_2]][z_1^{-1}]][[t]]\mod t^k,$$

and the lemma follows from Lemma 5.1. □

Thus we can *define* in the present setup the singular multiplication X_{z_1,z_2} to be the analytic continuation of the product $Y(a,z_1)Y(b,z_2)1$.

Alternatively, given the fields $Y(a,z)$ for any $a\in V$ we can define X_{z_1,z_2} as follows:

Definition 8.1. For any $a,b\in V$ define

$$X_{z_1,z_2}(a\otimes b)=e^{z_2D}Y_{z_1-z_2}S^{(z_2)}_{z_1-z_2,0}(a\otimes b).$$

The two definitions are equivalent:

Lemma 8.2. *If* X_{z_1,z_2} *is given by Definition* 8.1 *then*

$$i_{z_1;z_2}X_{z_1,z_2}(a\otimes b)=Y(a,z_1)e^{z_2D}b=Y(a,z_1)Y(b,z_2)1.$$

The proof follows from (8.1) and the vacuum axiom.

To obtain the axioms of Sec. 4 note that Lemma 8.2 implies the covariance axiom (4.3), by Lemma 6.2 and Remark 6.1. The rest of the axioms follow immediately.

Remark 8.1. We give in Definition 8.1 a direct construction of X_{z_1,z_2} in terms of Y_z, without using analytic continuation. It seems not so easy to give such an explicit formula for the singular multiplication $X_{z_1,\dots,z_n}$ of n elements of V, to be introduced in Theorem 11.1 using analytic continuation.

Remark 8.2. In the case of classical vertex algebras, as well as Etingof–Kazhdan (EK) quantum vertex operator algebras or Frenkel–Reshetikhin deformed chiral algebras, the translation map $S^{(\gamma)}_{z_1,z_2}$ is the identity, so that in this case

$$X_{z_1,z_2}(a\otimes b)=e^{z_2D}Y(a,z_1-z_2)b\in V[[z_1,z_2]][(z_1-z_2)^{-1}]. \tag{8.5}$$

In particular in these cases we can let $z_1=0$ as $X_{z_1,z_2}(a\otimes b)$ is not singular for $z_1=0$. That is no longer the case for the examples of vertex operators connected to symmetric polynomials. Therefore we have allowed for singular multiplication maps which are singular in z_1 (but not in z_2, if we want to be able to define Y_z fields as above). It is possible to modify the theory further to allow for singularities in both the variables, but we have not yet encountered examples which would call for such generalization.

The conclusion of this section is that we can start either with Y_z or with X_{z_1,z_2} as fundamental ingredient in the theory. Since there are by now hundreds of papers on vertex algebras written in terms of Y_z we have allowed ourselves to emphasize X_{z_1,z_2} in this paper.

9. Braiding and Skewsymmetry

An important fact of the theory of classical vertex algebras is that the singular multiplication maps X_{z_1,z_2} are "commutative", i.e. we have for any $a,b\in V$

$$X_{z_1,z_2}(a\otimes b)=X_{z_2,z_1}(b\otimes a).$$

In the case of H_D-quantum vertex algebras the singular multiplication maps X_{z_1,z_2} on $V^{\otimes 2}$ are no longer "commutative", but rather "braided commutative", as shown by the next lemma.

Lemma 9.1 (Braided Symmetry). *For any* $a,b\in V$

$$X_{z_1,z_2}(a\otimes b)=X_{z_2,z_1}S^{(\tau)}_{z_2,z_1}(b\otimes a).$$

Proof. Let $E = X_{z_1,z_2}(a \otimes b)$, $F = X_{z_2,z_1} S^{(\tau)}_{z_2,z_1}(b \otimes a)$. We have

$$\begin{aligned} i_{z_1;z_2} E &= X_{z_1,0}(1 \otimes X_{z_2})(a \otimes b) && \text{by Lemma 6.2} \\ &= X_{z_1,0}(1 \otimes X_{z_2,0})(a \otimes b \otimes 1) && \text{by (6.1).} \end{aligned} \tag{9.1}$$

On the other hand, by the same calculation,

$$i_{z_2;z_1} F = X_{z_2,0}(1 \otimes X_{z_1,0})(i_{z_2;z_1} S^{(\tau)}_{z_2,z_1}(b \otimes a) \otimes 1). \tag{9.2}$$

By the locality axiom (4.12) the right-hand sides of (9.1) and (9.2) are after multiplication by $(z_1 - z_2)^N$ equal modulo t^k. But then there is for all $k \geq 0$ an $N \geq 0$ such that for the left-hand sides we have

$$(z_1 - z_2)^N E \equiv (z_1 - z_2)^N F \mod t^k.$$

Since E and F both belong to $V[[z_1, z_2]][z_1^{-1}, z_2^{-1}, (z_1 - z_2)^{-1}]$ this implies that they are in fact equal. ☐

Corollary 9.1 (Skewsymmetry). *For any $a, b \in V$ we have*

$$e^{z_2 D} Y_{z_1 - z_2} \circ S^{(z_2)}_{z_1 - z_2, 0}(a \otimes b) = e^{z_1 D} Y_{z_2 - z_1} \circ S^{(z_1)}_{z_2 - z_1, 0} \circ S^{(\tau)}_{z_2, z_1}(b \otimes a).$$

The proof follows from Lemma 9.1 and Definition 8.1.

Remark 9.1. In the case of EK quantum vertex operator algebras the translation map $S^{(\gamma)}_{z_1,z_2}$ is the identity, *and* the braiding map depends on a single variable $z_1 - z_2$, therefore we can substitute $z_1 = 0$ and we get the EK braided skewsymmetry relation

$$e^{zD} Y(a, -z) b = Y_z \circ S^{(\tau)}_z (b \otimes a),$$

where $S^{(\tau)}_z = S^{(\tau)}_{z,0}$.

Note that we cannot substitute $z_1 = 0$ in general as $S^{(\tau)}_{z_2,z_1}$ might be singular at $z_1 = 0$, see Sec. 26. The skewsymmetry relation in Corollary 9.1 looks much less appealing than the braided symmetry relation in Lemma 9.1. Many of the properties of H_D-quantum vertex algebras look more symmetric in terms of the singular maps X_{z_1,z_2}, which was one of the reasons we prefer working with them, rather than the Y_z fields.

10. Braiding Maps for $n > 2$

The singular multiplication map X_{z_1,z_2} on $V^{\otimes 2}$ is invariant under simultaneous interchange of the variables z_1, z_2 and the factors in $V^{\otimes 2}$, up to insertion of the two variable braiding map $S^{(\tau)}_{z_1,z_2}$, according to the Lemma 9.1. In the next section we will construct for all $n \geq 1$ a singular multiplication map $X_{z_1,\ldots,z_n}$ on $V^{\otimes n}$, see Theorem 11.1. These are invariant under simultaneous permutation of the variables z_i and the factors in $V^{\otimes n}$, up to insertion of an n variable braiding map $S^f_{z_1,\ldots,z_n}$, see Corollary 11.1. In this section we construct these braiding maps.

Let $n \geq 2$, $I_n = \{1, 2, \ldots, n\}$ and let $\mathcal{S}_n$ be the permutation group of I_n, i.e. the group of bijections $\mathbf{f}\colon I_n \to I_n$. Let $\mathbf{w}_i = (i i+1) \in \mathcal{S}_n$ (where $i = 1, 2, \ldots, n$) be the simple transposition given on $j \in I_n$ by

$$\mathbf{w}_i(j) = \begin{cases} j & j \neq i, i+1 \\ i+1 & j = i \\ i & j = i+1. \end{cases}$$

Then $\mathcal{S}_n$ is generated by the $\mathbf{w}_i$, with as only relations

$$\mathbf{w}_i^2 = 1, \quad \mathbf{w}_i \mathbf{w}_{i+1} \mathbf{w}_i = \mathbf{w}_{i+1} \mathbf{w}_i \mathbf{w}_{i+1}, \tag{10.1}$$

and

$$\mathbf{w}_i \mathbf{w}_j = \mathbf{w}_j \mathbf{w}_i, \quad |i - j| \geq 2. \tag{10.2}$$

Now let V be a topologically free k-module, and define a *right* action for $\mathbf{f} \in \mathcal{S}_n$ on the n-fold tensor product of V by

$$\sigma_{\mathbf{f}}\colon V^{\otimes n} \to V^{\otimes n}, \quad A_n \mapsto a_{\mathbf{f}(1)} \otimes a_{\mathbf{f}(2)} \otimes \cdots \otimes a_{\mathbf{f}(n)},$$

where $A_n = a_1 \otimes a_2 \otimes \cdots \otimes a_n \in V^{\otimes n}$. Let $\tau\colon a \otimes b \mapsto b \otimes a \in V^{\otimes 2}$. Then

$$\sigma_{\mathbf{w}_i} = i^{i-1} \otimes \tau \otimes 1^{n-i-1}.$$

Here we write 1^k for $1_V \otimes 1_V \otimes \cdots \otimes 1_V$, the k-fold tensor product of the identity $1_V : V \to V$. We emphasize that if $\mathbf{f} = \mathbf{g}\mathbf{w}_i$ then $\sigma_{\mathbf{f}} = \sigma_{\mathbf{w}_i} \sigma_{\mathbf{g}}$.

Let $\mathrm{Rat}_{z_1, z_2, \ldots, z_n}$ be the space of rational functions in n variables. Then $\mathcal{S}_n$ acts on the *left* on $\mathrm{Rat}_{z_1, z_2, \ldots, z_n}$ by permutation the variables: if $\mathbf{f} \in \mathcal{S}_n$ and $A_{z_1, \ldots, z_n} \in \mathrm{Rat}_{z_1, z_2, \ldots, z_n}$, then we put

$$\mathbf{f}.A_{z_1, \ldots, z_n} = A_{\mathbf{f}(z_1, z_2, \ldots, z_n)},$$

where we write $\mathbf{f}(z_1, z_2, \ldots, z_n)$ for $z_{\mathbf{f}(1)}, z_{\mathbf{f}(2)}, \ldots, z_{\mathbf{f}(n)}$.

Now let $\mathrm{Map}_{z_1, z_2, \ldots, z_n}(V^{\otimes n})$ be the space of linear maps

$$V^{\otimes n} \to V^{\otimes n}[z_i^{\pm 1}, (z_i - z_j)^{-1}], \quad 1 \leq i < j \leq n.$$

We have an action of $\mathcal{S}_n$ on $\mathrm{Map}_{z_1, z_2, \ldots, z_n}(V^{\otimes n})$ combining the action of $\mathcal{S}_n$ on $V^{\otimes n}$ and on rational functions: if $\mathbf{f} \in \mathcal{S}_n$ and $A_{z_1, \ldots, z_n} \in \mathrm{Map}_{z_1, z_2, \ldots, z_n}(V^{\otimes n})$ then define

$$\mathbf{f} \cdot A_{z_1, \ldots, z_n} = \sigma_{\mathbf{f}}^{-1} \circ A_{\mathbf{f}(z_1, z_2, \ldots, z_n)} \circ \sigma_{\mathbf{f}}.$$

Now let V be an H_D-quantum vertex algebra. So we get, by definition, in particular a braiding map $S^{(\tau)}_{z_1, z_2} \in \mathrm{Map}_{z_1, z_2}(V^{\otimes 2})$. For simplicity we denote it by S_{z_1, z_2} in this section, as we will not use $S^{(\gamma)}_{z_1, z_2}$ here. It satisfies, see Definition 4.1,

$$S_{z_1, z_2} \circ \tau \circ S_{z_2, z_1} \circ \tau = 1_{V^{\otimes 2}}, \tag{10.3}$$

$$S^{12}_{z_1, z_2} S^{13}_{z_1, z_3} S^{23}_{z_2, z_3} = S^{14}_{z_1, z_3} S^{13}_{z_1, z_3} S^{12}_{z_1, z_2}. \tag{10.4}$$

We will use the braiding matrix S_{z_1, z_2} to define a map $\mathcal{S}_n \to \mathrm{Map}_{z_1, z_2, \ldots, z_n}(V^{\otimes n})$.

Definition 10.1 (Braiding Maps). Define for each $\mathbf{f} \in \mathcal{S}_n$ an element $S^f_{z_1,\ldots,z_n}$ of $\mathrm{Map}_{z_1,z_2,\ldots,z_n}(V^{\otimes n})$, called the braiding map associated to $\mathbf{f}$, by expanding $\mathbf{f}$ (in any way) in simple reflections w_i and using

$$S^{\mathbf{w}_i}_{z_1,\ldots,z_n} = 1^{i-1} \otimes S^{(\tau)}_{z_i,z_{i+1}} \otimes i^{n-i-1},$$

and

$$S^{\mathbf{fg}}_{z_1,\ldots,z_n} = S^{\mathbf{g}}_{z_1,\ldots,z_n} \sigma_{\mathbf{g}} S^{\mathbf{f}}_{\mathbf{g}^{-1}(z_1,\ldots,z_n)} (\sigma_{\mathbf{f}})^{-1}. \tag{10.5}$$

The point is that to define $S^{\mathbf{f}}_{z_1,\ldots,z_n}$ for $\mathbf{f} \in \mathcal{S}_n$, we can take *any* decomposition of $\mathbf{f}$ into simple transpositions $\mathbf{w}_i$, i.e. this definition is unambigious. The proof of this statement is discussed in Appendix B.

11. Analytic Continuation for $n > 2$

If V is an H_D-quantum vertex algebra, recall that we have the "singular" multiplications X_z and X_{z_1,z_2} of 1, respectively 2 elements of V, see (6.1) and Definition 4.1. We will in this section construct singular multiplications $X_{z_1,\ldots,z_n}$ of n elements of V.

Let $\mathbf{f}_n = \mathbf{w}_1\mathbf{w}_2 \ldots \mathbf{w}_{n-1}$ be the n-cycle $(123\ldots n)$ and consider the associated braiding matrix $S^{\mathbf{f}_n}_{z_1,\ldots,z_n}$. We have $\mathbf{f}_n = \mathbf{w}_1(23\ldots n)$. Writing $\mathbf{f}_{n-1} = (23\ldots n)$ and $\sigma_n = \sigma_{\mathbf{f}_n}$ we find from (10.5) that

$$S^{\mathbf{f}_n}_{z_2,z_3,\ldots,z_n,z_1} \sigma_n = (1 \otimes S^{\mathbf{f}_{n-1}}_{z_3,\ldots,z_n,z_1} \sigma_{n-1})(S^{(\tau)}_{z_2,z_1} \tau \otimes 1^{\otimes n-2}). \tag{11.1}$$

We will frequently use the abbreviation

$$p_n = p_n(z_1, z_2, \ldots, z_n) = \prod_{1 \le i < j \le n} z_i - z_j. \tag{11.2}$$

Theorem 11.1 (Analytic Continuation). *Let V be an H_D-quantum vertex algebra. There exist for all $n \ge 2$ maps*

$$X_{z_1,\ldots,z_n} \colon V^{\otimes n} \to V[[z_k]][z_i^{-1}, (z_i - z_j)^{-1}], \quad 1 \le i < j \le n, 1 \le k \le n$$

such that

$$i_{z_1;z_2,\ldots,z_n} X_{z_1,\ldots,z_n} = X_{z_1,0}(1 \otimes X_{z_2,\ldots,z_n}). \tag{11.3}$$

and

$$X_{z_1,z_2,\ldots,z_n} = X_{z_2,\ldots,z_n,z_1} S^{\mathbf{f}_n}_{z_2,\ldots,z_n,z_1} \sigma_n, \tag{11.4}$$

where $S^{\mathbf{f}_n}_{z_2,\ldots,z_n,z_1}$ is defined in Definition 10.1.

Proof. The theorem is true for $n = 2$ by Lemmas 6.2 and 9.1. Assume the theorem is true for ℓ, $2 \le \ell \le n_0$ and let $n = n_0 + 1$. The induction hypothesis implies that

$$i_{z_2;z_3,\ldots,z_n} X_{z_2,z_3,\ldots,z_n} = X_{z_2,0}(1 \otimes X_{z_3,z_4,\ldots,z_n}),$$

so that for every $k \geq 0$ there is an $N \geq 0$ such that

$$p_{n-1}^N X_{z_2,z_3,\dots,z_n} \equiv p_{n-1}^N X_{z_2,0}(1 \otimes X_{z_3,z_4,\dots,z_n}) \mod t^k. \tag{11.5}$$

Also we have

$$X_{z_1,z_3,z_4\dots,z_n} = X_{z_3,z_4,\dots,z_n,z_1} S^{\mathbf{f}_{n-1}}_{z_3,z_4,\dots,z_n,z_1} \sigma_{n-1}. \tag{11.6}$$

Consider $E = X_{z_1,0}(1 \otimes X_{z_2,z_3,\dots,z_n})(A_n)$, $A_n \in V^{\otimes n}$. This is an element of $V((z_1))[[z_2, z_3, \dots, z_n]][z_1^{-1}, \dots, z_{n-1}^{-1}, (z_i - z_j)^{-1}]$, $2 \leq i < j \leq n$, and we want to show E is in the image of the expansion $i_{z_1;z_2,\dots,z_n}$. For this it suffices to show that for every $k \geq 0$ there is an $N \geq 0$ such that $p_n^N E$ has at worst a finite order pole in z_1. This is a small calculation: for all $k \geq 0$ there is $N \geq 0$ such that modulo t^k we have

$$\begin{aligned} p_n^N E &= p_n^N X_{z_1,0}(1 \otimes (X_{z_2,0}(1 \otimes X_{z_3,\dots,z_n})))(A_n) && \text{by (11.5)} \\ &= p_n^N X_{z_2,0}(1 \otimes (X_{z_1,0}(1 \otimes X_{z_3,\dots,z_n})))(i_{z_2;z_1} S^{(\tau)}_{z_2,z_1} \tau \otimes 1^{n-2})(A_n) && \text{by (4.12)} \\ &= p_n^N X_{z_2,0}(1 \otimes (X_{z_1,z_3,\dots,z_n}))(S^{(\tau)}_{z_2,z_1} \tau \otimes 1^{n-2})(A_n) && \text{by (11.5)} \\ &= p_n^N X_{z_2,0}(1 \otimes (X_{z_3,z_4,\dots,z_n,z_1}))(1 \otimes S^{\mathbf{f}_{n-1}}_{z_3,z_4,\dots,z_n,z_1} \sigma_{n-1}) \\ &\quad \times (S^{(\tau)}_{z_2,z_1} \tau \otimes 1^{n-2})(A_n) && \text{by (11.6)} \\ &= p_n^N X_{z_2,0}(1 \otimes (X_{z_3,z_4,\dots,z_n,z_1})) S^{\mathbf{f}_n}_{z_2,z_3,\dots,z_n,z_1} \sigma_n(A_n) && \text{by (11.1).} \end{aligned}$$

We see from the last expression that $p_n^N E$ has at worst a finite order pole in z_1 and hence there is $X_{z_1,z_2,\dots,z_n}$ such that (11.3) holds.

