

A note on quantum vertex operators and associativity¹

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Abstract

The purpose of this note is to show that many of the examples of quantum vertex operators do *not* satisfy vertex operator associativity.

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1 Introduction

Vertex operators were introduced in the earliest days of string theory and axioms for vertex algebras were developed to incorporate these examples (see for instance [4, 9, 13]). Vertex algebras are known to have the so called “associativity” property, as well as locality (or “commutativity”) property. In [3] a notion of a field algebra was developed, which generalizes the notion of a vertex algebra. Briefly, field algebras only obey the associativity, but not the commutativity property (see section 2 for a precise definition).

Besides vertex operators, there are many examples in the literature of quantum vertex operators (see for instance [8, 7, 10, 2], and many others). A definition of a quantum vertex algebra should be such that it accommodates those examples of quantum vertex operators and their properties. There are several proposals for a definition of a quantum vertex algebra. They include Borcherds’ theory of (A, H, S)-vertex algebras, see [5], the Etingof-Kazhdan theory of quantum vertex algebras [6], the Frenkel-Reshetikhin theory of deformed chiral algebras, see [11]. (H. Li has developed the Etingof-Kazhdan theory further, see for example [14]). One of the major and well known differences between quantum vertex algebras and the usual nonquantized vertex algebras is that the quantum vertex operators can no longer satisfy a locality (or “commutativity”) axiom, and there is instead a braiding map controlling the failure of locality. In particular, the Etingof-Kazhdan definition of a quantum vertex algebra can be described as a field algebra with a braiding map. Thus their examples necessarily satisfy the associativity property [6]. Also, the author together with M. Bergvelt proposed in [1] a notion of an H_D -quantum vertex algebra (where $H_D = \mathbb{C}[D]$ is the Hopf algebra of infinitesimal translations), generalizing the Etingof-Kazhdan theory of quantum vertex algebras in various ways. In particular, the definition of an H_D -quantum vertex algebra introduces, besides the braiding map, a translation map controlling the failure of translation covariance. (As we will see in section 3 most examples of quantum vertex operators fail to satisfy the usual translation covariance). An H_D -quantum vertex algebra essentially specializes to an Etingof-Kazhdan (EK) quantum vertex algebra in the case when the translation map is identity. A major difference from EK quantum vertex algebras is that the H_D -quantum vertex algebras are in general *not* field algebras. Part of the motivation for such a definition of a quantum vertex algebra is explained in this paper: many of the examples of quantum vertex operators in the literature *cannot* belong to a field algebra, i.e., they do not satisfy the associativity property. Thus in order to incorporate those examples, one needs to give a more general definition, such as in [1].

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The outline of the paper is as follows. In the next section we recall the definition of a field algebra, vertex algebra and EK quantum vertex algebra. In particular we prove a technical lemma stating that associativity implies translation covariance (thus it is not necessary to include translation covariance as part of the definition of a field algebra). Next we summarize a class of examples of quantum vertex operators from the literature. We then show that they fail to satisfy the translation covariance, thus they do *not* obey the associativity property.

2 Field algebras and vertex algebras

We work over a field \mathbf{k} of characteristic zero containing the rationals. Let V be a vector space over \mathbf{k} .

Definition 2.1 (field [9, 13]). A field $a(z)$ on V is a series of the form

$$a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}, \quad a_{(n)} \in \text{End}(V), \quad \text{such that } a_{(n)} v = 0, \quad \forall v \in V, n \gg 0$$

Definition 2.2 (state-field correspondence [9, 13]). A state-field correspondence is a linear map Y from V to the space of fields that associates to any $a \in V$ a field $Y(a, z) = a(z)$. In this case we say that the field $Y(a, z) = a(z)$ is the vertex operator corresponding to the state $a \in V$.

Notation 2.1. For a meromorphic function $f(z, w)$ we denote by $i_{z,w} f(z, w)$ the expansion of $f(z, w)$ in the region $|z| \gg |w|$ (i.e., in powers of $\frac{w}{z}$), and similarly for $i_{w,z} f(z, w)$.

Definition 2.3 (field algebra [3, 13]). A field algebra consists of the following data:

- the space of states—a vector space V .
- the vacuum vector—a vector $|0\rangle \in V$.
- the space of fields and state-field correspondence.
- a distinguished operator $D : V \rightarrow V$.

