

DIMENSIONS OF \mathbb{R} -TREES AND SELF-SIMILAR FRACTAL SPACES OF NONPOSITIVE CURVATURE

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We study different dimensions of geodesic spaces with nonpositive curvature in Aleksandrov sense ($CAT(0)$ -spaces), in particular, \mathbb{R} -trees. We prove the following statements.

Theorem 1. *Geodesic metric space is \mathbb{R} -tree if and only if it is $CAT(K)$ -space for every $K \leq 0$ and contains more than one point.*

Theorem 2. *Geodesic metric space is \mathbb{R} -tree if and only if it is simply connected and its (small inductive) Urysohn-Menger dimension is equal to 1.*

Theorem 3. *1. There exists a compact \mathbb{R} -tree X , which satisfies the following inequalities*

$$\dim_T X = 1 < \dim_H X < \dim_E X = \dim_r X \quad (0.1)$$

for its topological $\dim_T X$, Hausdorff $\dim_H X$, entropy $\dim_E X$, and rough $\dim_r X$ dimensions. The value $\dim_H X$ can be arbitrary large, while $\dim_E X$ can be any desired real number more than $\dim_H X$.

2. There exists a compact \mathbb{R} -tree with infinite Hausdorff dimension.

Question 1. *In constructed examples $\dim_E X = \dim_r X$. Are there analogous \mathbb{R} -trees with additional inequality*

$$\dim_E X < \dim_r X?$$

Theorem 4. *There exists a compact $CAT(0)$ -space Y which satisfies all statements of Theorem 3 but with changing the equality $\dim_T X = 1$ by the equality $\dim_T Y = n$ for any natural number n .*

Let us separate especially simple construction of some \mathbb{R} -tree X . Denote by $\Delta_S(k)$, $k \geq 2$, self-similar fractal k -simplex of Serpiński supplied with induced inner (geodesic) metric and separate one of its $k + 1$ vertices v_0 . Then consider $X' = \Delta_S(k) - S(v_0)$, where $S(v_0)$ is the cut loci of the point v_0 in $\Delta_S(k)$. Then take the induced inner metric ρ on X' and define X as the metric completion of (X', ρ) . Then X will be compact self-similar \mathbb{R} -tree of the same Hausdorff dimensions as $\Delta_S(k)$. We have also

$$\dim_T X = 1 < \dim_H X = \dim_E X = \dim_r X = \log_2(k + 1). \quad (0.2)$$

The space X admits another description in the spirit of *symbolic dynamics* using sequences on $k + 1$ symbols. Notice in addition that this X is a *nested fractal* in the sense of Lindstrom, for which he constructed in [1] a theory of Brownian motion.

REFERENCES

- [1] Tom Lindstrom, *Brownian motion on nested fractals*, Memoirs of AMS, Providence, RI, USA: January 1990, Vol. 83, No. 420.