Next consider $F = X_{z_2,z_3,\dots,z_n,z_1} S^{f_n}_{z_2,z_3,\dots,z_n,z_1} \sigma_n(A_n)$ and $G = X_{z_1,\dots,z_n}(A_n)$. For all $k \geq 0$ there is an $N \geq 0$ such that modulo t^k

$$p_n^N E = p_n^N G.$$

By what we just proved we have

$$i_{z_2;z_3,\dots,z_n,z_1} F = X_{z_2,0}(1 \otimes X_{z_3,\dots,z_n,z_1}) i_{z_2;z_3,\dots,z_n,z_1} S^{\mathbf{f}_n}_{z_2,z_3,\dots,z_n,z_1} \sigma_n(A_n),$$

and so for all $k \geq 0$ there is $N \geq 0$ such that modulo t^k we have

$$p_n^N F = p_n^N X_{z_2,0}(1 \otimes X_{z_3,\dots,z_n,z_1}) i_{z_2;z_3,\dots,z_n,z_1} S^{\mathbf{f}_n}_{z_2,z_3,\dots,z_n,z_1} \sigma_n(A_n) = p_n^N E = p_n^N G.$$

Since G, F both belong to $V[[z_i]][z_i^{-1}, (z_i - z_j)^{-1}]$ this forces $G = F$, i.e. (11.4) holds. □

Corollary 11.1 (Analytic Continuation for Products of Fields). *For all $n \geq 2$ and $1 \leq i \leq n-1$ we have, if $A_n = a_1 \otimes a_2 \otimes \cdots \otimes a_n$, the expansion*

$$\begin{aligned} &i_{z_1;z_2;\dots,z_i;z_{i+1},z_{i+2},\dots,z_n} X_{z_1,\dots,z_n}(A_n) \\ &\quad = X_{z_1,0}(a_1 \otimes X_{z_2,0}(a_2 \otimes \cdots (a_{i-1} \otimes X_{z_i,0}(a_i \otimes X_{z_{i+1},z_{i+2},\dots,z_n} \\ &\qquad \times (a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_n) \cdots)))). \end{aligned}$$

In particular the case of $i = n-1$ of the corollary is (using the notation (7.1))

$$i_{z_1;z_2;\ldots;z_n} X_{z_1,z_2,\ldots,z_n}(A_n) = Y(a_1,z_1)Y(a_2,z_2)\cdots Y(a_n,z_n)1. \tag{11.7}$$

In other words the n-variable $X_{z_1,z_2,\ldots,z_n}(A_n)$ is the analytic continuation of the composition of n vertex operators acting on the vacuum.

12. Further Consequences

Lemma 12.1. *The infinitesimal forms of the H_D-covariance axioms* (4.4), (4.5) *are*

$$(1\otimes D + \partial_{z_2})S_{z_1,z_2} = S_{z_1,z_2}(1\otimes D), \tag{12.1}$$

$$DY(a,z)b = \partial_z Y(a,z)b + Y(a,z)Db - Y_z \circ \alpha_{z,0}(a\otimes b). \tag{12.2}$$

where α_{z_1,z_2} is defined to be $\partial_\gamma S^{(\gamma)}_{z_1,z_2}$ and satisfies the infinitesimal form of the vacuum axioms (4.2)

$$\alpha_{z_1,z_2}(a\otimes 1) = 0, \quad \alpha_{z_1,z_2}(1\otimes b) = 0,$$

Lemma 12.2. *For all $a,b\in V$*

$$X_{z_1,z_2}(Da\otimes b) = \partial_{z_1} X_{z_1,z_2}(a\otimes b), \tag{12.3}$$

Proof. By the previous Lemma 12.1

$$\begin{aligned}
X_{z_1,z_2}(Da\otimes b) &= X_{z_1,z_2}\tau(1\otimes D)(b\otimes a) \\
&= X_{z_2,z_1}S^{(\tau)}_{z_2,z_1}(1\otimes D)(b\otimes a) && \text{by Lemma 9.1} \\
&= X_{z_2,z_1}(1\otimes D+\partial_{z_2})S^{(\tau)}_{z_2,z_1}(b\otimes a) && \text{by (12.1)} \\
&= (\partial_{z_2}(X_{z_2,z_1})S^{(\tau)}_{z_2,z_1} + X_{z_2,z_1}\partial_{z_2}S^{(\tau)}_{z_2,z_1})(b\otimes a) && \text{by (4.3)} \\
&= \partial_{z_2}(X_{z_2,z_1}S^{(\tau)}_{z_2,z_1})(b\otimes a) \\
&= \partial_{z_2}X_{z_1,z_2}(a\otimes b) && \text{by Lemma 9.1,}
\end{aligned}$$

proving (12.3). □

Corollary 12.1. *For all $n\ge 2$, $1\le i\le n$ and $A_n = a_1\otimes a_2\otimes\cdots\otimes a_n \in V^{\otimes n}$ we have*

$$\partial_{z_i} X_{z_1,z_2,\ldots,z_n}(A_n) = X_{z_1,z_2,\ldots,z_n}(a_1\otimes\cdots\otimes Da_i\otimes\cdots\otimes a_n).$$

Proof. For $n=2$ and $i=2$ this is axiom (4.3), and for $i=1$ this is Lemma 12.2. Put for $n>2$

$$E = \partial_{z_i}X_{z_1,z_2,\ldots,z_n}(A_n), \quad F = X_{z_1,z_2,\ldots,z_n}(a_1\otimes\cdots\otimes Da_i\otimes\cdots\otimes a_n).$$

Since ∂_{z_i} commutes with expansions we have

$$\begin{aligned}
&i_{z_1;z_2;\ldots;z_i;z_{i+1},\ldots,z_n}E \\
&\quad = X_{z_1,0}(a_1 \otimes X_{z_2,0}(a_2 \otimes \cdots (a_{i-1} \otimes \partial_{z_i} X_{z_i,0}(a_i \otimes X_{z_{i+1},\ldots,z_n} \\
&\qquad \times (a_{i+1} \otimes \cdots \otimes a_n))\cdots))) && \text{by Corollary 11.1} \\
&\quad = X_{z_1,0}(a_1 \otimes X_{z_2,0}(a_2 \otimes \cdots (a_{i-1} \otimes X_{z_i,0}(Da_i \otimes X_{z_{i+1},\ldots,z_n} \\
&\qquad \times (a_{i+1} \otimes \cdots \otimes a_n)))\cdots)) && \text{by Lemma 12.2} \\
&\qquad = i_{z_1;z_2;\ldots;z_i;z_{i+1},\ldots,z_n}F && \text{by Corollary 11.1.}
\end{aligned}$$

Since both E and F belong to $V[[z_i]][z_i^{-1}, (z_i - z)^{-1}]$, $1 \leq i < j \leq n$, and have the same expansion they must be equal. □

Lemma 12.3. *For all $n \geq 1$ and $A_n = a_1 \otimes a_2 \otimes \cdots \otimes a_n \in V^n$ we have*

$$X_{z_1,z_2,\ldots,z_n,0}(A_n \otimes 1) = X_{z_1,z_2,\ldots,z_n}(A_n).$$

Proof. For $n = 1$ this is (6.2). Assume that the lemma is true for all ℓ, $1 \leq \ell \leq n_0$, and let $n = n_0 + 1$. Put $E = X_{z_1,z_2,\ldots,z_n,0}(A_n \otimes 1)$, $F = X_{z_1,z_2,\ldots,z_n}(A_n)$. By Theorem 11.1 and the induction hypothesis

$$\begin{aligned}
i_{z_1;z_2,\ldots,z_n}E &= X_{z_1,0}(a_1 \otimes X_{z_2,\ldots,z_n,0}(a_2 \otimes a_3 \otimes \cdots \otimes a_n \otimes 1)) \\
&= X_{z_1,0}(a_1 \otimes X_{z_2,\ldots,z_n}(a_2 \otimes a_3 \otimes \cdots \otimes a_n)) \\
&= i_{z_1;z_2,\ldots,z_n}F.
\end{aligned}$$

Since both E and F belong to $V[[z_i]][z_i^{-1}, (z_i - z)^{-1}]$, $1 \leq i < j \leq n$, and have the same expansion they must be equal. □

Lemma 12.4. *Suppose that $S^{(\gamma)}_{z_1,z_2}$ is the identity map on $V \otimes V$. Then the following is true*:

$$DX_{z_1,z_2} = (\partial_{z_1} + \partial_{z_2})X_{z_1,z_2} = X_{z_1,z_2}(D \otimes 1 + 1 \otimes D), \tag{12.4}$$

$$[D, Y(a,z)] = \partial_z Y(a,z), \tag{12.5}$$

$$X_{z_1,z_2} \circ (\partial_{z_1} + \partial_{z_2})S^{(\tau)}_{z_1,z_2} = 0. \tag{12.6}$$

Proof. The second property is a direct consequence of Lemma 12.1. The first equality follows from expanding both sides of (4.5) in powers of γ and comparing the coefficients in front of γ^1.

For the last part rewrite (12.4) as

$$e^{\gamma D} X_{z_1,z_2} = X_{z_1,z_2}\Delta(e^{\gamma D}),$$

where Δ is the coproduct of H_D, so that $\Delta(e^{\gamma D} = e^{\gamma D} \otimes e^{\gamma D})$. Similarly rewrite the H_D-covariance axiom (4.4) for the braiding as

$$(1 \otimes e^{-\gamma D})S_{z_1,z_2} = e^{\gamma(\partial_{z_1} + \partial_{z_2})} S_{z_1,z_2}(1 \otimes e^{-\gamma D}).$$

By differentiating with respect to z_1 the axiom (4.9) we obtain a similar equation involving ∂_{z_1} and $e^{-\gamma D} \otimes 1$, and we combine these as

$$\Delta(e^{-\gamma D}) S^{(\tau)}_{z_1,z_2} = e^{\gamma(\partial_{z_1}+\partial_{z_2})} S^{(\tau)}_{z_1,z_2} \Delta(e^{-\gamma D}).$$

Now we calculate

$$\begin{aligned} e^{-\gamma D} X_{z_2,z_1} &= e^{-\gamma D} X_{z_1,z_2} S_{z_1,z_2} \tau \\ &= X_{z_1,z_2} \Delta(e^{-\gamma D}) S_{z_1,z_2} \tau \\ &= X_{z_1,z_2} e^{\gamma(\partial_{z_1}+\partial_{z_2})} S^{(\tau)}_{z_1,z_2} \tau \Delta(e^{-\gamma D}). \end{aligned}$$

On the other hand

$$e^{-\gamma D} X_{z_2,z_1} = X_{z_1,z_2} S_{z_1,z_2} \tau \Delta(e^{-\gamma D}).$$

By multiplying by $\Delta(e^{\gamma D})\tau$ on the right we find

$$X_{z_1,z_2} e^{\gamma(\partial_{z_1}+\partial_{z_2})} S^{(\tau)}_{z_1,z_2} = X_{z_1,z_2} S^{(\tau)}_{z_1,z_2},$$

from which (12.6) follows. □

Remark 12.1. In the context of the lemma above it is natural to assume that $S^{(\tau)}_{z_1,z_2}$ is a function of just $z_1 - z_2$. In this case V is a quantum vertex operator algebra as defined by Etingof–Kazhdan, see [5] (except for the fact that they insist that the braiding is of the form $S^{(\tau)} = 1 + \mathcal{O}(t)$).

13. Braiding and Singular Multiplication

We have seen that the n-fold singular multiplication has cyclic symmetry: if $\mathbf{f}_n$ is the cyclic permuation $(123\ldots n)$, then

$$X_{z_1,\ldots,z_n} = X_{\mathbf{f}_n(z_1,\ldots,z_n)} S^{\mathbf{f}_n}_{\mathbf{f}_n(z,\ldots,z_n)} \sigma_n, \tag{13.1}$$

see Theorem 11.1. In this section we show that in fact the n-fold singular multiplication has arbitrary permutation symmetry: in (13.1) we can replace $\mathbf{f}_n$ by any $\mathbf{f} \in \mathcal{S}_n$.

Lemma 13.1. *For all $n \geq 2$ we have*

$$X_{z_1,z_2,\ldots,z_n} = X_{\mathbf{w}_1(z_1,z_2,\ldots,z_n)} S^{\mathbf{w}_1}_{z_1,\ldots,z_n} (\tau \otimes 1^{n-2}).$$

Proof. Let $E = X_{z_1,z_2,\ldots,z_n}(A_n)$, $F = X_{z_2,z_1,\ldots,z_n} \circ (S^{(\tau)}_{z_2,z_1}\tau \otimes 1^{n-2})(A_n)$, $A_n \in V^{\otimes n}$. Then there exist for all $k \geq 0$ an $N \geq 0$ such that modulo t^k

$$\begin{aligned} &p_n^N i_{z_1;z_2;z_3,z_4,\ldots,z_n} E \\ &\quad = p_n^N X_{z_1,0}(1 \otimes X_{z_2,0}(1 \otimes X_{z_3,z_4,\ldots,z_n}))(A_n) && \text{by Theorem 11.1} \\ &\quad = p_n^N X_{z_2,0}(1 \otimes X_{z_1,0}(1 \otimes X_{z_3,z_4,\ldots,z_n}))(S^{(\tau)}_{z_2,z_1} \otimes 1^{\otimes n-2})(A_n) && \text{by (4.12)} \\ &\quad = p_n^N i_{z_1;z_2;z_3,z_4,\ldots,z_n} F, \end{aligned}$$

by Theorem 11.1 again. Since both E and F belong to $V[[z_i]][z_i^{-1}, (z_i - z)^{-1}]$, $1 \leq i < j \leq n$, and have the same expansion they must be equal. □

Recall that the first simple transposition $\mathbf{w}_1$ and the cyclic permutation $f_n = (123\dots n)$ generate $\mathcal{S}_n$.

Corollary 13.1. *If* $\mathbf{f} \in \mathcal{S}_n$ *is a permutation of* $\{1, 2, \dots, n\}$ *and* $\sigma_{\mathbf{f}}(A_n) = a_{\mathbf{f}(1)} \otimes a_{\mathbf{f}(2)} \otimes \cdots \otimes a_{\mathbf{f}(n)}$, *then*

$$X_{z_1,\dots,z_n} = X_{\mathbf{f}(z_1,\dots,z_n)} S^{\mathbf{f}}_{\mathbf{f}(z_1,\dots,z_n)} \sigma_{\mathbf{f}}. \tag{13.2}$$

Proof. Suppose we have two elements $\mathbf{f}, \mathbf{g} \in \mathcal{S}_n$ such that (13.2) holds. Then, by (B.5),

$$\begin{aligned} X_{\mathbf{fg}(z_1,\dots,z_n)} S^{\mathbf{fg}}_{\mathbf{fg}(z_1,\dots,z_n)} \sigma_{\mathbf{fg}} &= X_{\mathbf{fg}(z_1,\dots,z_n)} S^{\mathbf{g}}_{\mathbf{fg}(z_1,\dots,z_n)} \sigma_{\mathbf{g}} S^{\mathbf{f}}_{\mathbf{g}(z_1,\dots,z_n)} \sigma_{\mathbf{f}} \\ &= \mathbf{f}.(X_{\mathbf{g}(z_1,\dots,z_n)} S^{\mathbf{g}}_{\mathbf{g}(z_1,\dots,z_n)} \sigma_{\mathbf{g}}) S^{\mathbf{f}}_{\mathbf{f}(z_1,\dots,z_n)} \sigma_{\mathbf{f}} \\ &= \mathbf{f}.(X_{(z_1,\dots,z_n)}) S^{\mathbf{f}}_{\mathbf{f}(z_1,\dots,z_n)} \sigma_{\mathbf{f}} \\ &= X_{z_1,\dots,z_n}. \end{aligned}$$

So if (13.2) holds for $\mathbf{f}$ and for $\mathbf{g}$ it holds for $\mathbf{fg}$. But we know that (13.2) holds for $\mathbf{f}_n$, by Theorem 11.1, and for $\mathbf{w}_1$, by Lemma 13.1, and these elements generate $\mathcal{S}_n$. So (13.2) holds for all $\mathbf{f} \in \mathcal{S}_n$. □

14. Expansions of $X_{z_1,z_2,0}$

We have seen that the expansion of $X_{z_1,\dots,z_n}$ (in the region $|z_1| > |z_2| > \cdots > |z_n|$) is expressed as a composition of 1-variable vertex operators. In particular, for $n = 3$ we get, if $A = a \otimes b \otimes c$,

$$i_{z_1;z_2} X_{z_1,z_2,0}(A) = Y(a, z_1) Y(b, z_2) c,$$

see (11.7). In this section we find other expansions of $X_{z_1,z_2,0}$ that have useful expressions in terms of Y_z.

First we need a variant of the analytic continuation Theorem 11.1.

Lemma 14.1.

$$X_{z_1,z_2}(1 \otimes X_{w,0} i_{z_2;w} S^{(z_2)}_{w,0}) = i_{z_1-z_2,z_2;w} X_{z_1,z_2+w,z_2}.$$

Proof.

$$\begin{aligned} &i_{z_1;z_2} X_{z_1,z_2}(1 \otimes X_{w,0} i_{z_2;w} S^{(z_2)}_{w,0}) \\ &\quad= X_{z_1,0}(1 \otimes e^{z_2 D} X_{w,0} i_{z_2;w} S^{(z_2)}_{w,0}) && \text{by Lemma 6.2} \\ &\quad= X_{z_1,0}(1 \otimes i_{z_2;w} X_{w+z_2,z_2}) && \text{by Axiom (4.5)} \\ &\quad= i_{z_2;w} i_{z_1;w+z_2,z_2} X_{z_1,z_2+w,z_2} && \text{by Theorem 11.1} \\ &\quad= i_{z_1;z_2} i_{z_1-z_2,z_2;w} X_{z_1,z_2+w,z_2}, \end{aligned}$$

since

$$i_{z_1;z_2} i_{z_1-z_2;w} f(z_1 - z_2 - w) = i_{z_2;w} i_{z_1;w+z_2} f(z_1 - z_2 - w).$$

The lemma follows then by canceling $i_{z_1;z_2}$. □

Next we need a variant of the compatibility with multiplication Axiom (4.8).

Lemma 14.2.

$$S^{(\tau)}_{z_1,z_2}(1\otimes X_{w,0}S^{(\gamma)}_{w,0}) = (1\otimes X_{w,0}S^{(\gamma)}_{w,0})i_{z_1-z_2,z_2;w}S^{(\tau)12}_{z_1;z_2+w}S^{(\tau)13}_{z_1,z_2}.$$

Proof. We need some simple identities. By Axiom (4.5)

$$X_{w,0}S^{(\gamma)}_{w,0} = e^{-\gamma D}X_{w+\gamma,\gamma}. \tag{14.1}$$

By Axiom (4.4)

$$S^{(\tau)}_{z_1,z_2}(1\otimes e^{-\gamma D}) = (1\otimes e^{-\gamma D})i_{z_1-z_2,z_2;\gamma}S^{(\tau)}_{z_1,z_2-\gamma}. \tag{14.2}$$

Finally, by Axiom (4.8)

$$S^{(\tau)}_{z_1,z_2}(1\otimes X_{w,0}) = (1\otimes X_{w,0})i_{z_1-z_2,z_2;w}S^{(\tau)12}_{z_1,z_2+w}S^{(\tau)13}_{z_1,z_2}. \tag{14.3}$$

Then

$$\begin{aligned}
&S^{(\tau)}_{z_1,z_2}(1\otimes X_{w,0}S^{(\gamma)}_{w,0})\\
&\quad = S^{(\tau)}_{z_1,z_2}(1\otimes e^{-\gamma D}X_{w+\gamma,\gamma}) && \text{by (14.1)}\\
&\quad = (1\otimes e^{-\gamma D})i_{z_1-z_2,z_2;\gamma}S^{(\tau)}_{z_1,z_2-\gamma}(1\otimes X_{w+\gamma,\gamma}) && \text{by (14.2)}\\
&\quad = (1\otimes e^{-\gamma D})(1\otimes X_{w+\gamma,\gamma})\\
&\qquad \times i_{z_1-z_2,z_2;\gamma}i_{z_1-z_2-\gamma,z_2-\gamma;w+\gamma,\gamma}S^{(\tau)12}_{z_1,z_2-\gamma+(w-\gamma)}S^{(\tau)13}_{z_1,(z_2-\gamma)+\gamma} && \text{by (14.3)}\\
&\quad = (1\otimes X_{w,0}S^{(\gamma)}_{w,0})\\
&\qquad \times i_{z_1-z_2,z_2;\gamma}i_{z_1-z_2-\gamma,z_2-\gamma;w+\gamma,\gamma}S^{(\tau)12}_{z_1,z_2-\gamma+(w-\gamma)}S^{(\tau)13}_{z_1,(z_2-\gamma)+\gamma} && \text{by (14.1)}\\
&\quad = (1\otimes X_{w,0}S^{(\gamma)}_{w,0})i_{z_1-z_2,z_2;w}S^{(\tau)12}_{z_1,z_2+w}S^{(\tau)13}_{z_1,z_2},
\end{aligned}$$

since

$$\begin{aligned}
i_{z_1-z_2-\gamma,z_2-\gamma;w+\gamma,\gamma}f((z-\gamma)+(w+\gamma)) &= i_{z_1-z_2,z_2;w}f(z+w)\\
i_{z_1-z_2-\gamma,z_2-\gamma;\gamma}f((z-\gamma)+\gamma) &= f(z).
\end{aligned}$$

□

Remark 14.1. Note that in Lemma 14.2 we establish the equality of two complicated expressions that depend on γ only via the powers $(w+\gamma)^n$. In particular we can take $\gamma = z_2$, and the equalities will still hold, although the proof of Lemma 14.2 breaks down in that case, as $S^{(\tau)}_{z_1,0}$ need not be defined.