These data should satisfy the following set of axioms

- **vacuum axioms:** $Y(|0\rangle, z) = Id_V$, $Y(a, z)|0\rangle = e^{zD} a$;
- **associativity axiom:** for all $a, b, c \in V$ there exists an element $X_{z,w,0} \in V[[z, w]][[z^{-1}, w^{-1}, (z-w)^{-1}]$ such that:

$$i_{z,w} X_{z,w,0}(a \otimes b \otimes c) = Y(a, z)Y(b, w)c \in V((z))((w)) \quad (2.1)$$

$$i_{w,(z-w)} X_{z,w,0}(a \otimes b \otimes c) = Y(Y(a, z-w)b, w)c \in V((w))((z-w)) \quad (2.2)$$

Definition 2.4 (vertex algebra [9, 13]). A vertex algebra is a field algebra which satisfies the commutativity axiom

$$X_{z,w,0}(a \otimes b \otimes c) = X_{w,z,0}(b \otimes a \otimes c)$$

Remark 2.1 (EK quantum vertex algebra [6]). An EK quantum vertex algebra (see [6] for a precise definition), is a field algebra with additional structure: the *braiding* map (satisfying certain properties)

$$S_{z-w}^{(\tau)} : V^{\otimes 2} \rightarrow V^{\otimes 2}[(z-w)^{\pm}][[t]]$$

such that we have the following *braided commutativity axiom*:

$$X_{z,w,0}(a \otimes b \otimes c) = X_{w,z,0}(S_{w-z}^{(\tau)}(b \otimes a) \otimes c)$$

In fact, EK quantum vertex algebra is defined differently, but in the case when the classical limit is a nondegenerate vertex algebra, the above definition is equivalent to the original one (see [6] for particulars).

Lemma 2.1. *Any vertex operator in a field algebra satisfies the following translation covariance properties:*

$$\partial_z Y(a, z) = DY(a, z) - Y(a, z)D \tag{2.3}$$

$$Y(Da, z) = \partial_z Y(a, z) \tag{2.4}$$

Proof. As for any a, b, c in V $X_{z,w,0}$ is in $V[[z, w]][z^{-1}, w^{-1}, (z-w)^{-1}]$, there exists $N \in \mathbb{N}$ such that

$$(z-w)^N i_{z,w} X_{z,w,0}(a \otimes b \otimes c) = (z-w)^N i_{w,z} X_{z,w,0}(a \otimes b \otimes c)$$

Change variables in equation (2.2) to get

$$i_{w,z} X_{z+w,w,0}(a \otimes b \otimes c) = Y(Y(a, z)b, w)c \in V((w))((z))$$

from whence it follows that

$$(z+w)^N i_{z,w} Y(a, z+w)Y(b, w)c = (z+w)^N Y(Y(a, z)b, w)c$$

Substitute the state c to be the vacuum $|0\rangle$ and use the vacuum axiom to get

$$i_{z,w} Y(a, z+w)e^{wD}b = e^{wD}(Y(a, z)b)$$

Now comparing the coefficients of w^1 we have: $\partial_z Y(a, z) + Y(a, z)Db = D(Y(a, z)b)$, which completes the proof of (2.3). The second property is proved similarly, by putting $b = |0\rangle$. \square

3 Quantum vertex operators and field algebras

Many of the quantum vertex operators in the literature can be presented in the following way: Let \mathbb{F} be a field extension of \mathbf{k} (for instance $\mathbf{k}(t)$ or $\mathbf{k}(q, t)$, where q, t are parameters).

Notation 3.1. Let $v_n \in \mathbb{F}$ for any $n \in \mathbb{N}$. Denote by \mathcal{H}_v the (deformed) Heisenberg algebra with generators h_n , $n \in \mathbb{Z}$, and relations

$$[h_m, h_n] = mv_{|m|}\delta_{m+n,0}1 \tag{3.1}$$

Let the vector space V_0 be the highest weight module with highest weight vector $|0\rangle = 1$, such that $h_n|0\rangle = 0$ for $n \in \mathbb{N}$, $h_0|0\rangle = 0$. Let $V = V_0 \otimes \mathbf{k}[\mathbb{Z}\alpha] = V_0 \otimes \mathbf{k}[e^\alpha, e^{-\alpha}]$. Denote by $\Psi_v(z)$ and $\Psi_v^{-1}(z)$ the following fields on V (called the *exponentiated bosons*):

$$\Psi_v(z) = \exp\left(\sum_{n \geq 1} \frac{h_{-n}}{n} z^n\right) \exp\left(-\sum_{n \geq 1} \frac{h_n}{n} z^{-n}\right) e^\alpha z^{\partial_\alpha} \tag{3.2}$$

$$\Psi_v^{-1}(z) = \exp\left(-\sum_{n \geq 1} \frac{h_{-n}}{n} z^n\right) \exp\left(+\sum_{n \geq 1} \frac{h_n}{n} z^{-n}\right) e^{-\alpha} z^{-\partial_\alpha} \tag{3.3}$$

Examples:

- $v_{|n|} = 1$ – corresponds to the Schur symmetric functions,
- $v_{|n|} = 1 - t^n$ – corresponds to the Hall-Littlewood symmetric functions,
- $v_{|n|} = \frac{1-t^n}{1-q^n}$ – Macdonald case,
- $v_{|n|} = \frac{(1-q^n)(1-(\frac{p}{q})^n)}{1+p^n}$ – the deformed Virasoro algebra.

For convenience we also introduce the reduced exponentiated boson field $\Phi(z)$ and its modes:

$$\Phi^v(z) = \exp\left(\sum_{n \geq 1} \frac{h_{-n}}{n} z^n\right) \exp\left(-\sum_{n \geq 1} \frac{h_n}{n} z^{-n}\right) = \sum_{n \in \mathbb{Z}} \Phi_n^v z^n$$

We recall the following results.

