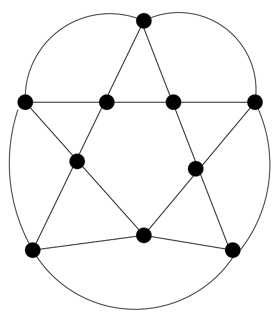
MATH 103: Contemporary Mathematics

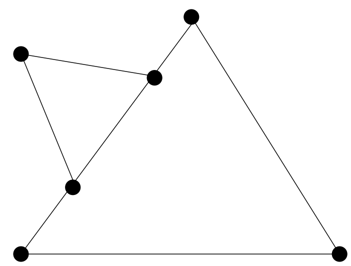
Study Guide for Chapter 1: Euler Circuits including Eulerization

Note: Test 3 will be mostly focused on Eulerization.

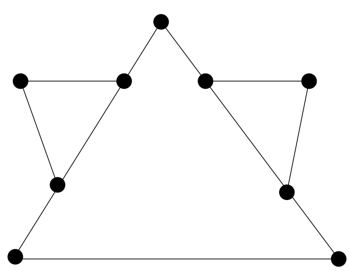
1. Use the graphs below to practice the following:
   (a) Determine which ones have Euler Circuits.
   (b) For the graphs which have an Euler Circuit, use Fleury's Algorithm (the method involving making a copy of the graph and erasing edges as you trace the route) to find such circuit.



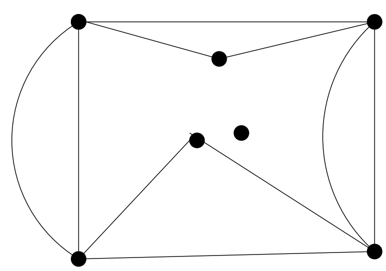
Graph 1



Graph 2



Graph 3



Graph 4

2. Review Chapter 1 as well as the class notes, beginning with the graph model for the the Königsberg bridge problem. Review the definitions of *edge*, *vertex*, *graph*, *valence* of a vertex, *path*, *loop*, *circuit*, *connected graph*. Do problems 1-18 of the Skills Check, p. 22-24.

3. Explain the difference between a path and a circuit.

4. Give the definition of Euler circuit.

5. What determines whether a given circuit **is** an Euler circuit?

6. What determines whether a given graph **has** an Euler curcuit?

7. You will now be asked to give examples of graphs with certain properties. You can look at the examples in your book and notes, but you must come up with your own graph (be creative!). (Note that every edge must terminate in two vertices.)

   (a) Give an example of a graph in which all the vertices have even valence.

   (b) Give an example of disconnected graph.

   (c) Give an example of a graph with at least one edge whose removal causes the graph to become disconnected (such an edge is called a *bridge*). Highlight this edge in your graph.

   (d) Draw a graph in which every edge, if removed, causes the graph to become disconnected (i.e. in which every edge is a bridge).

8. Before working the next problems, read Euler's First Theorem on page 9.

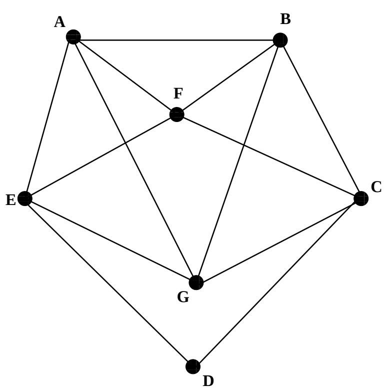
   (a) Use Euler's First Theorem to provide two examples of graphs which have no Euler circuit for two different reasons.

   (b) Use Euler's First Theorem to construct several graphs which have Euler circuits. (You do not need to know what these circuits are in order to construct such examples. To show this, make one of your graphs have many vertices.)

9. State here the famous (and useful!) Euler's Third Theorem (by the way, this is also known as the *Handshaking Theorem*.) Do you remember what reasoning we used to justify it? (It was our first rigorous proof.)

10. If you understand Euler's Third Theorem, complete the following sentences:
    (a) A graph with 33 edges has a total valence of ______________________________
    (b) In a graph with N vertices of which 2 vertices have valence 2 and all others have valence 1, there are ______________________ edges.
    (c) Statement:"*Each of the 7 vertices of a given graph has odd valence*".
    This statement is __________ (Justify.)