Proposition 14.1. *Let V be an H_D-quantum vertex algebra, and $A = a\otimes b\otimes c \in V^{\otimes 3}$. Then we have the following expansions*:

$$i_{z_1;z_2}X_{z_1,z_2,0}(A) = Y(a,z_1)Y(b,z_2)c, \tag{14.4}$$

$$i_{z_2;z_1}X_{z_1,z_2,0}(A) = Y_{z_2}(1\otimes Y_{z_1})i_{z_2;z_1}S^{(\tau),12}_{z_2,z_1}(b\otimes a\otimes c), \tag{14.5}$$

$$i_{z_2;z_3}X_{z_2+z_3,z_2,0}(A) = Y_{z_2}(Y_{z_3}\otimes 1)i_{z_2;z_3}S^{(z_2),12}_{z_3,0}(a\otimes b\otimes c). \tag{14.6}$$

Proof. (14.4) is (11.7) for $n = 3$ and $z_3 = 0$. By Corollary 13.1 (for $n = 3$ and $f = \mathbf{w}_1$) we have

$$X_{z_1,z_2,0}(A) = X_{z_2,z_1,0}(S^{(\tau)}_{z_2,z_1}(b \otimes a) \otimes c).$$

Expanding this equation by applying $i_{z_2;z_1}$ and using (14.4) and definition (7.1) gives (14.5).

For the last part, let $\mathbf{f} = (132) = \mathbf{w}_2\mathbf{w}_1$, so that $\sigma_{\mathbf{f}}(a \otimes b \otimes c) = c \otimes a \otimes b$. Then

$$\begin{aligned} S(\mathbf{f}) &= S^{\mathbf{f}}_{z_1,z_2,z_3}\sigma_{\mathbf{f}} = S^{\mathbf{w}_1}_{z_1,z_2,z_3}\tau_1 S^{\mathbf{w}_2}_{\mathbf{w}_1(z_1,z_2,z_3)}\tau_1 \\ &= S^{(\tau)12}_{z_1,z_2}S^{(\tau)13}_{z_1,z_3}\sigma_{\mathbf{f}}. \end{aligned}$$

Therefore

$$S^{\mathbf{f}}_{\mathbf{f}(z_2+w,z_2,z_1)} = S^{\mathbf{f}}_{z+1,z_2+w,z_2} = S^{(\tau)12}_{z_1,z_2+w}S^{(\tau)13}_{z_1,z_2}. \tag{14.7}$$

Let $E = X_{z_2,z_1}(X_{z_3,0}i_{z_2;z_3}S^{(z_2)}_{z_3,0} \otimes 1)(A)$. Then

$$\begin{aligned} E &= X_{z_1,z_2}S^{(\tau)}_{z_1,z_2}\tau(X_{z_3,0}i_{z_2;z_3}S^{(z_2)}_{z_3,0} \otimes 1)(A) && \text{by Lemma 9.1} \\ &= X_{z_1,z_2}S^{(\tau)}_{z_1,z_2}(1 \otimes X_{z_3,0}i_{z_2;z_3}S^{(z_2)}_{z_3,0})(c \otimes a \otimes b) \\ &= X_{z_1,z_2}(1 \otimes X_{z_3,0}i_{z_2;z_3}S^{(z_2)}_{z_3,0})i_{z_1-z_2,z_2;z_3}S^{(\tau)12}_{z_1,z_2+z_3}S^{(\tau)13}_{z_1,z_2}\sigma_{\mathbf{f}}(A) && \text{by Lemma 14.2} \\ & && \text{and Remark 14.1} \\ &= i_{z_1-z_2,z_2;z_3}(X_{z_1,z_2+z_3,z_2}S^{(\tau)12}_{z_1,z_2+z_3}S^{(\tau)13}_{z_1,z_2})\sigma_{\mathbf{f}}(A) && \text{by Theorem 11.1} \\ &= i_{z_1-z_2,z_2;z_3}(X_{z_1,z_2+z_3,z_2}S^{\mathbf{f}}_{\mathbf{f}(z_2+z_3,z_2,z_1)})\sigma_{\mathbf{f}}(A) && \text{by (14.7)} \\ &= i_{z_1-z_2,z_2;z_3}X_{z_2+z_3,z_2,z_1}(A) && \text{by Theorem 11.1} \end{aligned}$$

Putting $z_1 = 0$ proves then (14.6). □

15. The Braided Jacobi Identity

In one approach to the usual vertex algebras the Jacobi identity for vertex operators is the basic identity, see e.g., [15]. In this section we derive the braided analog in our context of H_D-quantum vertex algebras.

Introduce some more notation. If $f(z_1, z_2) \in \mathbb{C}[[z_1, z_2]][z_1^{-1}, z_2^{-1}, (z_1 - z_2)^{-1}]$ define the difference of expansions of f as

$$\delta(f(z_1, z_2)) = (i_{z_1;z_2} - i_{z_2;z_1})(f(z_1, z_2)). \tag{15.1}$$

For instance,

$$\delta\left(\frac{1}{z_1 - z_2}\right) = \delta(z_1, z_2) = \sum_{n\in\mathbb{Z}} z_1^n z_2^{-n-1}. \tag{15.2}$$

This is the usual *Dirac Delta Distribution*.

Recall that in this paper we are always expanding all expressions in positive powers of t. For instance, if we write $\frac{1}{z_1 - tz_2}$ we mean $\sum_{n\geq 0}(tz_2)^n/z_1^{n+1}$. Thus we have, for instance,

$$\delta\left(\frac{1}{z_1 - tz_2}\right) = 0. \tag{15.3}$$

Lemma 15.1. *For all* $f(z_1, z_2, z_3) \in \mathbb{C}[[z_1, z_2, z_3]][z_1^{-1}, z_2^{-1}, z_3^{-1}]$ *we have*

$$\begin{aligned} &i_{z_1;z_2}(\delta(z_1 - z_2, z_3)f(z_1, z_2, z_1 - z_2)) - i_{z_2;z_1}(\delta(z_1 - z_2, z_3)f(z_1, z_2, z_1 - z_2)) \\ &\quad = i_{z_2;z_3}(\delta(z_1, z_2 + z_3)f(z_2 + z_3, z_2, z_3)). \end{aligned}$$

Proof. See for example [15, Proposition 2.3.26]. □

Definition 15.1. We will write $a(z)$ for the 1-variable vertex operator $Y(a, z)$, the field associated to $a \in V$.

Theorem 15.1 (Braided Jacobi Identity). *Let V be an H_D-quantum vertex algebra. For all $a, b, c \in V$ we have the identity*:

$$\begin{aligned} &i_{z_1;z_2}\delta(z_1 - z_2, z_3)a(z_1)b(z_2)c - i_{z_2;z_1}\delta(z_1 - z_2, z_3)Y_{z_2}(1 \otimes Y_{z_1})S^{(\tau),12}_{z_2,z_1}(b \otimes a \otimes c) \\ &\quad = i_{z_2;z_3}\delta(z_1, z_2 + z_3)Y_{z_2}(Y_{z_3} \otimes 1)S^{(z_2),12}_{z_3,0}(a \otimes b \otimes c). \end{aligned}$$

Proof. Let V^* be the dual of V, fix $v^* \in V^*$ and let $\langle\, , \rangle$ be the pairing $V^* \otimes V \to k = \mathbb{C}[[t]]$. Then for all $A = a \otimes b \otimes c \in V^{\otimes 3}$ we have

$$\langle v^*, X_{z_1,z_2,0}(A)\rangle = \sum_{p\geq 0}\sum_{l,m,n\in\mathbb{Z}} \frac{g_{l,m,n,p}(z_1, z_2)}{(z_1 - z_2)^l z_1^m z_2^n} t^p,$$

for $g_{l,m,n,p}(z_1, z_2) \in \mathbb{C}[[z_1, z_2]]$. (The sum over l, m, n is finite, for each p.) Define then

$$F(z_1, z_2, z_3) = \sum_{p\geq 0}\sum_{l,m,n\in\mathbb{Z}} \frac{g_{l,m,n,p}(z_1, z_2)}{z_3^l z_1^m z_2^n} t^p \in \mathbb{C}[[z_1, z_2, z_3]][z_1^{-1}, z_2^{-1}, z_3^{-1}][[t]].$$

Then we have by Corollary 14.1

$$\begin{aligned} i_{z_1;z_2}F(z_1, z_2, z_1 - z_2) &= \langle v^*, a(z_1)b(z_2)c\rangle, \\ i_{z_2;z_1}F(z_1, z_2, z_1 - z_2) &= \langle v^*, Y_{z_2}(1 \otimes Y_{z_1})i_{z_2;z_1}S^{(\tau),12}_{z_2,z_1}(b \otimes a \otimes c)\rangle, \\ i_{z_2;z_3}F(z_2 + z_3, z_2, z_3) &= \langle v^*, Y_{z_2}(Y_{z_3} \otimes 1)i_{z_2;z_3}S^{(z_2),12}_{z_3,0}(a \otimes b \otimes c)\rangle. \end{aligned}$$

Then we get from Lemma 15.1 that

$$\begin{aligned} &\langle v^*, i_{z_1;z_2}\delta(z_1 - z_2, z_3)a(z_1)b(z_2)c\rangle \\ &\qquad - \langle v^*, i_{z_2;z_1}\delta(z_1 - z_2, z_3)Y_{z_2}(1 \otimes Y_{z_1})S^{(\tau),12}_{z_2,z_1}(b \otimes a \otimes c)\rangle \\ &\quad = \langle v^*, i_{z_2;z_3}\delta(z_1, z_2 + z_3)Y_{z_2}(Y_{z_3} \otimes 1)i_{z_2,z_3}S^{(z_2),12}_{z_3,0}(a \otimes b \otimes c)\rangle. \end{aligned}$$

Since this is true for all $v^* \in V^*$ the theorem follows. □

Remark 15.1. Suppose V is an H_D-quantum vertex algebra where $S^{(\tau)}_{z_1,z_2}$ and $S^{(\gamma)}_{z_1,z_2}$ both are the identity map on $V \otimes V$. Then the fields $a(z) = Y(a,z)$ satisfy the usual Jacobi identity:

$$\begin{aligned} &i_{z_1;z_2}\delta(z_1 - z_2, z_3)a(z_1)b(z_2) - i_{z_2;z_1}\delta(z_1 - z_2, z_3)b(z_2)a(z_1) \\ &\quad = i_{z_2;z_3}\delta(z_1, z_2 + z_3)Y(Y(a,z_3)b, z_2), \end{aligned}$$

and it follows that V is an ordinary vertex algebra, cf., [15].

Remark 15.2. In the usual Jacoby identity the coefficient of a monomial of the three variables z_1, z_2 and z_3 is in the vector space V. For a quantum vertex algebra that is no longer the case: we first expand in powers of the parameter t, and *only then* collect monomials in front of each power of t. An explicit example is shown in Sec. 26.

16. Braided Borcherds Identity

The original definition by Borcherds of vertex algebras was given in [2]. He took as starting point what later was called the Borcherds identity, instead of the Jacobi identity, cf. [9]. In this section we derive a braided version of the Borcherds Identity.

The following lemma is easy to check and well known (at least for $t = 0$).

Lemma 16.1. *Let W be a topologically free k-module and $f(z,w) \in W[[z_1,z_2]][z_1^{-1}, z_2^{-1}, (z_1 - z_2)^{-1}]$. Then*

$$\mathrm{Res}_{z_1}(\delta(f(z_1,z_2))) = \mathrm{Res}_{z_3}(i_{z_2;z_3}f(z_2 + z_3, z_2)).$$

Theorem 16.1 (Braided Borcherds Identity). *Let V be an H_D-quantum vertex algebra. Let $F \in \mathbb{C}[[z,w]][z^{-1}, w^{-1}, (z-w)^{-1}][[t]]$ and $a,b,c \in V$. Then we have the following identity*:

$$\begin{aligned} &\mathrm{Res}_{z_1}(Y(a,z_1)Y(b,z_2)c\, i_{z_1;z_2}F(z_1,z_2) \\ &\qquad - Y_{z_2}(1 \otimes Y_{z_1})i_{z_2;z_1}S^{(\tau),12}_{z_2,z_1}(b \otimes a \otimes c)F(z_1,z_2)) \\ &\quad = \mathrm{Res}_{z_3}(Y_{z_2}(Y_{z_3} \otimes 1)\, i_{z_2;z_3}(S^{(z_2),12}_{z_3,0}(a \otimes b \otimes c)F(z_2 + z_3, z_2))). \end{aligned}$$

Proof. Take in Lemma 16.1 $f(z_1,z_2) = X_{z_1,z_2,0}(a \otimes b \otimes c)F(z_1,z_2)$ and use Corollary 14.1 to relate expansions of f to products and iterates of one-variable vertex operators. □

Remark 16.1. In the case of the usual vertex algebras the Borcherds identity is often written in modes, by collecting the coefficients of the powers in z_2. As in Remark 15.2 we note that for a quantum vertex algebra we first expand in powers of the parameter t, and *only then* collect the powers of z_2 in the coefficients of each power of t. An explicit example is shown in Sec. 26.

17. The S-Commutator, Locality, and (n)-Products of Fields

Definition 17.1. Let V be an H_D-quantum vertex algebra, and let $a, b, c \in V$. The S-*commutator* of the fields associated to a, b is

$$[a(z_1), b(z_2)]_S c = \delta(X_{z_1,z_2,0}(a \otimes b \otimes c)).$$

Here δ is the difference of expansions, see (15.1). We can write the S-commutator using Corollary 14.1 explicitly as

$$[a(z_1), b(z_2)]_S c = a(z_1)b(z_2) - Y_{z_2}(1 \otimes Y_{z_1}) i_{z_2;z_1} S^{(\tau),12}_{z_2,z_1}(b \otimes a \otimes c).$$

Now the image of δ is a power series in t with coefficients (finite) sums of derivatives of the Dirac distribution (15.2) with coefficients V-valued distributions in z_2. So we can write the commutator as

$$\begin{aligned} [a(z_1), b(z_2)]_S &= \sum_{k>0} t^k \left(\sum_n \gamma_{n,k}(z_2) \partial^{(n)}_{z_2} \delta(z_1, z_2) \right) \\ &= \sum_{n \geq 0} \gamma_n(z_2; t) \partial^{(n)}_{z_2} \delta(z_1, z_2). \end{aligned} \tag{17.1}$$

This implies that for all $k \geq 0$ there is an $N > 0$ such that

$$(z_1 - z_2)^N [a(z_1), b(z_2)]_S \equiv 0 \mod t^k, \tag{17.2}$$

and we see that the S-commutator of $a, b \in V$ is a local distribution $\mod t^k$, see [9]. (The S-commutator is of course not necessarily itself local. There might be no uniform N such that $(z_1 - z_2)^N [a(z_1), b(z_2)]_S = 0$.)

Definition 17.2. For all $n \in \mathbb{Z}$ the (n)-product of fields associated to $a, b \in V$ is

$$a(z_2)_{(n)} b(z_2) c = \mathrm{Res}_{z_1}(\delta(X_{z_1,z_2,0}(a \otimes b \otimes c)(z_1 - z_2)^n)).$$

This definition allows us to write the S-commutator in terms of the (n)-product of fields, for $n \geq 0$.

Theorem 17.1. *Let V be an H_D-quantum vertex algebra. For all $a, b \in V$*

$$[a(z_1), b(z_2)]_S = \sum_{n \geq 0} a(z_2)_{(n)} b(z_2) \partial^{(n)}_{z_2} \delta(z_1, z_2).$$

Proof. By the usual calculus of local distributions, see e.g. [9], it follows from (17.1) that

$$\begin{aligned} \gamma_n(z_1; t) &= \mathrm{Res}_{z_2}([a(z_1), b(z_2)]_S (z_1 - z_2)^n) \\ &= \mathrm{Res}_{z_2}(\delta(X_{z_1,z_2,0}(a \otimes b \otimes -)(z_1 - z_2)^n)), \end{aligned}$$

by Definition 17.1. Then the lemma follows from Definition 17.2. ◻

18. (n)-Products of States

We will call an element of V also a *state*. We define the (n)-product of states (as opposed to that of fields) in V in the usual way:

$$a_{(n)}b = \mathrm{Res}_z(Y(a,z)bz^n),$$

so that

$$Y(a,z) = \sum_{n\in\mathbb{Z}} a_{(n)}z^{-n-1}. \tag{18.1}$$

We also have

$$a_{(n)}b = 0, \quad n \gg 0. \tag{18.2}$$

In contrast to the usual vertex algebras the state-field correspondence $a \mapsto a(z)$ is not quite a homomorphism of the corresponding (n)-products: in general

$$a_{(n)}b(z) \neq a(z)_{(n)}b(z).$$

Indeed, introduce the generating series $\mathcal{Y}_{\mathcal{F}}$ of the (n)-products of fields by

$$\mathcal{Y}_{\mathcal{F}}(a(z),w) = \sum_{n\in\mathbb{Z}} a(z)_{(n)}w^{-n-1}.$$

Then, if the state-field correspondence were a homomorphism we would have

$$\mathcal{Y}_{\mathcal{F}}(a(z),w)b(z) = Y(Y(a,w)b,z). \tag{18.3}$$

But this in general not true: the translation map $S^{(\gamma)}_{z_1,z_2}$ is the obstruction to (18.3) being true. More precisely we have the following theorem.

Theorem 18.1.

$$\mathcal{Y}_{\mathcal{F}}(a(z),w)b(z)c = Y_z(Y_w \otimes 1)i_{z;w}S^{(z)}_{w,0}(a\otimes b)\otimes c.$$

Proof. By definition of the (n)-product of fields, Lemma 16.1 and Proposition 14.1 we have

$$\begin{aligned}\mathcal{Y}_{\mathcal{F}}(a(z),w)b(z)c &= \mathrm{Res}_{z_1}(\delta(X_{z_1,z_2,0}(a\otimes b\otimes c)(z_1-z)^n)w^{-n-1})\\ &= \mathrm{Res}_{z_3}(i_{z;z_3}(X_{z+z_3,z_2,0}(a\otimes b\otimes c)\delta(z_3,w))\\ &= i_{z;w}X_{z+w,z,0}(a\otimes b\otimes c)\\ &= Y_z(Y_w\otimes 1)i_{z;w}S^{(z)}_{w,0}(a\otimes b)\otimes c. \qquad \square\end{aligned}$$

Suppose that the translation map $S^{(z_3)}_{z_2,0}$ is such that there exists $N \in \mathbb{Z}$ such that for all $a, b \in V$

$$i_{z_2;z_3}S^{(z_2)}_{z_3,0}(a\otimes b) = \sum_{k\geq -N}\left(\sum_i a_{i,k}\otimes b_{i,k}\right)s_k(z_2)z_3^k, \quad s_k(z_2)\in\mathbb{C}((z_2)), \tag{18.4}$$

where for fixed k the summation over i is finite.

