College of Charleston
Department of Mathematics

Name:

Examination in: Discrete Mathematics

Math Course Number	Math 207
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Total number of problems: 8

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1. Let $f(n) = 1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3$ for $n \in \mathbb{Z}^+$.

a) Is f a one-to-one function from the set of positive integers to the set of positive integers? Is f an onto function from the set of positive integers to the set of positive integers? Explain the reason behind your answers.

Sln: $f(n)$ is one-to-one: Suppose $f(m) = f(n)$ with $m > n > 0$.

$$\begin{aligned} f(n) = f(m) &\to 1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = 1^3 + 2^3 + 3^3 + 4^3 + \cdots + m^3 \\ &\to 0 = (n+1)^3 + \cdots + m^3 > 0 \text{ which is impossible.} \end{aligned}$$

Hence $m = n$.

For $m > 1$ we have $f(m) = 1^2 + 2^3 + \cdots + m^3 \geq 1 + 8 = 9$. Hence $2 \notin \mathrm{im} f$ and f can not be onto.

b) Show that $1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3$ is $O(n^4)$. Be sure to specify the values of the witnesses C and k.

Sln: One way to do this is to recall from calculus that for $n \geq 1$,

$$1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4} = \frac{n^4 + 2n^3 + n^2}{4} \leq \frac{n^4 + 2n^4 + n^4}{4} = n^4$$

and then the later is $\mathcal{O}(n^4)$. Here the witnesses would be $C = 1$ and $k = 1$. Another way is

$$1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 \leq n^3 + n^3 + n^3 + n^3 + \cdots + n^3 = n^4 \qquad n \text{ summands}$$

here the witness would be $k = 1$ and $C = 1$.

2. Describe an algorithm for finding the second largest integer in a finite sequence of distinct integers.

Sln:

procedure $secondlargest(a_1, \ldots, a_n:$ integers)
$largest := a_1$
$secondlargest := a_2$
if $a_2 > a_1$ **then**
  begin
    $secondlargest := a_1$
    $largest := a_2$
  end
if $n = 2$ **then**
  stop
for $i := 3$ **to** n
if $a_i > largest$ **then**
  begin
    $secondlargest := largest$
    $largest := a_i$
  end
if ($a_i > secondlargest$ **and** $a_i \leq largest$) **then** $secondlargest := a_i$.

3. Let m be a positive integer, and let a, b, and c be integers. Show that if $a \equiv b \mod m$, then $a - c \equiv b - c \mod m$.

Proof:

$$\begin{aligned} a \equiv b \mod m &\leftrightarrow m|a-b \\ &\to \exists k \in \mathbb{Z}\,(mk = a-b) \\ &\to \exists k \in \mathbb{Z}\,(mk = a-c-b+c = a-c-(b-c)) \\ &\to \exists k \in \mathbb{Z}\,(m|a-c-(b-c)) \\ &\to a-c \equiv b-c \mod m. \end{aligned}$$

4. Let $A = \begin{pmatrix} 1 & -1 \\ 2 & 0 \\ 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 \\ 2 & 1 \\ 3 & 2 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & -3 \\ 2 & -1 \end{pmatrix}$.

a) Find $A+B$.

Sln:

$$A + B = \begin{pmatrix} 3 & 0 \\ 4 & 1 \\ 5 & 5 \end{pmatrix}$$

b) Find AC.

Sln:

$$AC = \begin{pmatrix} -1 & -2 \\ 2 & -6 \\ 8 & -9 \end{pmatrix}.$$

c) If $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, find $A \odot C$.

$$A \odot C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} (1\wedge 1)\vee(1\wedge 0) & (1\wedge 1)\vee(1\wedge 1) \\ (0\wedge 1)\vee(0\wedge 0) & (0\wedge 1)\vee(0\wedge 1) \\ (0\wedge 1)\vee(1\wedge 0) & (0\wedge 1)\vee(1\wedge 1) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

5. Find the prime factorization of 111111.

Sln: This is not divisible by 2, but $3 \cdot 37037 = 111111$. Now $37 \cdot 1001 = 37037$, hence we just need to see if 1001 factors. We need only check primes less then or equal to $\sqrt{1001}$ which is approximately 33. $2, 3$ and 5 don't divide it. $7 \cdot 143 = 1001$. Next $2, 3, 5$ and 7 don't divide 143, but $11 \cdot 13 = 143$ Hence it is prime and $111111 = 3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$.

6. Find an integer k such that $0 \le k < 17$ such that $2^{18} \equiv k \mod 17$.

Sln: $2^4 = 16 \equiv -1 \mod 17$. Hence $2^{16} = (2^4)^4 \equiv (-1)^4 \mod 17$ so that $2^{18} = (2^4)^4 \cdot 2^2 \equiv 4 \mod 17$.

7. Use the Euclidean algorithm to find

a) $\gcd(201, 302)$,

Sln: $\gcd(201, 302) = 1$

b) $\gcd(144, 233)$.

Sln:$\gcd(144, 233) = 1$.

8. a) What is the binary expansion of $(1010101)_2 + (1111111)_2$?

Sln: $(11010100)_2$.

b) What is the binary expansion of $(10101)_2 \cdot (11111)_2$?

Sln: $(1010001011)_2$.