Fact (undeformed case). Let $v_n = 1$ for any $n \in \mathbb{N}$. If λ be a partition, $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$, then

$$\Phi_{\lambda_1}^1 \Phi_{\lambda_2}^1 \dots \Phi_{\lambda_k}^1 |0\rangle = S_\lambda$$

where S_λ is the Schur function corresponding to the partition λ . (We identify $h_{-n}1$ with the power symmetric function p_n .) Moreover the fields $\Psi_1(z)$ and $\Psi_1^{-1}(z)$ generate a vertex operator super-algebra (see [13]) corresponding to the rank one odd lattice.

Fact (Hall-Littlewood case). Let $v_{|n|} = 1 - t^n$. If λ is a partition, $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$, then

$$\Phi_{\lambda_1}^t \Phi_{\lambda_2}^t \dots \Phi_{\lambda_k}^t |0\rangle = H_\lambda$$

where H_λ is the Hall-Littlewood function corresponding to the partition λ (due to Naihuan Jing, [12]). Clearly, the exponentiated boson doesn't belong to a super vertex algebra, as it doesn't satisfy the commutativity axiom. In fact we have

$$(1 - tz_1/z_2)\Phi^t(z_1)\Phi^t(z_2) = -(1 - tz_2/z_1)\Phi^t(z_2)\Phi^t(z_1)$$

The quantum fields $\Psi_t(z)$ and $\Psi_t^{-1}(z)$ are the simplest quantum deformation of the fields $\Psi_1(z)$ and $\Psi_1^{-1}(z)$ (the last can be obtained from $\Psi_t(z)$ and $\Psi_t^{-1}(z)$ by letting $t = 0$). One can ask, does it belong to a field algebra? Below we prove that this is not possible.

Theorem 3.1. *The exponentiated boson fields $\Psi(z)$ and $\Psi^{-1}(z)$ defined in (3.2) and (3.2) can not be vertex operators in a field algebra, except in the undeformed case, $v_n = 1$ in (3.1) for any $n \in \mathbb{N}$.*

Proof. Consider for simplicity the deformed case where $v_1 \neq 1$. The other deformed cases (we have to have at least for one $\tilde{n} \in \mathbb{N}$ $v_{\tilde{n}} \neq 1$) are proved similarly.

Suppose the opposite is true, i.e., that the fields $\Psi(z)$ and $\Psi^{-1}(z)$ belong to a field algebra. Then by the creation axiom we have $\Psi(z) = Y(e^\alpha, z)$ for the state e^α of some field algebra, and $\Psi^{-1}(z) = Y(e^{-\alpha}, z)$ for the state $e^{-\alpha}$. Then both those fields should satisfy the equation (2.3) and the equation (2.4).

Note that by application to the vacuum itself the vacuum axioms immediately imply $D|0\rangle = 0$. For any vertex operator $Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}$ we have

$$\partial_z Y(a, z)|0\rangle = D(Y(a, z)|0\rangle)$$

which implies $Da = a_{(-2)}1$. In particular for the state e^α we have $De^\alpha = h_{-1}e^\alpha$. If we apply (2.3) we should have

$$Y(e^{-\alpha}, z)De^\alpha = -\partial_z Y(e^{-\alpha}, z)e^\alpha + D(Y(e^{-\alpha}, z)e^\alpha) \quad (3.4)$$

We have

$$\begin{aligned}\partial_z Y(e^{-\alpha}, z)e^\alpha &= \partial_z \left(z^{-1} \exp \left(\sum_{n \geq 1} \frac{h_{-n}|0\rangle}{n} z^n \right) \right) \\ Y(e^{-\alpha}, z)De^\alpha &= z^{-2}[h_{-1}, h_1] \exp \left(\sum_{n \geq 1} \frac{h_{-n}|0\rangle}{n} z^n \right)\end{aligned}$$

As $D(Y(e^{-\alpha}, z)e^\alpha)$ has no term z^{-2} , a simple comparison of both sides of (3.4) shows that this could only be valid when $[h_{-1}, h_1] = -1$, i.e., we have a contradiction. \square

Thus if we want to construct a system of axioms for a quantum vertex algebra containing the fields $\Psi(z)$ and $\Psi^{-1}(z)$ as vertex operators for some elements of the space of states we have to go beyond field algebras. In particular we have to relax the translation covariance axiom. This is what we do in our definition of H_D -quantum vertex algebras in [1]. H_D -quantum vertex algebras satisfy (2.4), but not (2.3). Their properties do not include associativity, but rather *braided* associativity, as we want to include the fields $\Psi(z) = Y(e^\alpha, z)$ and $\Psi^{-1}(z) = Y(e^{-\alpha}, z)$ as vertex operators among our examples. The braided associativity reflects the failure of the translation covariance property (2.3) (see [1] for particulars).

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