Note that in a general H_D-quantum vertex algebra such expansion need not exist. In the main example (see Sec. 26) this condition *is* satisfied, however.

Corollary 18.1. *Assume that* (18.4) *holds in* V. *Then for all* $a, b \in V$ *and* $n \in \mathbb{Z}$

$$a(z)_{(n)}b(z) = \sum_{k\geq -N}\left(\sum_i Y((a_{i,k})_{(k+n)}b_{i,k}, z)\right)s_k(z).$$

Proof. This is the case $F = (z_1 - z_2)^n$ of the braided Borcherds identity, Theorem 16.1. Indeed, in this case the left-hand side is just the (n)-product of the fields $a(z_2)$ and $b(z_2)$ acting on c, see Definition 17.2 and Corollary 14.1. On the other hand the right-hand side of the braided Borcherds identity is in this case

$$\begin{aligned}
&\mathrm{Res}_{z_3}\left(Y_{z_2}(Y_{z_3}\otimes 1)\sum_{k\geq -N}\left(\sum_i a_{i,k}\otimes b_{i,k}\otimes c\right)s_k(z_2)z_3^{k+n}\right)\\
&\quad = \mathrm{Res}_{z_3}\left(Y\left(\sum_{k\geq -N}\left(\sum_i (a_{i,k})_{(m)}b_{i,k}\right)z_3^{-m-1}, z_2\right)cs_k(z_2)z_3^{k+n}\right)\\
&\quad = \sum_{k\geq -N}\left(\sum_i Y((a_{i,k})_{(k+n)}b_{i,k}, z_2)c\right)s_k(z_2).
\end{aligned}$$

The proof is concluded by the substitution $z_2 \mapsto z$. □

19. Normal Ordered Products and Operator Product Expansion

We have used the (n)-product (of fields) for $n \geq 0$ to calculate the S-commutator, see Theorem 17.1. The (n)-products for $n \leq -1$ are also of course important.

Definition 19.1. The normal ordered product of fields $a(z_1)$ and $b(z_2)$ is given by

$$:a(z_1)b(z_2):_S c = \mathrm{Res}_z\left(\delta\left(X_{z,z_2,0}(a\otimes b\otimes c)\frac{1}{z-z_1}\right)\right).$$

We introduce projections on singular and holomorphic parts of a formal distribution as usual by

$$\mathrm{Sing}_{z_1}(f(z_1, z_2, \ldots)) = -\mathrm{Res}_z\left(f(z, z_2, \ldots)i_{z_1;z}\frac{1}{z-z_1}\right),$$

$$\mathrm{Hol}_{z_1}(f(z_1, z_2, \ldots)) = \mathrm{Res}_z\left(f(z, z_2, \ldots)i_{z;z_1}\frac{1}{z-z_1}\right).$$

In particular, if f does not depend on $z_2, \ldots$ we write

$$f_{\mathrm{Sing}}(z_1) = \mathrm{Sing}_{z_1}(f(z_1)), \quad f_{\mathrm{Hol}}(z_1) = \mathrm{Hol}_{z_1}(a(z_1)).$$

Then we can rewrite the definition of the normal ordered product as

$$:a(z_1)b(z_2):_S = a_{\mathrm{Hol}}(z_1)b(z_2) + \mathrm{Sing}_{z_1}(Y_{z_2}(1\otimes Y_{z_1})i_{z_2;z_1}S^{(\tau)}_{z_2,z_1}(b\otimes a)).$$

Comparing this with Definition 17.2 we see that

$$a(z_2)_{(-1)}b(z_2) = :a(z_2)b(z_2):_S,$$

and more generally

$$a(z_2)_{(-n-1)}b(z_2) = :\partial_{z_2}^{(n)}a(z_2)b(z_2):_S\ .$$

This gives the *Operator Product Expansion* of fields $a(z_1)$, $b(z_2)$:

$$\begin{aligned} a(z_1)b(z_2) &= :a(z_1)b(z_2):_S + \mathrm{Sing}_{z_1}([a(z_1), b(z_2)]_S) \\ &= :a(z_1)b(z_2):_S + \sum_{n\geq 0} a(z_2)_{(n)}b(z_2) i_{z_1;z_2}\left(\frac{1}{(z_1-z_2)^{n+1}}\right). \end{aligned}$$

Of course, using Corollary 18.1 we can express the operator product expansion in terms of the (n)-product of states, but this seems rather messy.

20. Weak Associativity

Two basic ingredients in the usual theory of vertex algebras are locality and associativity. For H_D-quantum vertex algebras the analog of locality is S-locality, (17.2). In this section we derive the analog of associativity. It involves the translation map $S^{(\gamma)}_{z_1,z_2}$.

Theorem 20.1 (Weak Associativity). *Let V be an H_D-quantum vertex algebra. For all $a, b, c \in V$ and for all powers t^k there is an $N \geq 0$ such that*

$$\begin{aligned} &(z_2+z_3)^N i_{z_3;z_2} a(z_2+z_3)b(z_2)c \\ &\qquad \equiv (z_2+z_3)^N Y_{z_2}(Y_{z_3}\otimes 1) i_{z_2;z_3}(S^{(z_2)}_{z_3,0}(a\otimes b)\otimes c) \mod t^k. \end{aligned}$$

Proof. Take Res_{z_1} in the braided Jacobi identity of Theorem 15.1 to find

$$\begin{aligned} &i_{z_3;z_2}a(z_2+z_3)b(z_2)c - Y_{z_2}(Y_{z_3}\otimes 1)(i_{z_2;z_3}S^{(z_2)}_{z_3,0}(a\otimes b)\otimes c) \\ &\quad = -\mathrm{Res}_{z_1}(i_{z_2;z_1}\delta(z_1-z_2,z_3)Y_{z_2}(1\otimes Y_{z_1})(S^{(\tau)}_{z_2,z_1}(b\otimes a)\otimes c)) \\ &\quad = -\mathrm{Res}_{z_1}\left(\sum_{k=0}^{\infty}(-z_1)^k\partial_{z_2}^{(k)}\delta(-z_2,z_3)Y_{z_2}(1\otimes Y_{z_1})i_{z_2;z_1}(S^{(\tau)}_{z_2,z_1}(b\otimes a)\otimes c)\right). \end{aligned}$$

Expanding the right-hand side observe that the coefficient of each power of t is after taking the residue a finite sum of z_2 derivatives of $\delta(-z_2, z_3)$, hence vanishes if multiplied by a suitable power of $z_2 + z_3$. □

Remark 20.1. For ordinary vertex algebras the power of N in weak associativity depends only on a and c, not on b. The above proof in the case of H_D-quantum vertex algebras does not allow us to conclude the same, because of the appearance of the braiding $S^{(\tau)}_{z_2,z_1}(b\otimes a)$.

21. The H_D-Bialgebra V

In the rest of the paper we will construct a class of examples of H_D-quantum vertex algebras, using bicharacters on the underlying space V. To define bicharacters we need to assume that V has extra structure: we will assume that V is a commutative and cocommutative k-bialgebra, or even a Hopf algebra. The coproduct and counit of V will be denoted by Δ and ϵ. We assume also that V has a compatible H_D-action. This means that

- $D(ab) = (Da)b + aDb$, $a, b \in V$.
- $\Delta(Da) = \Delta_{H_D}(D)\Delta(a)$, $a \in V$.
- $\epsilon(Da) = \epsilon_{H_D}(D)\epsilon(a) = 0$.

We will call a V as above an H_D-bialgebra. The identity element $1 = 1_V$ will be the vacuum of V.

22. Bicharacters

Let W_2 be the algebra of power series in t, with coefficients rational functions in z_1, z_2 with poles at $z_1 = 0$, $z_2 = 0$ or $z_1 = z_2$:

$$W_2 = k[z_1^{\pm 1}, z_2^{\pm 1}, (z_1 - z_2)^{\pm 1}] = \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, (z_1 - z_2)^{\pm 1}][[t]]. \tag{22.1}$$

We extract some results from [4] on *bicharacters*. A W_2-valued bicharacter on an H_D-bialgebra V is a linear map

$$r_{z_1,z_2} \colon V^{\otimes 2} \to W_2,$$

satisfying

- (**Vacuum**) $r_{z_1,z_2}(a \otimes 1) = r_{z_1,z_2}(1 \otimes a) = \epsilon(a)$, $a \in V$.
- (**Multiplication**) For all $a, b, c \in V$ we have $r_{z_1,z_2}(a \otimes bc) = \sum r_{z_1,z_2}(a' \otimes b)r_{z_1,z_2}(a'' \otimes c)$ and $r_{z_1,z_2}(ab \otimes c) = \sum r_{z_1,z_2}(a \otimes c')r_{z_1,z_2}(b \otimes c'')$.

Here and below we use the notation $\Delta(a) = \sum a' \otimes a''$ for the coproduct of $a \in V$. Often we will also omit the summation symbol, to unclutter the formulas.

In case the bicharacter additionally satisfies

- (***$H_D \otimes H_D$*-covariance**) $r_{z_1,z_2}(D^k a \otimes D^\ell b) = \partial_{z_1}^k \partial_{z_2}^\ell r_{z_1,z_2}(a \otimes b)$, $a, b \in V$,

we call the bicharacter $H_D \otimes H_D$-*covariant*.

We can multiply bicharacters:

$$(r * s)_{z_1,z_2}(a \otimes b) = r_{z_1,z_2}(a' \otimes b')s_{z_1,z_2}(a'' \otimes b''). \tag{22.2}$$

The unit bicharacter is

$$\epsilon_{z_1,z_2}(a \otimes b) = \epsilon(a)\epsilon(b). \tag{22.3}$$

The collection of bicharacters on an H_D-bialgebra forms then a commutative monoid.

In case V is an H_D-Hopf algebra, i.e. comes with an antipode compatible with the H_D-action, all bicharacters are invertible, with inverse given by

$$r^{-1}_{z_1,z_2}(a \otimes b) = r_{z_1,z_2}(S(a) \otimes b).$$

In this case the set of bicharacters forms an Abelian group.

The transpose of a bicharacter is defined by

$$r^{\tau}_{z_1,z_2}(a \otimes b) = r_{z_2,z_1}(b \otimes a).$$

The transpose is an involution of the monoid of bicharacters:

$$(r * s)^{\tau}_{z_1,z_2} = (r^{\tau} * s^{\tau})_{z_1,z_2}.$$

If r is an invertible bicharacter with inverse r^{-1} we relate the transpose r^{τ} to r by

$$r^{\tau}_{z_1,z_2} = r_{z_1,z_2} * R_{z_1,z_2}, \tag{22.4}$$

where

$$R_{z_1,z_2} = r^{-1}_{z_1,z_2} * r^{\tau}_{z_1,z_2}. \tag{22.5}$$

We will call R_{z_1,z_2} the *braiding bicharacter* associated to r_{z_1,z_2}. It is the obstruction to r being *symmetric*: $r = r^{\tau}$. It will control the braiding in the quantum vertex algebra we are going to construct from r_{z_1,z_2} in Sec. 23 below. The braiding bicharacter R_{z_1,z_2} is *unitary*:

$$R^{\tau}_{z_1,z_2} = R^{-1}_{z_1,z_2}. \tag{22.6}$$

Define for a bicharacter r_{z_1,z_2} a shift

$$r^{\gamma}_{z_1,z_2} = r_{z_1+\gamma,z_2+\gamma}. \tag{22.7}$$

The shift $r^{\gamma}_{z_1,z_2}$ is again a bicharacter. If r_{z_1,z_2} is $H_D \otimes H_D$-covariant we have the following expansion:

$$i_{z_1,z_2;\gamma} r^{\gamma}_{z_1,z_2} = r_{z_1,z_2} \circ \Delta(e^{\gamma D}).$$

In case the bicharacter is invertible we relate the shift r^{γ} to r by

$$r^{\gamma}_{z_1,z_2} = r_{z_1,z_2} * R^{\gamma}_{z_1,z_2}, \quad R^{\gamma}_{z_1,z_2} = r^{-1}_{z_1,z_2} * r^{\gamma}_{z_1,z_2}. \tag{22.8}$$

We call $R^{\gamma}_{z_1,z_2}$ the *translation bicharacter* associated to r_{z_1,z_2}. It is the obstruction to r being shift invariant (i.e. to r being a function just of $z_1 - z_2$).

23. H_D-Quantum Vertex Algebras from Bicharacters

Suppose now that V is an H_D-bialgebra with invertible bicharacter r_{z_1,z_2}. In general, a bicharacter on V takes values in W_2, see (22.1). For the purpose of the construction of vertex operators we need to make an extra assumption: that r_{z_1,z_2} can be evaluated at $z_2 = 0$. More precisely, we make the following

Definition 23.1. A bicharacter r_{z_1,z_2} satisfies the *Vertex Operator Assumption* if it is a map

$$r_{z_1,z_2} \colon V^{\otimes 2} \to \mathbb{C}[z_1^{\pm 1}, z_2, (z_1 - z_2)^{-1}][[t]]. \tag{VO assumption}$$

In the sequel we will use ρ_{z_1,z_2} to denote an arbitrary W_2-valued bicharacter, and we will write r_{z_1,z_2} for a bicharacter satisfying the VO assumption.

Following the general philosophy of Borcherds, [4], (but not the technical details) we define in this section, given an invertible bicharacter r_{z_1,z_2} satisfying the VO assumption, an H_D-quantum vertex algebra structure on V. The final result is summarized in Theorem 23.1 below.

We define for any bicharacter ρ_{z_1,z_2} on V a map $S^{\rho_{z_1,z_2}}$ on $V\otimes V$ by

$$S^{\rho_{z_1,z_2}}(a\otimes b)=a'\otimes b'\rho_{z_1,z_2}(a''\otimes b''). \tag{23.1}$$

In particular, to a bicharacter r_{z_1,z_2} satisfying the VO assumption with braiding bicharacter R_{z_1,z_2}, see (22.5), we associate the map

$$S^{(\tau)}_{z_1,z_2}=S^{R_{z_1,z_2}}\colon V\otimes V\to V\otimes V[z_1^{\pm1},z_2^{\pm1},(z_1-z_2)^{\pm1}][[t]], \tag{23.2}$$

and associated to the translation bicharacter (22.8) we get a map

$$S^{(\gamma)}_{z_1,z_2}=S^{R^{\gamma}_{z_1,z_2}}\colon V\otimes V\to V\otimes V[z_1^{\pm1},z_2,(z_1+\gamma)^{\pm1},(z_2+\gamma),(z_1-z_2)^{\pm1}][[t]], \tag{23.3}$$

Lemma 23.1. (1) *If ϵ is the unit bicharacter on V, then $S^{\epsilon}=1_{V^{\otimes2}}$.*
(2) *If $\rho_{z_1,z_2},\sigma_{z_1,z_2}$ are bicharacters on V, then $S^{\rho_{z_1,z_2}*\sigma_{z_1,z_2}}=S^{\rho_{z_1,z_2}}\circ S^{\sigma_{z_1,z_2}}$.*
(3) *If ρ_{z_1,z_2} is a bicharacter, then $\tau\circ S^{\rho_{z_1,z_2}}\circ\tau=S^{\rho^{\tau}_{z_1,z_2}}$.*

Define then, for given invertible bicharacter r_{z_1,z_2} satisfying the VO assumption, singular multiplication maps

$$X_{z_1,z_2}\colon V^{\otimes2}\to V\otimes[[z_1,z_2]][z_1^{-1},(z_1-z_2)^{-1}][[t]].$$

by

$$X_{z_1,z_2}=m_2\circ(e^{z_1D}\otimes e^{z_2D})\circ S^{r_{z_1,z_2}}, \tag{23.4}$$

where m_2 is the (nonsingular) multiplications of the (associative) algebra V. More explicitly (dropping here and below the nonsingular multiplication m_2 on V):

$$X_{z_1,z_2}(a\otimes b)=e^{z_1D}a'e^{z_2D}b'r_{z_1,z_2}(a''\otimes b'').$$

Lemma 23.2. *For any bicharacter ρ_{z_1,z_2} on V we have for $a\in V$*

$$S^{\rho_{z_1,z_2}}(a\otimes1)=a\otimes1,\quad S^{\rho_{z_1,z_2}}(1\otimes a)=a\otimes1.$$

Proof. Since ρ_{z_1,z_2} is a bicharacter we have

$$\rho_{z_1,z_2}(a\otimes1)=\epsilon(a)=\rho_{z_1,z_2}(1\otimes a).$$

In any bialgebra we have $a'\epsilon(a'')=a$, and the Lemma follows from the definition of $S^{\rho_{z_1,z_2}}$, see (23.1). □

Corollary 23.1. *The vacuum axioms* (4.1) *and* (4.2) *hold for X_{z_1,z_2} defined by* (23.4) *and for $S^{(\tau)}_{z_1,z_2},S^{(\gamma)}_{z_1,z_2}$ defined by* (23.2) *and* (23.3).

Lemma 23.3. *For any $H_D \otimes H_D$-covariant bicharacter ρ_{z_1,z_2} we have*

$$[S^{\rho_{z_1,z_2}}, 1 \otimes D] = \partial_{z_2} S^{\rho_{z_1,z_2}}, \quad [S^{\rho_{z_1,z_2}}, D \otimes 1] = \partial_{z_1} S^{\rho_{z_1,z_2}}.$$

Proof. By assumption on V we have $\Delta(Db) = Db' \otimes b'' + b' \otimes Db''$. By assumption on the bicharacter we have $\rho_{z_1,z_2}(a \otimes Db) = \partial_{z_2}\rho_{z_1,z_2}(a \otimes b)$. Then, for $a, b \in V$

$$\begin{aligned} S^{\rho_{z_1,z_2}}(a \otimes Db) &= a' \otimes Db' \rho_{z_1,z_2}(a'' \otimes b'') + a' \otimes b' \rho_{z_1,z_2}(a'' \otimes Db'') \\ &= (1 \otimes D) S^{\rho_{z_1,z_2}} + \partial_{z_2} S^{\rho_{z_1,z_2}}(a \otimes b), \end{aligned}$$

proving the first part. The second part is similar. □

Corollary 23.2. *The H_D-covariance axiom* (4.3) *holds for X_{z_1,z_2} defined by* (23.4) *and the H_D-covariance axiom* (4.4) *holds for $S^{(\tau)}_{z_1,z_2}, S^{(\gamma)}_{z_1,z_2}$ defined by* (23.2) *and* (23.3).

Lemma 23.4. *The H_D-covariance axiom* (4.5) *holds for X_{z_1,z_2} defined by* (23.4).

Proof. We have

$$\begin{aligned} X_{z_1+\gamma,z_2+\gamma}(a \otimes b) &= e^{(z_1+\gamma)D} a' e^{(z_2+\gamma)D} b' r_{z_1+\gamma,z_2+\gamma}(a'' \otimes b'') \\ &= e^{\gamma D}(e^{z_1 D} a' e^{(z_2)D} b') r_{z_1,z_2}(a''' \otimes b''') R^{\gamma}_{z_1,z_2}(a'''' \otimes b'''') \\ &= e^{\gamma D} X_{z_1,z_2} \circ S^{(\gamma)}_{z_1,z_2}(a \otimes b). \end{aligned}$$ □

Lemma 23.5. *For any bicharacter ρ_{z_1,z_2} the map $S^{\rho_{z_1,z_2}}$ satisfies the Yang–Baxter equation* (4.6).

Proof. This follows form the combined cocommutativity and coassociativity identity

$$\tau^{23}(\Delta \otimes 1)\Delta = (\Delta \otimes 1)\Delta.$$ □

Corollary 23.3. *The maps $S^{(\tau)}_{z_1,z_2}, S^{(\gamma)}_{z_1,z_2}$ defined by* (23.2) *and* (23.3) *satisfy the Yang–Baxter axiom* (4.6).