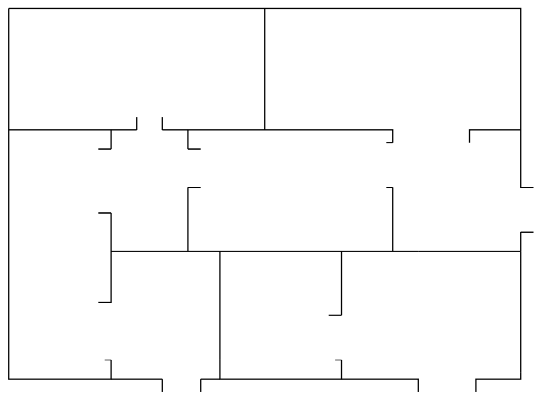
11. Use Fleury's algorithm to find an Euler circuit for the graph below. Practice problems 27, 28, 43 pp 28–31.You may demonstrate that you found an Euler's circuit by either
    (a) numbering the edges in the order in which you travel along them, or
    (b) listing the vertices in the order in which you travel through them.



12. In Fleury's algorithm for finding an Euler circuit we do not travel an edge which, if removed, will disconnect the untravelled portion of the graph.
    (a) What would happen if we travelled such an edge? Why do we avoid travelling such an edge?
    (b) If we travelled such an edge, would we be "cut off" from the starting vertex? (careful, the correct answer requires some thinking and maybe an example). Explain carefully.

(c) Think about the difference in the phrases "edge that disconnects the graph, if removed" and "edge that disconnects the untravelled portion of the graph, if removed". The word **untravelled** in the second phrase is very important in Fleury's algorithm.

13. We want to answer the following question regarding a suite of rooms in a building: *Can we enter the suite, go through each door (including the second door to the hallway) only once and get back to the door with started at?*



Given the illustration of the suite shown above, work through the following steps:

(a) Construct a graph that models the situation.

(b) Does your graph have an Euler Circuit? Explain your answer.

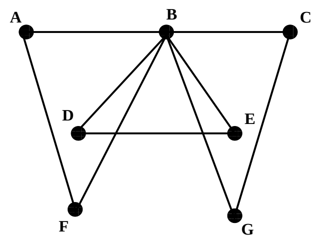
(c) What is the minimum number of doors that you would need to lock in order for the answer to the original question to be yes?

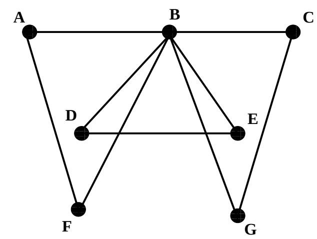
Problems on Eulerization

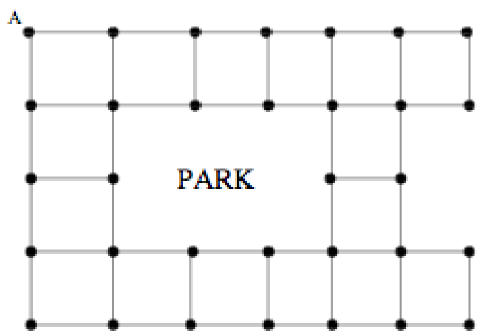
14. Practice Eulerizing graphs. Remember that when you "duplicate existing" edges, "duplication" must be interpreted as travelling twice along an edge of the graph given initially. Remember also that you cannot "invent" entirely new edges, you can only duplicate already existing ones. Good problems to practice are 31, 35, 36, 38, 46 on pages 30-32.

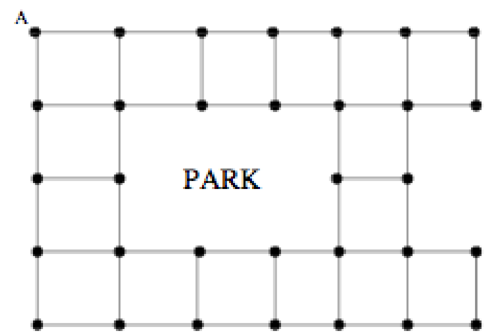
15. Work problem 35 p. 30 in the space below.

16. Eulerize the following graphs. Each graphs comes in two copies, on one copy show a non-optimal eulerization, on the other show an optimal one.









17. For the first of the two set of graphs above, use Fleury's algorithm to find a circuit that retraces the least number of edges. List this circuit below as a list of vertices. (Hint: you will have to work on the optimally eulerized graph.)

18. Review the concept of **optimal** Eulerization and then answer the following question: "Suppose a given graph has some vertices of odd valence and some of even valence. We count 24 odd valence vertices and we proceed to find its Eulerization. If, to do that, we have to duplicate 12 edges, is our Eulerization optimal? Explain."

19. When weights are added to the edges of a graph and you are asked for the "minimal weight" route that covers all the edges, the method of Eulerization is still used to solve this problem in the case the graph has no Euler circuit. Be aware though that an optimal Eulerization may not necessarily produce the "minimal weight" route.

   (a) Work problem 42 p. 31 in the space below.

   (b) Work problem 40 page 31.

(Problem 41 on the same page is another problem of this type.)