Lemma 23.6. *For any bicharacter ρ_{z_1,z_2} the map $S^{\rho_{z_1,z_2}}$ is compatible with the singular multiplication*:

$$\begin{aligned} S^{\rho_{z_1,z_2}}(X_{w_1,w_2} \otimes 1) &= (X_{w_1,w_2} \otimes 1) i_{z_1,z_1-z_2;w_1,w_2} S^{\rho_{z_1+w_1,z_2},23} S^{\rho_{z_1+w_2,z_2},13}, \\ S^{\rho_{z_1,z_2}}(1 \otimes X_{w_1,w_2}) &= (1 \otimes X_{w_1,w_2}) i_{z_1-z_2,z_2;w_1,w_2} S^{\rho_{z_1,z_2+w_1},12} S^{\rho_{z_1,z_2+w_2},13}. \end{aligned}$$

Proof. For $a, b, c \in V$ we have

$$\begin{aligned} &S^{\rho_{z_1,z_2}}(X_{w_1,w_2} \otimes 1)(a \otimes b \otimes c) \\ &\quad = S^{\rho_{z_1,z_2}}(e^{w_1 D} a' e^{w_2 D} b' \otimes c) r_{w_1,w_2}(a'' \otimes b'') \\ &\quad = (e^{w_1 D} a' e^{w_2 D} b')' c' \rho_{z_1,z_2}((e^{w_1 D} a' e^{w_2 D} b')'' \otimes c'') r_{w_1,w_2}(a'' \otimes b'') \end{aligned}$$

$$= e^{w_1 D}a''e^{w_2 D}b''c'i_{z_1,z_1-z_2;w_1,w_2}\rho_{z_1+w_1,z_2}(a'''\otimes c''')\rho_{z_1+w_2,z_2}(b'''\otimes c'''')$$
$$\times r_{w_1,w_2}(a''\otimes b'')$$
$$= i_{z_1,z_1-z_2;w_1,w_2}(X_{w_1,w_2}\otimes 1)S^{\rho_{z_1+w_1,z_2},23}S^{\rho_{z_1+w_2,z_2},13}(a\otimes b\otimes c)$$

The proof of the other part is similar. □

Corollary 23.4. *$S^{(\tau)}_{z_1,z_2}, S^{(\gamma)}_{z_1,z_2}$ defined by* (23.2) *and* (23.3) *satisfy the compatibility with multiplication axioms* (4.7) *and* (4.8).

Corollary 23.5. *$S^{(\tau)}_{z_1,z_2}$ defined by* (23.2) *satisfies the unitarity axiom* (4.9).

Proof. For unitarity, recall that $S^{(\tau)}_{z_1,z_2} = S^{R_{z_1,z_2}}$, so that by Lemma 23.1 we have $\tau\circ S^{(\tau)}_{z_1,z_2}\circ\tau = S^{R^{\tau}_{z_1,z_2}}$ so that by Lemma 23.1 again and (22.6) we find

$$S^{(\tau)}_{z_1,z_2}\circ\tau\circ S^{(\tau)}_{z_1,z_2}\circ\tau = S^{R_{z_1,z_2}}\circ S^{R^{\tau}_{z_1,z_2}} = S^{\epsilon} = 1_{V\otimes V}.$$ □

Corollary 23.6. *$S^{(\gamma)}_{z_1,z_2}$ defined by* (23.3) *satisfies the group axioms* (4.11) *and* (4.10).

Proof. Since $R^{(\gamma=0)}_{z_1,z_2} = \epsilon$, the unit bicharacter, axiom (4.11) follows.

Now

$$\begin{aligned} R^{\gamma_1+\gamma_2}_{z_1,z_2} &= r^{-1}_{z_1,z_2} * r^{\gamma_1+\gamma_2}_{z_1,z_2} \\ &= r^{-1}_{z_1,z_2} * r^{\gamma_1}_{z_1,z_2} * r^{\gamma_1,-1}_{z_1,z_2} * r^{\gamma_1+\gamma_2}_{z_1,z_2} \\ &= R^{\gamma_1}_{z_1,z_2} * R^{\gamma_2}_{z_1+\gamma_1,z_2+\gamma_1}, \end{aligned}$$

so that axiom (4.10) follows from Lemma 23.1 and definition (23.3). □

Lemma 23.7. *X_{z_1,z_2} and $S^{(z_2)}_{w_1,w_2}$ defined by* (23.4) *and* (23.3) *satisfy the locality axiom* (4.12).

Proof. Define

$$E = e^{z_1 D}a'e^{z_2 D}b'c'r_{z_1,z_2}(a'''\otimes b''')r_{z_1,0}(a''''\otimes c''')r_{z_2,0}(b''''\otimes c'''').$$

Then

$$\begin{aligned} &X_{z_1,0}(1\otimes X_{z_2,0})(A)\\ &\quad= e^{z_1 D}a'(e^{z_2 D}b'c')'r_{z_1,0}(a''\otimes(e^{z_2 D}b'c')'')r_{z_2,0}(b''\otimes c'')\\ &\quad= e^{z_1 D}a'e^{z_2 D}b''c''r_{z_1,0}(a'''\otimes e^{z_2 D}b''')r_{z_1,0}(a''''\otimes c''')r_{z_2,0}(b''\otimes c'')\\ &\quad= e^{z_1 D}a'e^{z_2 D}b'c''i_{z_1;z_2}r_{z_1,z_2}(a'''\otimes b''')r_{z_1,0}(a''''\otimes c''')r_{z_2,0}(b''''\otimes c'''')\\ &\quad= i_{z_1;z_2}E. \end{aligned}$$

On the other hand

$$\begin{aligned} &X_{z_2,0}(1\otimes X_{z_1,0})i_{z_2;z_1}S^{(\tau)12}_{z_2,z_1}(b\otimes a\otimes c)\\ &\quad= X_{z_2,0}(1\otimes X_{z_1,0})(b'\otimes a'\otimes c)i_{z_2;z_1}R^{(\tau)}_{z_2,z_1}(b''\otimes a'') \end{aligned}$$

$$
\begin{aligned}
&= X_{z_2,0}(b' \otimes e^{z_1 D}a''c')r_{z_1,0}(a''' \otimes c'')i_{z_2;z_1}R^{(\tau)}_{z_2,z_1}(b'' \otimes a'') \\
&= e^{z_2 D}b''(e^{z_1 D}a''c')'r_{z_2,0}(b''' \otimes (e^{z_1 D}a''c')'') \\
&\quad \times r_{z_1,0}(a''' \otimes c'')i_{z_2;z_1}R^{(\tau)}_{z_2,z_1}(b'' \otimes a'') \\
&= e^{z_2 D}b'e^{z_1 D}a'c'r_{z_2,0}(b'''' \otimes e^{z_1 D}a'''')r_{z_2,0}(b''' \otimes c''') \\
&\quad \times r_{z_1,0}(a'''' \otimes c'''')i_{z_2;z_1}R^{(\tau)}_{z_2,z_1}(b''''' \otimes a''''') \\
&= e^{z_2 D}b'e^{z_1 D}a'c'i_{z_2,z_1}(r_{z_2,z_1}(b'''' \otimes a'''')R^{(\tau)}_{z_2,z_1}(b''''' \otimes a''''')) \\
&\quad \times r_{z_2,0}(b''' \otimes c''')r_{z_1,0}(a'''' \otimes c'''') \\
&= e^{z_2 D}b'e^{z_1 D}a'c'i_{z_2,z_1}(r_{z_1,z_2}(a''' \otimes b'''))r_{z_2,0}(b''' \otimes c''') \\
&\quad \times r_{z_1,0}(a'''' \otimes c'''') \\
&= i_{z_2,z_1}E.
\end{aligned}
$$

Since

$$(z_1 - z_2)^N i_{z_1;z_2}E = (z_1 - z_2)^N i_{z_2;z_1}E$$

the locality axiom (4.12) follows. □

The results in this section are summarized in the following theorem.

Theorem 23.1. *Let V be an H_D-bialgebra with invertible bicharacter r_{z_1,z_2}, satisfying the VO assumption of Definition* 23.1*. Then the singular multiplications X_{z_1,z_2}, X_{z_1,z_2,z_3} and maps $S^{(\tau)}_{z_1,z_2}$, $S^{(\gamma)}_{z_1,z_2}$ defined by* (23.4), (23.2) *and* (23.3) *give V the structure of an H_D-quantum vertex algebra as in Definition* 4.1.

24. Bicharacters and EK-quantum Vertex Operator Algebras

Let V be an H_D-bialgebra with invertible bicharacter, so that we have on V by Theorem 23.1 an H_D-quantum vertex algebra structure. In case the bicharacter satisfies

$$(\partial_{z_1} + \partial_{z_2})r_{z_1,z_2} = 0 \tag{24.1}$$

the bicharacter is really just a function of $z_1 - z_2$: r_{z_1,z_2} takes values in $\mathbb{C}[(z_1 - z_2)^{\pm 1}][[t]]$. In this case the translation bicharacter $R^{\gamma}_{z_1,z_2}$ is the unit bicharacter on V.

In this situation we can evaluate the bicharacter r_{z_1,z_2}, the vertex operator X_{z_1,z_2} and the braiding S_{z_1,z_2} both at $z_1 = 0$ and at $z_2 = 0$.

We have in this case $r_{0,z} = r_{-z,0}$ so that

$$X_{0,z}(a \otimes b) = e^{zD}(e^{-zD}a'b')r_{0,z}(a'' \otimes b'') = e^{zD}Y(a, -z)b.$$

The braided commutativity Lemma 9.1 gives, by putting $z_2 = 0$,

$$Y(a', z)b'R_{z,0}(a'' \otimes b'') = e^{zD}Y(b, -z)a.$$

We emphasize that in general H_D-quantum vertex algebras one does not have a similar braided skew-symmetry, since the braiding $S^{(\tau)}_{z_1,z_2}$ cannot be evaluated at $z_2 = 0$.

The H_D-covariance axiom (4.5) reduces to the familiar formula

$$e^{\gamma D} Y(a,z) e^{-\gamma D} = i_{z;\gamma} Y(a, z+\gamma). \tag{24.2}$$

Infinitesimally this gives another familiar formula: by differentiating with respect to γ we obtain

$$[D, Y(a,z)] = \partial_z Y(a,z). \tag{24.3}$$

Bicharacters satisfying condition (24.1) give rise to quantum vertex operator algebras in the sense of Etingof–Kazhdan [5]. In case the bicharacter satisfies (24.1) and is also symmetric:

$$r^{\tau}_{z_1,z_2} = r_{z_1,z_2},$$

we obtain vertex operators of a vertex algebra as is usually defined (see [6, 9]). This is a special case of a more general result of Borcherds, see [4, Theorem 4.2].

The condition (24.1) is not satisfied in the case we are interested in, see Sec. 26.

25. Bicharacter Expansions and S-Commutator

We continue to assume that V has an H_D-quantum vertex algebra structure via a bicharacter r_{z_1,z_2}, see Theorem 23.1. In this section we show how an expansion of the bicharacter leads to a closed formula for the S-commutator of fields.

Consider the vectorspace $V \otimes W(z)$, where $W(z)$ is some space of functions (or power series) in z. Then we get an action of H_D on this vector space by using the coproduct:

$$D^{(k)}(a \otimes f(z)) = \sum_{p+q=k} D^{(p)} a \otimes \partial_z^{(q)} f(z).$$

Theorem 25.1. *Let $a, b \in V$ and suppose that*

$$\delta(r_{z_1,z_2}(a \otimes b)) = \sum_{k \geq 0} d_k(a \otimes b; t) \partial_{z_2}^{(k)} \delta(z_1, z_2),$$

where $d_k(a \otimes b; t) \in \mathbb{C}[[z_1^{\pm 1}, z_2^{\pm 1}]][[t]]$. Then we have

$$[a(z_1), b(z_2)]_S = \sum_{k \geq 0} d_k(a' \otimes b'; t) \sum_{p+q=k} Y([D^{(p)} a'']b'', z_2) \partial_{z_2}^{(q)} \delta(z_1, z_2).$$

Proof. The right-hand side of the S-commutator of the fields of a and b acting on c is

$$\begin{aligned} &e^{z_1 D} a' (e^{z_2 D} b') c' \delta(r_{z_1,z_2}(a''' \otimes b''')) r_{z_1,0}(a'''' \otimes c'') r_{z_2,0}(b'''' \otimes c''') \\ &\quad = (e^{z_2 D} b') c' \sum_{k \geq 0} d_k(a'' \otimes b''; t) \partial_{z_2}^{(k)} ([e^{z_2 D} a'] r_{z_2,0}(a''' \otimes c'') \delta(z_1, z_2)) \\ &\qquad \times r_{z_2,0}(b''' \otimes c''') \end{aligned}$$

$$= (e^{z_2 D} b')c' \sum_{k\geq 0} d_k(a'' \otimes b''; t) \sum_{p+q+r=k} e^{z_2 D}(D^{(p)} a')$$

$$\times r_{z_2,0}(D^{(q)} a''' \otimes c'') r_{z_2,0}(b''' \otimes c''') \partial_{z_2}^{(r)} \delta(z_1, z_2)$$

$$= \sum_{k\geq 0} \sum_{p+q+r=k} d_k(a' \otimes b'; t)[e^{z_2 D}(D^{(p)} a''') b''' c'] r_{z_2,0}([D^{(q)} a''''] b'' \otimes c'')$$

$$\times \partial_{z_2}^{(r)} \delta(z_1, z_2)$$

$$= \sum_{k\geq 0} \sum_{p+q=k} d_k(a' \otimes b'; t) Y((D^{(p)} a'') b'', z_2) c \partial_{z_2}^{(q)} \delta(z_1, z_2).$$

□

26. The Main Example

For the rest of the paper we will study a particular example of an H_D-quantum vertex algebra V obtained from a bicharacter as in Theorem 23.1. Recall that we work over $k = \mathbb{C}[[t]]$. As a vector space V is the topologically free k-module $V_L[[t]]$, where V_L is the underlying space of the usual lattice vertex algebra based on the rank 1 lattice $\mathbb{Z}$ with pairing $(m, n) \mapsto mn$, cf., [9, Sec. 5.4].

To define a bicharacter on V we need an H_D-bialgebra structure. As H_D-bialgebra V is generated by group-like elements $e^{\alpha}, e^{-\alpha}$, so that

$$\Delta(e^{m\alpha}) = e^{m\alpha} \otimes e^{m\alpha}, \quad \epsilon(e^{m\alpha}) = 1, \quad m \in \mathbb{Z}.$$

If we write $h = (De^{\alpha})e^{-\alpha}$ then h is primitive: we have $\Delta(h) = h \otimes 1 + 1 \otimes h$, $\epsilon(h) = 0$. Then

$$V = \bigoplus_{m\in\mathbb{Z}} V_m, \quad V_m = k[D^n h]_{n\geq 0} \otimes e^{m\alpha}.$$

In fact V is a Hopf algebra, with antipode $S\colon e^{\alpha} \mapsto e^{-\alpha}$. We define in this case a bicharacter on V by putting on generators

$$r_{z_1,z_2}(e^{m\alpha} \otimes e^{n\alpha}) = \sigma^{mn}, \quad \sigma = \frac{z_1 - z_2}{1 - tz_2/z_1}, \tag{26.1}$$

and extend to all of V by using the properties of bicharacters. Here (and below) we will expand any rational expression in t in *positive* powers of t. Note that r_{z_1,z_2} satisfies the VO assumption of Definition 23.1. So by Theorem 23.1 V has an H_D-quantum vertex algebra structure.

The bicharacter r_{z_1,z_2} of this example is implicit in the paper by Jing [8]. By putting $t = 0$ we obtain a bicharacter $r^0_{z_1,z_2}$ which is implicit in the usual construction of a lattice vertex algebra from the lattice $\mathbb{Z}$ with pairing $(m, n) \mapsto mn$.

We will collect for later reference some values of this bicharacter and of its associated braiding and translation bicharacters. First a simple lemma.

Lemma 26.1. *For any bicharacter ρ_{z_1,z_2} on V we have, if $\rho_{z_1,z_2}(e^{m\alpha} \otimes e^{n\alpha}) = \rho^{mn}$,*

$$\rho_{z_1,z_2}(h \otimes e^{m\alpha}) = m\partial_{z_1} \ln(\rho), \quad \rho_{z_1,z_2}(h \otimes h) = \partial_{z_2}\partial_{z_1} \ln(\rho).$$

Lemma 26.2.

$$r_{z_1,z_2}(h \otimes e^{m\alpha}) = m\left(\frac{1}{z_1 - z_2} - \frac{tz_2/z_1}{z_1 - tz_2}\right),$$

$$r_{z_1,z_2}(h \otimes h) = \frac{1}{(z_1 - z_2)^2} - \frac{t}{(z_1 - tz_2)^2}.$$

The bicharacter r_{z_1,z_2} is invertible (V being a Hopf algebra), with inverse on generators given by

$$r^{-1}_{z_1,z_2}(e^{m\alpha} \otimes e^{n\alpha}) = \sigma^{-mn}.$$

Lemma 26.3. *The braiding bicharacter R_{z_1,z_2} of r_{z_1,z_2} is given on the generators by*

$$R_{z_1,z_2}(e^{m\alpha} \otimes e^{n\alpha}) = \Sigma^{mn}, \quad \Sigma = \Sigma_{z_1,z_2} = -\frac{1 - tz_2/z_1}{1 - tz_1/z_2}, \tag{26.2}$$

and we have

$$R_{z_1,z_2}(h \otimes e^{m\alpha}) = m\left(\frac{tz_2/z_1}{z_1 - tz_2} + \frac{t}{z_2 - tz_1}\right), \tag{26.3}$$

and

$$R_{z_1,z_2}(h \otimes h) = \frac{t}{(z_1 - tz_2)^2} - \frac{t}{(z_2 - tz_1)^2}. \tag{26.4}$$

Lemma 26.4. *The translation bicharacter R^γ of r_{z_1,z_2} is given on generators by*

$$R^\gamma_{z_1,z_2}(e^{m\alpha} \otimes e^{n\alpha}) = \Pi^{mn}, \quad \Pi = \Pi_{z_1,z_2} = \frac{1 - tz_2/z_1}{1 - t\dfrac{z_2+\gamma}{z_1+\gamma}}, \tag{26.5}$$

and we have

$$R^\gamma_{z_1,z_2}(h \otimes e^{m\alpha}) = \frac{mtz_2/z_1}{z_1 - tz_2} - \frac{mt(z_2+\gamma)/(z_1+\gamma)}{(z_1+\gamma) - t(z_2+\gamma)}, \tag{26.6}$$

and

$$R^\gamma_{z_1,z_2}(h \otimes h) = \frac{t}{(z_1 - tz_2)^2} - \frac{t}{((z_1+\gamma) - t(z_2+\gamma))^2}. \tag{26.7}$$

Here we want to make some explicit calculations to illustrate what is involved in the case of quantum vertex algebras as opposed to classical vertex algebras. We will show particular examples of Jacobi and Borcherds identities, as well as calculate some (n)-products of states and of fields in V.

We will first start with showing an example of the Jacobi identity, Theorem 15.1, in the case of $a = b = e^\alpha$.

First, note that $r_{z,0}(e^\alpha \otimes e^\alpha) = z$. This implies that by (26.1)

$$Y(e^\alpha, z)e^\alpha = (e^{zD}e^\alpha)e^\alpha r_{z,0}(e^\alpha \otimes e^\alpha),$$

and following the proof of Theorem 25.1 we have:

$$\begin{aligned}
&Y(e^\alpha, z_1)Y(e^\alpha, z_2)c \\
&\quad = (e^{z_1 D}e^\alpha)(e^{z_2 D}e^\alpha)c' r_{z_1,z_2}(e^\alpha \otimes e^\alpha) r_{z_1,0}(e^\alpha \otimes c'') r_{z_2,0}(e^\alpha \otimes c''') \\
&\quad = (e^{z_1 D}e^\alpha)(e^{z_2 D}e^\alpha)c' \frac{z_1 - z_2}{1 - tz_2/z_1} r_{z_1,0}(e^\alpha \otimes c'') r_{z_2,0}(e^\alpha \otimes c''').
\end{aligned}$$

Similarly, taking into account

$$S^{(\tau)}_{z_1,z_2}(e^\alpha \otimes e^\alpha) = -\frac{1 - tz_2/z_1}{1 - tz_1/z_2}(e^\alpha \otimes e^\alpha),$$

we have

$$\begin{aligned}
&Y_{z_2}(1 \otimes Y_{z_1}) S^{(\tau),12}_{z_2,z_1}(e^\alpha \otimes e^\alpha \otimes c) \\
&\quad = Y(e^\alpha, z_2)Y(e^\alpha, z_1)c \left(-\frac{1 - tz_1/z_2}{1 - tz_2/z_1}\right) \\
&\quad = (e^{z_1 D}e^\alpha)(e^{z_2 D}e^\alpha)c' \frac{z_1 - z_2}{1 - tz_2/z_1} r_{z_1,0}(e^\alpha \otimes c'') r_{z_2,0}(e^\alpha \otimes c''').
\end{aligned}$$

And last, by Lemma 26.4 we have $R^{z_2}_{z_3,0}(e^\alpha \otimes e^\alpha) = \frac{1}{1 - t\frac{z_2}{z_2+z_3}}$, so that

$$S^{(z_2)}_{z_3,0}(e^\alpha \otimes e^\alpha) = e^\alpha \otimes e^\alpha \frac{1}{\left(1 - t\dfrac{z_2}{z_2 + z_3}\right)}.$$

Thus in the right-hand side of the Jacobi identity we have

$$\begin{aligned}
&Y_{z_2}(Y_{z_3} \otimes 1) S^{(z_2),12}_{z_3,0}(e^\alpha \otimes e^\alpha \otimes c) \\
&\quad = Y(Y(e^\alpha, z_3)e^\alpha, z_2)c \left(\frac{1}{\left(1 - t\dfrac{z_2}{z_2 + z_3}\right)}\right) \\
&\quad = (e^{(z_2+z_3)D}e^\alpha)(e^{z_2 D}e^\alpha)c' \left(\frac{z_3}{\left(1 - t\dfrac{z_2}{z_2 + z_3}\right)}\right) r_{z_2+z_3,0}(e^\alpha \otimes c'') r_{z_2,0}(e^\alpha \otimes c''').
\end{aligned}$$

Here, of course, we have to expand in powers of t:

$$\frac{1}{1 - tz_2/z_1} = \sum_{n=0}^{\infty} \frac{z_2^n}{z_1^n} t^n, \quad \frac{1}{\left(1 - t\dfrac{z_2}{z_2 + z_3}\right)} = \sum_{n=0}^{\infty} \frac{z_2^n}{(z_2 + z_3)^n} t^n.$$

Now, we see that if we denote $g_n(z_1, z_2, z_1 - z_2)$ the coefficient in front of t^n in

$$\frac{z_1 - z_2}{1 - tz_2/z_1} r_{z_1,0}(e^\alpha \otimes c'') r_{z_2,0}(e^\alpha \otimes c'''),$$

we have then that $g_n(z_2+z_3, z_2, z_3)$ is the coefficient of t^n in

$$\left(\frac{z_3}{\left(1 - t\dfrac{z_2}{z_2+z_3}\right)}\right) r_{z_2+z_3,0}(e^\alpha \otimes c'')r_{z_2,0}(e^\alpha \otimes c''').$$

The Jacobi identity is then a comparison between

$$\begin{aligned} & i_{z_1;z_2}\delta(z_1 - z_2, z_3)(e^{z_1 D}e^\alpha)(e^{z_2 D}e^\alpha)c' g_n(z_1, z_2, z_1 - z_2)t^n \\ & \quad - i_{z_2;z_1}\delta(z_1 - z_2, z_3)(e^{z_1 D}e^\alpha)(e^{z_2 D}e^\alpha)c' g_n(z_1, z_2, z_1 - z_2)t^n \end{aligned}$$

and

$$i_{z_2;z_3}\delta(z_1, z_2+z_3)(e^{(z_2+z_3)D}e^\alpha)(e^{z_2 D}e^\alpha)c' g_n(z_2+z_3, z_2, z_3)t^n.$$

Thus the Jacobi identity is just the direct application of Lemma 15.1, but for *each* of the coefficients of $t^n, n \geq 0$.

Similarly, let us show how one calculates the residues in the Borcherds identity, Theorem 16.1. We will take $a = e^\alpha$, $b = e^{-\alpha}$, $c = e^\alpha$, and $F \equiv 1$.

We have $r_{z,0}(e^\alpha \otimes e^{-\alpha}) = \frac{1}{z}$, thus

$$\begin{aligned} Y(e^\alpha, z_1)Y(e^{-\alpha}, z_2)e^\alpha &= (e^{z_1 D}e^\alpha)(e^{z_2 D}e^{-\alpha})e^\alpha i_{z_1;z_2}\left(\frac{1 - tz_2/z_1}{z_1 - z_2}\right)\frac{z_1}{z_2} \\ &= (e^{z_1 D}e^\alpha)(e^{z_2 D}e^{-\alpha})e^\alpha\left(\sum_{n=0}^\infty \frac{z_2^{n-1}}{z_1^n} - t\sum_{n=0}^\infty \frac{z_2^n}{z_1^{n+1}}\right), \end{aligned}$$

and

$$\begin{aligned} & Y_{z_2}(1 \otimes Y_{z_1})i_{z_2;z_3}S^{(\tau),12}_{z_2,z_1}(e^{-\alpha} \otimes e^\alpha \otimes e^\alpha) \\ & \quad = Y(e^{-\alpha}, z_2)Y(e^\alpha, z_1)e^\alpha i_{z_2;z_1}\left(-\frac{1 - tz_2/z_1}{1 - tz_1/z_2}\right) \\ & \quad = (e^{z_2 D}e^{-\alpha})(e^{z_1 D}e^\alpha)e^\alpha i_{z_2;z_1}\left(-\frac{1 - tz_2/z_1}{1 - tz_1/z_2}\right) \\ & \qquad \times r_{z_2,z_1}(e^{-\alpha} \otimes e^\alpha)r_{z_2,0}(e^{-\alpha} \otimes e^\alpha)r_{z_1,0}(e^\alpha \otimes e^\alpha) \\ & \quad = (e^{z_2 D}e^{-\alpha})(e^{z_1 D}e^\alpha)e^\alpha i_{z_2;z_1}\left(-\frac{1 - tz_2/z_1}{1 - tz_1/z_2}\right)\left(\frac{1 - tz_1/z_2}{z_2 - z_1}\right)\frac{z_1}{z_2} \\ & \quad = (e^{z_1 D}e^\alpha)(e^{z_2 D}e^{-\alpha})e^\alpha i_{z_2;z_1}\left(\frac{1 - tz_2/z_1}{z_1 - z_2}\right)\frac{z_1}{z_2}. \end{aligned}$$

And last, we have

$$Y(e^\alpha, z)e^{-\alpha} = (e^{zD}e^\alpha)e^{-\alpha}r_{z,0}(e^\alpha \otimes e^{-\alpha}) = \frac{1}{z} + h + \mathcal{O}(z), \tag{26.8}$$

and by Lemma 26.4 we have $R^{z_2}_{z_3,0}(e^\alpha \otimes e^{-\alpha}) = 1 - t\frac{z_2}{z_2+z_3}$, so that

$$i_{z_2;z_3} S^{(z_2)}_{z_3,0}(e^\alpha \otimes e^{-\alpha}) = e^\alpha \otimes e^{-\alpha} i_{z_2;z_3}\left(1 - t\frac{z_2}{z_2+z_3}\right) \tag{26.9}$$

$$= e^\alpha \otimes e^{-\alpha}\left(1 + t\sum_{k=0}^{\infty}(-1)^k\left(\frac{z_3}{z_2}\right)^k\right). \tag{26.10}$$

So we calculate

$$\begin{aligned}
&Y_{z_2}(Y_{z_3} \otimes 1) i_{z_2;z_3} S^{(z_2),12}_{z_3,0}(e^\alpha \otimes e^{-\alpha} \otimes e^\alpha)\\
&\quad = Y(Y(e^\alpha, z_3)e^{-\alpha}, z_2)e^\alpha i_{z_2;z_3}\left(1 - t\frac{z_2}{z_2+z_3}\right)\\
&\quad = Y((e^{z_3D}e^\alpha)e^{-\alpha}, z_2)e^\alpha i_{z_2;z_3}\left(1 - t\frac{z_2}{z_2+z_3}\right) r_{z_3,0}(e^\alpha \otimes e^{-\alpha})\\
&\quad = e^{z_2D}((e^{z_3D}e^\alpha)e^{-\alpha})e^\alpha r_{z_2,0}((e^{z_3D}e^\alpha)e^{-\alpha}) \otimes e^\alpha) i_{z_2;z_3}\\
&\qquad \times \left(1 - t\frac{z_2}{z_2+z_3}\right) r_{z_3,0}(e^\alpha \otimes e^{-\alpha})\\
&\quad = (e^{(z_2+z_3)D}e^\alpha)(e^{z_2D}e^{-\alpha})e^\alpha i_{z_2;z_3}\left(1 - t\frac{z_2}{z_2+z_3}\right)\\
&\qquad \times i_{z_2;z_3} r_{z_2+z_3,0}(e^\alpha \otimes e^\alpha) r_{z_2,0}(e^\alpha \otimes e^{-\alpha}) r_{z_3,0}(e^\alpha \otimes e^{-\alpha})\\
&\quad = (e^{(z_2+z_3)D}e^\alpha)(e^{z_2D}e^{-\alpha})e^\alpha \frac{z_2+z_3}{z_2z_3}\left(1 + t\sum_{k=0}^{\infty}(-1)^k\left(\frac{z_3}{z_2}\right)^k\right).
\end{aligned}$$

Now we have

$$\begin{aligned}
&\operatorname{Res}_{z_1}(Y(e^\alpha, z_1)Y(e^{-\alpha}, z_2)e^\alpha)\\
&\quad = \operatorname{Res}_{z_1}\left((e^{z_1D}e^\alpha)(e^{z_2D}e^{-\alpha})e^\alpha\left(\sum_{n=0}^{\infty}\frac{z_2^{n-1}}{z_1^n} - t\sum_{n=0}^{\infty}\frac{z_2^n}{z_1^{n+1}}\right)\right)\\
&\quad = (e^{z_2D}e^\alpha)(e^{z_2D}e^{-\alpha})e^\alpha(1-t) = e^\alpha - te^\alpha.
\end{aligned}$$

Here we took into account that

$$\operatorname{Res}_{z_1}\left(e^{z_1D}\sum_{n=0}^{\infty}\frac{z_2^n}{z_1^{n+1}}\right) = e^{z_2D}e^\alpha.$$

Similarly

$$\begin{aligned}
&\operatorname{Res}_{z_3}(Y_{z_2}(Y_{z_3} \otimes 1) i_{z_2;z_3} S^{(z_2),12}_{z_3,0}(e^\alpha \otimes e^{-\alpha} \otimes e^\alpha))\\
&\quad = \operatorname{Res}_{z_3}\left((e^{(z_2+z_3)D}e^\alpha)(e^{z_2D}e^{-\alpha})e^\alpha \frac{z_2+z_3}{z_2z_3}\left(1 + t\sum_{k=0}^{\infty}(-1)^k\left(\frac{z_3}{z_2}\right)^k\right)\right)\\
&\quad = (e^{z_2D}e^\alpha)(e^{z_2D}e^{-\alpha})e^\alpha(1-t) = e^\alpha - te^\alpha.
\end{aligned}$$

We now calculate some (n)-products of states and of fields in V. From (26.8) we have the following products of states:

$$e^{\alpha}_{(-1)}e^{-\alpha} = h, \quad e^{\alpha}_{(0)}e^{-\alpha} = 1, \quad e^{\alpha}_{(k)}e^{-\alpha} = 0, \quad k > 0. \tag{26.11}$$

Note that this are the same (n)-products as for the lattice vertex algebra corresponding to the bicharacter $r^0_{z_1,z_2}$ (obtained by putting $t = 0$).

Next we want to use Corollary 18.1 to calculate (n)-products of fields. We have from (26.10), Corollary 18.1 and (26.11)

$$\begin{aligned} e^{\alpha}(z)_{(-1)}e^{-\alpha}(z) &= (1+t)Y(e^{\alpha}_{(-1)}e^{-\alpha}, z) - Y(e^{\alpha}_{(0)}e^{-\alpha}, z)\frac{t}{z} \\ &= (1+t)h(z) - \frac{t}{z}. \end{aligned}$$

Now, see Sec. 19, $e^{\alpha}(z)_{(-1)}e^{-\alpha}(z) = :e^{\alpha}(z)e^{-\alpha}(z):_S$, and this normal ordered product of fields is *not* a vertex operator $Y(a, z)$ for any $a \in V$, since the action of $:e^{\alpha}(z)e^{-\alpha}(z):_S$ on the vacuum is not regular in z, contradicting the vacuum axiom (4.1). This is in contrast to the situation in the usual vertex algebras.

27. S-Commutators and Commutators

In this section we calculate some S-commutators of fields by expanding the bicharacter in our main example and express this in terms of commutators, using Theorem 25.1.

We have

$$\delta(r_{z_1,z_2}(e^{m\alpha} \otimes e^{n\alpha})) = \begin{cases} 0 & mn \geq 0, \\ (1 - tz_2/z_1)^{k+1}\partial^{(k)}_{z_2}\delta(z_1, z_2) & mn = -k-1 < 0, \end{cases}$$

which follows from the definition (26.1). Then

$$\begin{aligned} &[e^{m\alpha}(z_1), e^{n\alpha}(z_2)]_S \\ &\quad = \begin{cases} 0 & mn \geq 0 \\ \left(1 - \dfrac{tz_2}{z_1}\right)^{k+1} \sum Y(v^p_{m,n}, z_2)\partial^{(q)}_{z_2}\delta(z_1, z_2) & mn = -k-1 < 0 \end{cases} \end{aligned}$$

where $v^p_{m,n} = D^{(p)}(e^{m\alpha})e^{n\alpha} \in V$ and the sum is over all $p, q \geq 0$ such that $p + q = k$. In particular

$$[e^{\alpha}(z_1), e^{-\alpha}(z_2)]_S = \left(1 - t\frac{z_2}{z_1}\right)\delta(z_1, z_2) = (1-t)\delta(z_1, z_2).$$

So

$$e^{\alpha}(z)_{(0)}e^{-\alpha}(z) = 1 - t, \quad e^{\alpha}(z)_{(k)}e^{-\alpha}(z) = 0, \quad k > 0.$$

In the same way

$$\delta(r_{z_1,z_2}(h \otimes e^{m\alpha})) = m\delta(z_1, z_2),$$

which follows from Lemma 26.2, see also (15.3). Hence

$$[h(z_1), e^{m\alpha}(z_2)]_S = me^{m\alpha}(z_2)\delta(z_1, z_2). \tag{27.1}$$

Finally, using Lemma 26.2 again, we find

$$\delta(r_{z_1,z_2}(h \otimes h)) = \partial_{z_2}\delta(z_1, z_2),$$

so that,

$$[h(z_1), h(z_2)]_S = \partial_{z_2}\delta(z_1, z_2). \tag{27.2}$$

It is sometimes useful to express the S-commutators of fields in terms of the usual commutators. We give some examples.

We have by definition of the S-commutator

$$\begin{aligned}[h(z_1), e^{m\alpha}(z_2)]_S &= h(z_1)e^{m\alpha}(z_2) - e^{m\alpha}(z_2)h'(z_1)R_{z_2,z_1}(e^{m\alpha} \otimes h'') \\ &= [h(z_1), e^{m\alpha}(z_2)] - e^{m\alpha}(z_2)R_{z_2,z_1}(e^{m\alpha} \otimes h) \\ &= [h(z_1), e^{m\alpha}(z_2)] - e^{m\alpha}(z_2)m\partial_{z_1}\ln(\Sigma_{z_2,z_1}),\end{aligned}$$

where Σ is defined in Lemma 26.3. Combining this with (27.1) gives

$$[h(z_1), e^{m\alpha}(z_2)] = me^{m\alpha}(z_2)(\delta(z_1, z_2) + \partial_{z_1}\ln(\Sigma_{z_2,z_1})). \tag{27.3}$$

Now

$$\operatorname{Res}_{z_1}(z_1^n\partial_{z_1}\ln(\Sigma_{z_2,z_1})) = \begin{cases} 0 & n = 0, \\ -t^{|n|}z_2^n & n \neq 0. \end{cases}$$

Hence

$$[h_{(n)}, e^{m\alpha}(z_2)] = \begin{cases} me^{m\alpha}(z_2) & n = 0, \\ mz_2^n(1 - t^{|n|})e^{m\alpha}(z_2) & n \neq 0. \end{cases} \tag{27.4}$$

Similarly,

$$\begin{aligned}[h(z_1), h(z_2)]_S &= [h(z_1), h(z_2)] - R_{z_2,z_1}(h \otimes h) \\ &= [h(z_1), h(z_2)] - \left(\frac{t}{(tz_1 - z_2)^2} - \frac{t}{(tz_2 - z_1)^2}\right).\end{aligned}$$

Note that here we see that the ordinary commutator of $h(z)$ with itself is not killed by any power of $z_1 - z_2$, whereas the S-commutator is killed by $(z_1 - z_2)^2$, see (27.2).

By (27.2)

$$[h(z_1), h(z_2)] = \partial_{z_2}\delta(z_1, z_2) + R_{z_2,z_1}(h \otimes h).$$

Now

$$\operatorname{Res}_{z_1}(z_1^n R_{z_2,z_1}(h \otimes h)) = -nt^{|n|}z_2^{n-1},$$

and we have

$$[h_{(m)}, h(z_2)] = m z_2^{m-1}(1 - t^{|m|}) \tag{27.5}$$

and

$$[h_{(m)}, h_{(n)}] = m(1 - t^{|m|})\delta_{m+n,0}. \tag{27.6}$$

We see therefore that the coefficients of $h(z)$ generate a *deformed Heisenberg algebra* $\mathcal{H}_t$. As a Lie algebra $\mathcal{H}_t$ is isomorphic to the usual Heisenberg Lie algebra $\mathcal{H} = \mathcal{H}_{t=0}$. In particular the representation theory of $\mathcal{H}_t$ is the same as in the undeformed case. We have a decomposition

$$V = \oplus_{m\in\mathbb{Z}} V_m, \quad V_m = k[D^n h]e^{m\alpha},$$

where each V_m is an irreducible $\mathcal{H}_t$-module, with action given by

$$h_{(m)} = \begin{cases} \text{multiplication by } D^k h/k! & m = -k-1 < 0, \\ \partial_\alpha & m = 0, \\ m(1-t^m)\dfrac{\partial}{\partial h_{(-m-1)}} & m > 0. \end{cases}$$

The case $m = 0$ follows from Corollary 18.1 and (27.1).

28. Braided Bosonization

Define

$$\Gamma_+(z) = \exp\left(\sum_{n>0} h_{(-n)} z^n/n\right), \quad \Gamma_-(z) = \exp\left(-\sum_{n>0} h_{(n)} z^{-n}/n\right).$$

By (27.6) we have for $m \neq 0$

$$[h_{(\pm m)}, \Gamma_\pm(z)] = \pm z^{\pm m}(1 - t^{|m|})\Gamma_\pm(z), \quad [h_{(\mp m)}, \Gamma_\pm(z)] = 0.$$

Then we see that

$$\Sigma_n(z) = \Gamma_+^{-n}(z) e^{n\alpha}(z) \Gamma_-^{n\alpha}(z) e^{n\alpha}$$

commutes with the deformed boson:

$$[h(z_1), \Sigma_n(z_2)] = 0,$$

and by the usual arguments using the representation theory of the deformed Heisenberg algebra (see e.g. [9]) one finds the bosonization formula

$$e^{n\alpha}(z) = \Gamma_+^n(z)\Gamma_-^{-n}(z) e^{n\alpha} z^{n\partial_\alpha}.$$

This formula (for $n = \pm 1$) can by found in Jing's paper [8], with a slightly different notation.

29. Hall–Littlewood Polynomials

In this section we recall the Macdonald definition of Hall–Littlewood symmetric polynomials [16]. Also we explain how the bosonized vertex operators described in the previous section (as considered by Jing [8]), serve as generating functions for the Hall–Littlewood polynomials.

Denote by $\Lambda_{\mathbf{C}}[[t]]$ the ring of symmetric functions over $\mathbf{C}[[t]]$ in countably many independent variables $x_i,\ i \geq 0$.

Let λ be a partition, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k, \ldots),\ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq \cdots$. Let $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_k + \cdots$.

Denote $z_\lambda = \prod_{i\geq 0} i^{m_i} \cdot m_i!$, where $m_i = m_i(\lambda)$ is the number of parts of λ equal to i.

We call a family (a_λ) of elements in a ring indexed by partitions *multiplicative* if $a_\lambda = \prod a_{\lambda_i}$.

For any partition α we use the vector notation x^α for $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_k^{\alpha_k} \ldots$. We will use the basis (m_λ) of monomial symmetric functions:

$$m_\lambda = \sum_\alpha x^\alpha, \tag{29.1}$$

where the sum is over distinct permutations of λ, as well as the multiplicative basis generated by the power sums $p_n = \sum_{i\geq 0} x_i^n,\ p_0 = 1$.

Define a scalar product $\langle\ ,\ \rangle_t$ on $\Lambda_{\mathbf{C}}[[t]]$ by putting for the power functions

$$\langle p_\lambda, p_\mu \rangle_t = \delta_{\lambda\mu} z_\lambda v_\lambda,$$

for any partitions λ, μ, where the mulitplicative family v_λ is defined by $v_n = \frac{1}{1-t^n}$. Define a set of symmetric functions $\{H_\lambda\}$ indexed by partitions by the following two (over-determining) conditions:

$$\langle H_\lambda, H_\mu \rangle_t = 0 \qquad \text{for } \lambda \neq \mu,$$
$$H_\lambda = m_\lambda + \sum_{\mu<\lambda} u_{\lambda\mu} m_\mu, \quad u_{\lambda\mu} \in \mathbf{C}[[t]].$$

Here $\mu < \lambda$ is with respect to the usual partial order on partitions.

It is proved in [16] that such symmetric functions $\{H_\lambda\}$ exist. Denote also by Q_λ the dual of H_λ, i.e. $\langle H_\lambda, Q_\mu \rangle_t = \delta_{\lambda,\mu}$. Note that when $t = 0$ both the H_λ and the Q_λ reduce to the Schur polynomials (Schur polynomials are self dual).

We can view the nth power symmetric function p_n as an operator acting on $\Lambda_{\mathbf{C}}[[t]]$ by multiplication. Define also for given multiplicative family (v_λ) the operators $p_n^\perp$ by requiring

$$\langle p_n^\perp f, g \rangle_t = \langle f, p_n g \rangle_t,$$

for any $f, g \in \Lambda_{\mathbf{C}}[[t]]$.

Lemma 29.1. *The operators* $\{h_{(n)} | n \in \mathbf{Z}\}$ *given by* $h_{(n)} = -(1 - t^n) p_n^\perp$, $h_{(-n)} = (1-t^n) p_n$ *for* $n \in \mathbf{N}$, $h_{(0)} = 0$ *generate a representation of the deformed Heisenberg* *algebra* $\mathcal{H}_t$ *on* $\Lambda_{\mathbf{C}}[[t]]$, *i.e.*

$$[h_{(m)}, h_{(n)}] = m(1 - t^{|m|})\delta_{m+n,0}. \tag{29.2}$$

The proof is based on the undeformed case ($v_n = 1$), which can be found in [16].

From the fact that the power symmetric functions form a basis of $\Lambda_{\mathbf{C}}[[t]]$, it follows that $\Lambda_{\mathbf{C}}[[t]]$ is a highest weight module for $\mathcal{H}_t$, and is thus an irreducible $\mathcal{H}_t$ module. Therefore we have that $\Lambda_{\mathbf{C}}[[t]]$ is isomorphic as a module and as an algebra to V_0 (V_0 was defined in Sec. 26). Thus we can identify $(1 - t^n)p_n$ with $D^{(n-1)}h$ $(n > 0)$.

The following theorem [8] explains the connection between the Hall–Littlewood symmetric functions and the vertex operators considered in the previous section:

Theorem 29.1. *Let* $\tilde{m}$ *is a partition of length* l, $\tilde{m} = (m_1, m_2, \ldots, m_l, 0, \ldots)$, *and let* ρ *be the partition defined by* $\rho = (l, l-1, \ldots, 1, 0, \ldots)$. *The constant term of* $Y(D^{(m_1)}e^\alpha, z_1)Y(D^{(m_2)}e^\alpha, z_2)\ldots Y(D^{(m_1)}e^\alpha, z_l)1$ *is* $Q_{\tilde{m}-\rho}e^{l\alpha}$, *where* $Q_{\tilde{m}-\rho}$ *is the dual Hall–Littlewood polynomial corresponding to the partition* $\tilde{m} - \rho$.

The proof is straightforward modification of the main theorem in [8] using the properties of the vertex operators.

Thus the vertex operators $Y(D^{(m)}e^\alpha, z)$ (as described in Sec. 26) and the coefficients of their products are very important in the theory of the Hall–Littlewood polynomials. This makes them an important example of quantum vertex operators, and they are the main motivation for our definition of H_D-quantum vertex algebras. The previous definitions of quantum vertex algebras were not general enough to incorporate the Hall–Littlewood vertex operators.

Appendix A. Braided Algebras with Symmetry

A.1. *Introduction*

To motivate the rather complicated definition of an H_D-quantum vertex algebra in Sec. 4 we discuss in this appendix braided algebras (with symmetry). The idea is that a vertex algebra has a singular multiplication, and that it is good to understand the nonsingular case first.

A.2. *Commutative associative algebras*

As a preliminary, note that an efficient way to describe commutative associative unital algebras is as follows. Let M be a vector space and $1 \in M$ a distinguished element, and let

$$m\colon M^{\otimes 2} \to M$$

be a multiplication for which 1 is the unit:

$$m(a \otimes 1) = m(1 \otimes a) = a, \quad a \in M. \tag{A.1}$$

We need some notation. If a is a linear map on $M^{\otimes 2}$, then a^{23} is the operator on $M^{\otimes 3}$ acting on the 2nd and 3rd factor (so $a^{23} = 1 \otimes a$). The other superscripts have a similar meaning. Let $\tau\colon M^{\otimes 2} \to M^{\otimes 2}$ be the flip $a \otimes b \mapsto b \otimes a$. Let $m_3 = m(1 \otimes m)\colon M^{\otimes 3} \to M$. Then we impose

$$m_3 = m_3 \tau^{12}, \qquad \textbf{(Commutativity/Associativity Axiom)}.$$

In other words, writing $m_3(a \otimes b \otimes c) = a(bc)$, we require $a(bc) = b(ac)$. Then one easily checks that $(M, m, 1)$ is in fact commutative ($m = m\tau$) and associative ($m(1 \otimes m) = m(m \otimes 1)$).

A.3. *Braided algebras*

We are next interested in non commutative algebras where the noncommutativity is controlled by a braiding map.

Definition A.1. A *braided algebra* is a unital algebra $(M, m, 1)$ with a braiding $S\colon M^{\otimes 2} \to M^{\otimes 2}$ such that

(1) (**Vacuum Axiom**) $S(a \otimes 1) = a \otimes 1$, and $S(1 \otimes a) = 1 \otimes a$.
(2) (**Braiding Axiom**) $m_3 S^{12} = m_3 \tau^{12}$.
(3) (**Unitarity Axiom**) $S \circ \tau \circ S \circ \tau = 1_{M^{\otimes 2}}$.
(4) (**Yang–Baxter Axiom**) $S^{12} S^{13} S^{23} = S^{23} S^{13} S^{12}$.
(5) (**Compatibility with Multiplication Axiom**) $S m^{12} = m^{12} S^{23} S^{13}$ and $S m^{23} = m^{23} S^{12} S^{13}$.

The Compatibility with Multiplication Axiom allows us to express the braiding involving a product in terms of a product of the braidings of the factors. Also, together with the braiding axiom it gives associativity, as we now proceed to show.

Lemma A.1 (Braided Commutativity).

$$mS = m\tau.$$

Proof. Apply the Braiding Axiom to $a \otimes b \otimes 1$, using $m_3(a \otimes b \otimes 1) = m(a \otimes b)$. □

Theorem A.1 (Associativity). *A braided algebra is associative*:

$$m(1 \otimes m) = m(m \otimes 1).$$

Proof. Let $A = a \otimes b \otimes c$. Then

$$\begin{aligned} mm^{12}(A) &= mS\tau m^{12}(A) = mSm^{23}(c \otimes a \otimes b) \\ &= mm^{23} S^{12} S^{13}(c \otimes a \otimes b) = mm^{23} \tau^{12} S^{13}(c \otimes a \otimes b) \\ &= mm^{23} S^{23}(a \otimes c \otimes b) = mm^{23} \tau^{23}(a \otimes c \otimes b) \\ &= mm^{23}(A) \end{aligned}$$

□

We used the Compatibility with Multiplication Axiom to derive associativity. If we do not impose this axiom, we can only derive *braided associativity*, (also called *quasi-associativity* cf. [5]):

$$mm^{23}S^{23}S^{13} = mSm^{12}, \quad mm^{12}S^{12}S^{13} = mSm^{23}.$$

We have not yet used the unitarity and Yang–Baxter axioms. They are used to describe the behavior under permutations of the arguments of the n-fold multiplication $m_n\colon M^n \to M$ (defined recursively by $m_n = m(1 \otimes m_{n-1})$) as we now proceed to explain.

Lemma A.2.

$$m_n \tau^{ii+1} = m_n S^{ii+1}.$$

Proof. We can use associativity to write

$$m_n = m_3 \circ (m_{i-1} \otimes m_2 \otimes m_{n-i-1}).$$

The Lemma follows from Braided Commutativity, Lemma A.1 . □

Remark A.1. Note that if i, j are not adjacent, then it is in general not true that the transposition τ^{ij} does act on m_n by multiplication by S^{ij}.

For instance, a simple example of a non trivial braided algebra is a super commutative algebra $M = M_{\bar{0}} \oplus M_{\bar{1}}$. The braiding is given (for homogeneous elements) by $S(a \otimes b) = (-1)^{|a||b|} a \otimes b$. It is then clear that the braiding corresponding to the permutation $\tau^{13}\colon a \otimes b \otimes c \mapsto c \otimes b \otimes a$ is given by

$$S^{\tau^{13}}(a \otimes b \otimes c) = S^{12}S^{13}S^{23}(a \otimes b \otimes c) = (-1)^{|a||b|}(-1)^{|a||c|}(-1)^{|b||c|}(a \otimes b \otimes c),$$

whereas

$$S^{13}(a \otimes b \otimes c) = (-1)^{|a||c|} a \otimes b \otimes c.$$

One knows that the symmetric group $\mathcal{S}_n$ is generated by the simple transpositions $w_i = (ii+1)$, $i = 1, 2, \ldots, n-1$, see Sec. 10. Then define a map $S\colon \mathcal{S}_n \to \mathrm{GL}(M^{\otimes n})$ by

$$S(w_i) = 1^{i-1} \otimes S\tau \otimes 1^{n-i-1},$$

and extend this as an anti-homomorphism:

$$S(f) = S(w_{i_k})S(w_{i_{k-1}}) \ldots S(w_{i_1}),$$

in $f = w_{i_1} w_{i_2} \ldots w_{i_k} \in \mathcal{S}_n$. Then the unitarity and the Yang–Baxter axioms and Lemma A.2 imply

Theorem A.2. *The braiding map $S^f : M^{\otimes n} \to M^{\otimes n}$ is independent of the representation of σ in terms of simple reflections. Furthermore*

$$m_n \circ S(f) = m_n.$$

for all $f \in \mathcal{S}_n$.

This concludes our discussion of braided algebras *an sich*.

A.4. *Braided algebras with symmetry*

We now assume that we have additionally an action of a group G on the braided algebra M. If $g \in G$ we write $\Delta(g) = g \otimes g \in G \otimes G$ for the coproduct of g.

Definition A.2. Let $(M, m, 1, S)$ be a braided algebra, with a G-action on M. We call this a braided G-algebra in case for each $g \in G$ there is a map

$$S^g : M^{\otimes 2} \to M^{\otimes 2},$$

such that

- (**Vacuum Axiom**) $S^g(a \otimes 1) = a \otimes 1$, and $S^g(1 \otimes a) = 1 \otimes a$.
- (G**-Symmetry**) $gmS^g = m\Delta(g)$.
- (**Multiplicativity**) $S^{gh} = S^h \circ \Delta(h^{-1}) \circ S^g \circ \Delta(h)$
- (G**-Yang-Baxter**) $S^{g,12}S^{g,13}S^{g,23} = S^{g,23}S^{g,13}S^{g,12}$.
- (**Compatibility with Multiplication Axiom**) $S^g m^{12} = m^{12}S^{g,23}S^{g,13}$ and $S^g m^{23} = m^{23}S^{g,12}S^{g,13}$.

Of course, the simplest case is where $S^g = 1 \otimes 1$ for all $g \in G$. Then the multiplication intertwines the action of G on $M^{\otimes 2}$ and M; usually M is then called a module-algebra.

Lemma A.3. *Define* $\Sigma_n, \tilde{\Sigma}_n \colon M^{n+1} \to M^{n+1}$ *by* $\Sigma_n = S^{12}S^{13}\dots S^{1n+1}$, $\tilde{\Sigma}_n = S^{1n+1}\dots S^{13}S^{12}$. *Then we have compatibility with the higher multiplications*:

$$S(1 \otimes m_n) = (1 \otimes m_n)\Sigma_n, \quad S(m_n \otimes 1) = (m_n \otimes 1)\tilde{\Sigma}_n.$$

Proof. For $n = 2$ the lemma is just the compatibility with multiplication axiom. Assume the lemma is true for $n = k - 1$. Then

$$\begin{aligned} S(1 \otimes m_k) &= S(1 \otimes m)(1 \otimes 1 \otimes m_{k-1}) \\ &= (1 \otimes m)S^{12}S^{13}(1 \otimes 1 \otimes m_{k-1}) \\ &= (1 \otimes m)(1 \otimes 1 \otimes m_{k-1})S^{12}\Sigma_{n-1}^{13\dots nn+1}, \end{aligned}$$

Noting that $\Sigma_n = S^{12} \circ \Sigma_{n-1}^{13\dots nn+1}$ the first equation of the lemma follows. The second one is proved similarly. □

Now define

$$S_n^g = \Sigma_{n-1} \circ (1 \otimes S_{n-1}^g).$$

Theorem A.3. *We have* $S_n^g = \tilde{\Sigma}_{n-1} \circ (S_{n-1}^g \otimes 1)$ *and*

$$gm_n S_n^g = m_n \Delta_n(g).$$

A.5. *Bicharacters*

Let M be a commutative and cocommutative Hopf algebra. A bicharacters on M is a linear map

$$r \colon M^{\otimes 2} \to \mathbb{C},$$

satisfying

- (**Vacuum**) $r(a \otimes 1) = r(1 \otimes a) = \epsilon(a)$, $a \in M$.
- (**Multiplication**) For all $a, b, c \in M$ we have $r(a \otimes bc) = \sum r(a' \otimes b)r(a'' \otimes c)$ and $r(ab \otimes c) = \sum r(a \otimes c')r(b \otimes c'')$.

Here and below we use the notation $\Delta(a) = \sum a' \otimes a''$ for the coproduct for $a \in V$. Often we will also omit the summation symbol, to unclutter the formulas.

We can multiply bicharacters: if r, s are bicharacters and $a, b \in M$ then

$$(r * s)(a \otimes b) = r(a' \otimes b')s(a'' \otimes b''). \tag{A.2}$$

The unit bicharacter is

$$\epsilon(a \otimes b) = \epsilon(a)\epsilon(b). \tag{A.3}$$

Since M is a Hopf algebra it comes with an antipode, and all bicharacters are invertible, with inverse given by

$$r^{-1}(a \otimes b) = r(S(a) \otimes b).$$

The set of bicharacters forms an Abelian group.

The transpose of a bicharacter is defined by

$$r^{\tau}(a \otimes b) = r(b \otimes a).$$

The transpose is an involution of the algebra of bicharacters:

$$(r * s)^{\tau} = (r^{\tau} * s^{\tau}).$$

If r is an invertible bicharacter with inverse r^{-1} we define another bicharacter

$$R = r^{\tau} * r^{-1}, \tag{A.4}$$

We will call R the *braiding bicharacter* associated to r. It will control the braiding in the braided algebra we are going to construct from r below. The braiding bicharacter R is *unitary*:

$$R^{\tau} = R^{-1}. \tag{A.5}$$

Also we have

$$r * R = r^{\tau}. \tag{A.6}$$

For any bicharacter ρ on M we define a map

$$S^{(\rho)} \colon M^{\otimes 2} \to M^{\otimes 2}, \quad a \otimes b \mapsto a' \otimes b' \rho(a'' \otimes b''). \tag{A.7}$$

Lemma A.4. (1) *If ϵ is the unit bicharacter on M, then $S^{(\epsilon)} = 1_{M^{\otimes 2}}$.*
(2) *If ρ, σ are bicharacters on M, then $S^{(\rho * \sigma)} = S^{(\rho)} \circ S^{(\sigma)}$.*
(3) *If ρ is a bicharacter, then $\tau \circ S^{(\rho)} \circ \tau = S^{(\rho^{\tau})}$.*

Lemma A.5. *For all $a \in M$ and bicharacters ρ on M we have*

(1) (*Vacuum*) $S^{(\rho)}(a \otimes 1) = a \otimes 1$ *and* $S^{(\rho)}(1 \otimes a) = 1 \otimes a$.
(2) (*Yang–Baxter*) $S^{(\rho),12}S^{(\rho),13}S^{(\rho),23} = S^{(\rho),23}S^{(\rho),13}S^{(\rho),12}$.

Now we fix a bicharacter r on M, and define a twisting of the multiplication m on M:

$$m_r = m \circ S^{(r)} \colon M^{\otimes 2} \to M.$$

Lemma A.6. *For any bicharacter ρ the map $S^{(\rho)}$ is compatible with the twisted multiplication m_r:*

$$S^{(\rho)}(m_r \otimes 1) = (m_r \otimes 1)S^{(\rho),23}S^{(\rho),13}, \quad S^{(\rho)}(1 \otimes m_r) = (1 \otimes m_r)S^{(\rho),12}S^{(\rho),13}$$

Proof. For $a, b, c \in M$ we have

$$\begin{aligned} S^{(\rho)}(m_r \otimes 1)(a \otimes b \otimes c) &= S^{(\rho)}(a'b' \otimes c)r(a'' \otimes b'') \\ &= (a'b')' \otimes c'\rho((a'b')'' \otimes c'')r(a'' \otimes b'') \\ &= a''b'' \otimes c'\rho(a''' \otimes c''')\rho(b''' \otimes c'''')r(a'' \otimes b'') \end{aligned}$$

Now by coassociativity and cocommutativity of M we have $a'' \otimes a''' \otimes a'' = a'' \otimes$ $\otimes a'' \otimes a'''$, so that we get

$$\begin{aligned} &= a''b''r(a''' \otimes b''') \otimes c'\rho(a'' \otimes c''')\rho(b'' \otimes c'''') \\ &= (m_r \otimes 1)S^{(\rho),23}S^{(\rho),13}(a \otimes b \otimes c). \end{aligned}$$

The proof of the other part is similar. □

Recall the braiding bicharacter $R = r^{-1} * r^{\tau}$ associated to r, and write $S = S^{(R)}$.

Proposition A.1. *For any bicharacter r on M the twist $(M, m_r, 1, S)$ is a braided algebra.*

Proof. We need to check the axioms in Definition A.1. The vacuum and Yang–Baxter axioms are dealt with in Lemma A.5. For unitarity we have $\tau \circ S \circ \tau = S^{(R^{\tau})}$ so that by Lemma 23.1 and (A.5)

$$S \circ \tau \circ S \circ \tau = S^{(R)} \circ S^{(R^{\tau})} = S^{(R * R^{\tau})} = S^{(\epsilon)} = 1_{M^{\otimes 2}}.$$

Compatibility of S with the multiplication m_r is the case $\rho = R$ of Lemma A.6.

Now m_r is braided commutative:

$$m_r S = m \circ S^{(r)} \circ S^{(R)} = m \circ S^{(r^{\tau})} = m \circ \tau \circ S^{(r)} \circ \tau = m_r \circ \tau,$$

by Lemma 23.1, (A.6) and the fact that m is commutative. From compatibility of S with multiplication m_r and the Yang–Baxter equation it follows that m_r is associative. The braiding axiom for $m_{r,3} = m_r(1 \otimes m_r) = m_r(m_r \otimes 1)$ follows from this. □

A.6. *Bicharacters and group action*

Now we assume that we have an action of a group G on the commutative and cocommutative Hopf algebra M compatible with the multiplication and the comultiplication:

$$gm = m \circ \Delta(g), \quad \Delta(gm) = \Delta_G(g)\Delta(m).$$

Define for any bicharacter r on M and $g \in G$

$$r^g = r \circ \Delta(g).$$

It is easy to check that r^g is again a bicharacter, so that we can write

$$r^g = r * R^g, \quad R^g = r^{-1} * r^g. \tag{A.8}$$

Also R^g is then a bicharacter. Define $S^g = S^{(R^g)}$.

Lemma A.7. *For all $g \in G$ and bicharacters r on M*

$$gm_r S^g = m_r \Delta(g).$$

Proof. By Lemma 23.1 and (A.6)

$$\begin{aligned} gm_r S^g &= gm \circ S^{(r)} \circ S^{(R^g)} = gm \circ S^{(r*R^g)} \\ &= gm \circ S^{(r^g)} = m \circ \Delta(g) \circ S^{(r^g)} = m \circ S^{(r)}\Delta(g). \end{aligned}$$

Here we use

$$\Delta(g)S^{(r^g)} = S^{(r)}\Delta(g), \tag{A.9}$$

which follows from the definition of $S^{(r)}$, see (A.7). □

Corollary A.1. *Let r be a bicharacter on a commutative and cocommutative Hopf algebra M with an action of a group G. Then $(M, m_r, 1, S)$ is a braided G-algebra for the maps*

$$S^g = S^{(R^g)}, \quad g \in G,$$

where R^g is defined in (A.8).

Proof. We need to check the axioms in Definition A.2. The G-symmetry axiom is verified in the previous Lemma A.7. The Vacuum Axiom and G-Yang–Baxter Axiom for S^g are verified in Lemma A.5, as $S^g = S^{(R^g)}$ and R^g is a bicharacter. The compatibility of S^g with multiplication m_r is the case $\rho = R^g$ of Lemma A.6. For multiplicativity

$$\begin{aligned} S^{gh} &= S^{(r^{-1}*r^{gh})} = S^{(r^{-1})} \circ S^{(r^{gh})} \\ &= S^{(r^{-1})} \circ \Delta(gh)^{-1} \circ S^{(r)} \circ \Delta(gh) \quad \text{(by (A.9))} \\ &= S^{(r^{-1})} \circ \Delta(h)^{-1} \circ S^{(r)} \circ S^{(r^{-1})} \circ S^{(r^g)} \circ \Delta(h) \\ &= S^{(r^{-1})} \circ S^{(r^h)} \circ \Delta(h)^{-1} \circ S^g \circ \Delta(h) \\ &= S^h \circ \Delta(h^{-1}) \circ S^g \circ \Delta(h). \end{aligned}$$

□

Remark A.2. In a braided G-algebra we implement the action of G by a system of maps S^g satisfying

$$gmS^g = m\Delta(g).$$

In the bicharacter case of a twisted multiplication $m_r = m \circ S^{(r)}$ we can also implement the group action by twisting the coproduct on G: we have

$$\begin{aligned} gm_r &= gm \circ S^{(r)} = m \circ \Delta(g) \circ S^{(r)} \\ &= m \circ S^{(r)} \circ S^{(r^{-1})} \circ \Delta(g) \circ S^{(r)} \\ &= m_r \Delta_r(g), \end{aligned}$$

where the twisted coproduct is

$$\Delta_r(g) = S^{(r^{-1})} \circ \Delta(g) \circ S^{(r)}.$$

The fact that the two approaches are equivalent,

$$\Delta(g)(S^g)^{-1} = \Delta_r(g),$$

follows from (A.9). It is at this point not clear whether one can replace in an arbitrary braided G-algebra the maps S^g by a twist of the coproduct.

Appendix B. Braiding Maps

Let V be a topologically free k-module and let $\mathrm{Map}_{z_1,z_2,\ldots,z_n}(V^{\otimes n})$ be the space of linear maps

$$V^{\otimes n} \to V^{\otimes n}[z_i^{\pm 1}, (z_i - z_j)^{-1}][[t]], \quad 1 \le i < j \le n.$$

Suppose we are given $S_{z_1,z_2} \in \mathrm{Map}_{z_1,z_2}(V^{\otimes 2})$ that satisfies

$$S_{z_1,z_2} \circ \tau \circ S_{z_2,z_1} \circ \tau = 1_{V^{\otimes 2}}, \tag{B.1}$$

$$S^{12}_{z_1,z_2} S^{13}_{z_1,z_3} S^{23}_{z_2,z_3} = S^{14}_{z_1,z_3} S^{13}_{z_1,z_3} S^{12}_{z_1,z_2}. \tag{B.2}$$

We then define for each $\mathbf{f} \in \mathcal{S}_n$ an element $S^{\mathbf{f}}_{z_1,\ldots,z_n} \in \mathrm{Map}_{z_1,z_2,\ldots,z_n}(V^{\otimes n})$ as follows. First, for $\mathbf{w}_i \in \mathcal{S}_n$ a simple transposition, define

$$S^{\mathbf{w}_i}_{z_1,\ldots,z_n} = 1^{i-1} \otimes S^{(\tau)}_{z_i,z_{i+1}} \otimes i^{n-i-1},$$

and extend this to $\mathbf{f} \in \mathcal{S}_n$ by expanding it in simple transpositions and using

$$S^{\mathbf{fg}}_{z_1,\ldots,z_n} = S^{\mathbf{g}}_{z_1,\ldots,z_n} \sigma_{\mathbf{g}} S^{\mathbf{f}}_{\mathbf{g}^{-1}(z_1,\ldots,z_n)} (\sigma_{\mathbf{f}})^{-1}. \tag{B.3}$$

The problem is that the expansion of $\mathbf{f}$ is not unique, because of the relations (10.1) and (10.2) in $\mathcal{S}_n$.

To address this problem introduce the free monoid $\mathcal{F}_n$ generated by symbols $\tilde{\mathbf{w}}_i$, $i = 1, 2, \ldots, n-1$. In $\mathcal{F}_n$ any element $\tilde{\mathbf{f}}$ has a unique expression in terms of the $\tilde{\mathbf{w}}_i$s. Consider the semi-direct product $\mathrm{Map}_{z_1,z_2,\ldots,z_n}(V^{\otimes n}) \rtimes \mathcal{S}_n$: elements of the semi-direct product are pairs $(A_{z_1,\ldots,z_n}, \mathbf{f})$, with product

$$(A_{z_1,\ldots,z_n}, \mathbf{f}).(B_{z_1,\ldots,z_n}, \mathbf{g}) = (A_{z_1,\ldots,z_n} \circ \sigma_{\mathbf{f}}^{-1} \circ B_{\mathbf{f}(z_1,z_2,\ldots,z_n)} \circ \sigma_{\mathbf{f}}, \mathbf{fg}). \tag{B.4}$$

We have a homomorphism $\mathcal{F}_n \to \mathcal{S}_n$, which maps generator $\tilde{\mathbf{w}}_i$ to simple transposition $\mathbf{w}_i$. Let

$$\phi\colon \mathcal{F}_n \to \mathrm{Map}_{z_1,z_2,\ldots,z_n}(V^{\otimes n}) \rtimes \mathcal{S}_n$$

be given on generators by

$$\phi(\tilde{\mathbf{w}}_i) = (S^{\mathbf{w}_i}_{z_1,\ldots,z_n}, \mathbf{w}_i),$$

and we extend this to all of $\mathcal{F}_n$ as an *anti-homomorphism* of monoids.

We need some more notation. If $\tilde{\mathbf{f}} = \tilde{\mathbf{w}}_{i_1}\tilde{\mathbf{w}}_{i_2}\ldots\tilde{\mathbf{w}}_{i_k} \in \mathcal{F}_n$, and the corresponding permutation is $\mathbf{f} = \mathbf{w}_{i_1}\mathbf{w}_{i_2}\ldots\mathbf{w}_{i_k} \in \mathcal{S}_n$, then introduce

$$\mathbf{g}_\ell = \mathbf{w}_{i_k}\mathbf{w}_{i_{k-1}}\ldots\mathbf{w}_{i_{\ell+1}}, \quad \ell = 1,2,\ldots,k-1,$$

and $\mathbf{g}_k = 1$.

Lemma B.1. *Let* $\tilde{\mathbf{f}} = \tilde{\mathbf{w}}_{i_1}\tilde{\mathbf{w}}_{i_2}\ldots\tilde{\mathbf{w}}_{i_k}$ *and* $\mathbf{f} = \mathbf{w}_{i_1}\mathbf{w}_{i_2}\ldots\mathbf{w}_{i_k}$. *Then*

$$\phi(\tilde{\mathbf{f}}) = (S^{\mathbf{f}}_{z_1,\ldots,z_n}, \mathbf{f}^{-1}) \in \mathrm{Map}_{z_1,z_2,\ldots,z_n}(V^{\otimes n}) \rtimes \mathcal{S}_n,$$

where

$$S^{\mathbf{f}}_{z_1,\ldots,z_n} = S^k S^{k-1}\ldots S^1 \sigma_{\mathbf{f}}^{-1}, \quad S^\ell = S^{\mathbf{w}_{i_\ell}}_{\mathbf{g}_\ell(z_1,z_2,\ldots,z_n)}\tau_{i_\ell}.$$

Furthermore, for $\tilde{\mathbf{f}}, \tilde{\mathbf{g}} \in \mathcal{F}_n$

$$S^{\mathbf{fg}}_{z_1,\ldots,z_n}\sigma_{\mathbf{fg}} = S^{\mathbf{g}}_{z_1,\ldots,z_n}\sigma_{\mathbf{g}} S^{\mathbf{f}}_{\mathbf{g}^{-1}(z_1,z_2,\ldots,z_n)}\sigma_{\mathbf{f}}. \tag{B.5}$$

Proof. Using the anti-homomorphism property of ϕ and the multiplication (B.4) we have

$$\begin{aligned}
\phi(\tilde{\mathbf{f}}) &= \phi(\tilde{\mathbf{w}}_{i_k})\phi(\tilde{\mathbf{w}}_{i_{k-1}})\ldots\phi(\tilde{\mathbf{w}}_{i_1}) \\
&= (S^{\mathbf{w}_{i_k}}_{z_1,\ldots,z_n}, \mathbf{w}_{i_k}).(S^{\mathbf{w}_{i_{k-1}}}_{z_1,\ldots,z_n}, \mathbf{w}_{i_{k-1}})\ldots(S^{\mathbf{w}_{i_1}}_{z_1,\ldots,z_n}, \mathbf{w}_{i_1}) \\
&= (S^{\mathbf{w}_{i_k}}_{z_1,\ldots,z_n}\tau_{i_k} S^{\mathbf{w}_{i_{k-1}}}_{\mathbf{w}_{i_k}(z_1,\ldots,z_n)}\tau_{i_k}, \mathbf{w}_{i_k}\mathbf{w}_{i_{k-1}}) \\
&\quad \cdot (S^{\mathbf{w}_{i_{k-2}}}_{z_1,\ldots,z_n}, \mathbf{w}_{i_{k-2}})\ldots(S^{\mathbf{w}_{i_1}}_{z_1,\ldots,z_n}, \mathbf{w}_{i_1}) \\
&= (S^k S^{k-1}\ldots S^1\sigma_{\mathbf{f}}^{-1}, \mathbf{f}^{-1}).
\end{aligned}$$

This proves the first part. Then

$$\begin{aligned}
\phi(\tilde{\mathbf{f}}\tilde{\mathbf{g}}) &= \phi(\tilde{\mathbf{g}})\phi(\tilde{\mathbf{f}}) \\
&= (S^{\mathbf{g}}_{z_1,\ldots,z_n}, \mathbf{g}^{-1}) \cdot (S^{\mathbf{f}}_{z_1,\ldots,z_n}, \mathbf{f}^{-1}) \\
&= (S^{\mathbf{g}}_{z_1,\ldots,z_n}\sigma_{\mathbf{g}} S^{\mathbf{f}}_{\mathbf{g}^{-1}(z_1,\ldots,z_n)}\sigma_{\mathbf{g}}^{-1}, \mathbf{g}^{-1}\mathbf{f}^{-1}).
\end{aligned}$$

Since $\sigma_{\mathbf{g}}^{-1}\sigma_{\mathbf{fg}} = \sigma_{\mathbf{f}}$ (B.5) follows. □

The observant reader might object to the notation $S^{\mathbf{f}}_{z_1,\ldots,z_n}$ used in the above Lemma: this map depends *a priori* on the element $\tilde{\mathbf{f}} \in \mathcal{F}_n$, not just on its image in $\mathcal{S}_n$. The following Lemma justifies the notation.

Lemma B.2. *The map* $\phi\colon \mathcal{F}_n \to \mathrm{Map}_{z_1,z_2,\dots,z_n}(V^{\otimes n}) \rtimes \mathcal{S}_n$ *factors through the canonical map* $\mathcal{F}_n \to \mathcal{S}_n$.

Proof. We need to check that the relations (10.1) and (10.2) (with $\mathbf{w}_i$ replaced by $\tilde{\mathbf{w}}_i$) belong to the kernel of ϕ. But we have

$$\begin{aligned}\phi(\tilde{\mathbf{w}}_i^2) &= \phi(\tilde{\mathbf{w}}_i)\phi(\tilde{\mathbf{w}}_i)\\ &= (S^{\mathbf{w}_i}_{z_1,\dots,z_n}, \mathbf{w}_i).(S^{\mathbf{w}_i}_{z_1,\dots,z_n}, \mathbf{w}_i)\\ &= (S^{\mathbf{w}_i}_{z_1,\dots,z_n}\tau_i S^{\mathbf{w}_i}_{\mathbf{w}_i(z_1,\dots,z_n)}\tau_i, \mathbf{w}_i\mathbf{w}_i)\\ &= (1,1)\end{aligned}$$

by the definition (B.3) and the property (10.3). Next, by the definition (B.3) and (10.2) we have, if $|i-j| \geq 2$,

$$\phi(\tilde{\mathbf{w}}_i\tilde{\mathbf{w}}_j) = \phi(\tilde{\mathbf{w}}_j)\phi(\tilde{\mathbf{w}}_i) = \phi(\tilde{\mathbf{w}}_i)\phi(\tilde{\mathbf{w}}_j) = \phi(\tilde{\mathbf{w}}_j\tilde{\mathbf{w}}_i).$$

Finally

$$\phi(\tilde{\mathbf{w}}_i\tilde{\mathbf{w}}_{i+1}\tilde{\mathbf{w}}_i) = \phi(\tilde{\mathbf{w}}_{i+1}\tilde{\mathbf{w}}_i\tilde{\mathbf{w}}_{i+1})$$

follows from the Yang–Baxter equation (10.4). □

The conclusion is that Definition 10.1 of $S^{\mathbf{f}}_{z_1,\dots,z_n}$ is well defined.

References

[1] I. I. Anguelova, Bicharacter constructions of quantum vertex algebras, Ph.D. thesis, University of Illinois, Urbana-Champaign (2006).

[2] R. E. Borcherds, Vertex algebras, Kac–Moody algebras, and the monster, *Proc. Nat. Acad. Sci. USA* **83**(10) (1986) 3068–3071; MR MR843307 (87m:17033).

[3] ———, Vertex algebras, in *Topological Field Theory, Primitive Forms and Related Topics* (*Kyoto, 1996*), Progr. Math., Vol. 160 (Birkhäuser Boston, Boston, MA, 1998), pp. 35–77; MR MR1653021 (99m:17034).

[4] ———, Quantum vertex algebras, in *Taniguchi Conference on Mathematics Nara '98*, Adv. Stud. Pure Math., Vol. 31 (Math. Soc. Japan, Tokyo, 2001), pp. 51–74; MR MR1865087 (2002k:17054).

[5] P. Etingof and D. Kazhdan, Quantization of Lie bialgebras. V. Quantum vertex operator algebras, *Selecta Math.* (*N.S.*) **6**(1) (2000) 105–130; MR 2002i:17022.

[6] I. Frenkel, J. Lepowsky and A. Meurman, *Vertex Operator Algebras and the Monster*, Pure and Applied Mathematics, Vol. 134 (Academic Press Inc., Boston, MA, 1988); MR MR996026 (90h:17026).

[7] E. Frenkel and N. Reshetikhin, Towards deformed chiral algebras, in *Proc. Quantum Group Symposium at the XXIth International Colloquium on Group Theoretical Methods in Physics*, Goslar, Germany (1996), 18pp.; arXiv:q-alg/9706023.

[8] N. H. Jing, Vertex operators and Hall–Littlewood symmetric functions, *Adv. Math.* **87**(2) (1991) 226–248; MR MR1112626 (93c:17039).

[9] V. Kac, *Vertex Algebras for Beginners*, University Lecture Series, Vol. 10, 2nd edn. (American Mathematical Society, Providence, RI, 1998); MR MR1651389 (99f:17033).

[10] C. Kassel, *Quantum Groups*, Graduate Texts in Mathematics, Vol. 155, (Springer-Verlag, New York, 1995); MR MR1321145 (96e:17041).

[11] H. Li, Axiomatic G_1-vertex algebras, *Commun. Contemp. Math.* **5**(2) (2003) 281–327; MR MR1966260 (2004e:17026).

[12] ———, Nonlocal vertex algebras generated by formal vertex operators, *Selecta Math.* (*N.S.*) **11**(3–4) (2005) 349–397; MR MR2215259.

[13] ———, Constructing quantum vertex algebras, *Internat. J. Math.* **17**(4) (2006) 441–476; MR MR2220654.

[14] ———, A new construction of vertex algebras and quasi-modules for vertex algebras, *Adv. Math.* **202**(1) (2006) 232–286; MR MR2218823.

[15] J. Lepowsky and H. Li, *Introduction to Vertex Operator Algebras and Their Representations*, Progress in Mathematics, Vol. 227 (Birkhäuser Boston Inc., Boston, MA, 2004); MR MR2023933 (2004k:17050).

[16] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Oxford Mathematical Monographs, 2nd edn. (The Clarendon Press Oxford University Press, New York, 1995); With contributions by A. Zelevinsky; MR MR1354144 (96h:05207).

[17] C. T. Snydal, Equivalence of Borcherds G-vertex algebras and axiomatic vertex algebras, arXiv:math.QA/9904104.